Problem 1.

Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

Solution

Define the mapping \( f : \mathbb{R} \to (-1, 1) \) as

\[
f(x) = \frac{x}{1 + |x|}
\]

\( f \) is a continuous bijection (note \( f(0) = 0 \)), where the inverse \( f^{-1} : (-1, 1) \to \mathbb{R} \) is

\[
f^{-1}(y) = \frac{y}{1 - |y|}
\]

which is also continuous at all \(-1 < y < 1\) (again note \( f^{-1}(0) = 0 \)). Thus \( f \) is a homeomorphism. With the usual metric, \( \mathbb{R} \) is complete but \((-1, 1)\) is incomplete.
Problem 2.

Let \((E, d)\) be a metric space and \(S \subset E\) a subset. Show that \(A \subset S\) is open relative to \(S\) if and only if \(A = S \cap U\) for some \(U \subset E\) open.\(^1\)

Solution

(⇒) Suppose \(A \subset S\) is open relative to \(S\). This means that \(\forall x \in A, \exists r_x > 0\) such that \(B_{r_x}(x) \cap S \subset A\). Define \(U = \bigcup_{x \in A} B_{r_x}(x)\). Since \(U\) is the union of open balls, it is open.

Note that \(A \subset U\) and \(A \subset S\). Then, \(A \subset S \cap U\). Also, \(S \cap U = \bigcup_{x \in A} B_{r_x}(x) \cap S\). And since \(A\) is open relative to \(S\), \(B_{r_x}(x) \cap S \subset A\), so is the union. Then, \(S \cap U \subset A\). Then, \(A = S \cap U\).

(⇐) If \(A = S \cap U\) for an open subset \(U \subset E\), then \(\forall x \in A, \exists r_x > 0\) such that \(B_{r_x}(x) \subset U\). This implies that \((B_{r_x}(x) \cap S) \subset (U \cap S) = A\). Then, \(A\) is open relative to \(S\).

\(^1\) \(A \subset S\) is open relative to \(S\) if \(\forall x \in A \exists r_x > 0\) such that \(B_{r_x}(x) \cap S \subset A\).
Problem 3.

Let \((X, d)\) be a metric space. Assume \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) are uniformly continuous on \((X, d)\) and \((\mathbb{R}, | \cdot |)\), with \(| \cdot |\) the absolute-value norm.

(a) Show that \(f + g : X \to \mathbb{R}\) is uniformly continuous, where \((f + g)(x) = f(x) + g(x)\).

(b) Show that \(\max\{f, g\} : X \to \mathbb{R}\) is uniformly continuous, where \(\max\{f, g\}(x) = \max\{f(x), g(x)\}\).

(c) Give a counterexample to the following statement: \(f \cdot g : X \to \mathbb{R}\) is uniformly continuous on \((X, d)\) and \((\mathbb{R}, | \cdot |)\), where \(f \cdot g = f(x) \cdot g(x)\).

Solution

(a) Since \(f\) and \(g\) are uniformly continuous, we know that \(\forall \epsilon' > 0\) and \(\forall x_0 \in X\), \(\exists \delta' > 0\) such that if \(d(x, x_0) < \delta'\), then \(|f(x) - f(x_0)| < \epsilon'\) and \(|g(x) - g(x_0)| < \epsilon'\). Given \(\epsilon'\), let \(\epsilon = 2\epsilon'\) and \(\delta = \delta'\). Assume \(d(x, x_0) < \delta\). Then

\[
| (f + g)(x) - (f + g)(x_0) | = |(f(x) + g(x)) - (f(x_0) + g(x_0)) | = |(f(x) - f(x_0)) + (g(x) - g(x_0)) | \\
\leq |f(x) - f(x_0) | + |g(x) - g(x_0) | < \epsilon' + \epsilon' = \epsilon.
\]

Then, \(f + g\) is uniformly continuous.

(b) Let \(x, y \in X\). Without loss of generality, suppose \(\max(f(x), g(x)) \geq \max(f(y), g(y))\) (otherwise switch the roles of \(x\) and \(y\) here). Then

\[
f(x) - \max(f(y), g(y)) \leq f(x) - f(y)
\]

and

\[
g(x) - \max(f(y), g(y)) \leq g(x) - g(y)
\]

So

\[
0 \leq \max(f(x), g(x)) - \max(f(y), g(y)) = \max(f(x) - \max(f(y), g(y)), g(x) - \max(f(y), g(y))) \leq \max(|f(x) - f(y)|, |g(x) - g(y)|) \leq \max(|f(x)|, |g(x)|) \leq |f(x)| + |g(x)| \\
\]

From here, the argument can be finished using the uniform continuity of \(f\) and \(g\).

(c) Take \(f(x) = g(x) = x\), so \((f \cdot g)(x) = x^2\), which is not uniformly continuous. To see this, consider the following example. Let \(\epsilon = 1\). Fix \(\delta > 0\); without loss of generality take \(\delta \leq 1\). Then let \(x = 2/\delta\) and \(y = 2/\delta - \delta\). So

\[
|x - y| = \delta \text{ and } x + y = \frac{4}{\delta} - \delta = \frac{4 - \delta^2}{\delta}
\]

So

\[
|x^2 - y^2| = |x + y||x - y| = 4 - \delta^2 > 1 = \epsilon
\]
Problem 4.
A function $f : X \rightarrow Y$ is open if $\forall A \subset X$ such that $A$ is open, $f(A)$ is open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotonic.

Solution
Towards a contradiction, assume $f$ is not strictly monotonic. This means that for some $a < c < b \in \mathbb{R}$, either (i) $f(a) \leq f(c) \leq f(b)$ or (ii) $f(a) \geq f(c) \leq f(b)$, is true.

Consider case (i). Since $[a, b]$ is compact and $f$ is continuous, then $f([a, b])$ is compact. Because of the extreme value theorem, $M \equiv \sup f([a, b]) \in f([a, b])$. Since $f(a) \leq f(c) \leq f(b)$, there are two options
- $f(a) = M$ or $f(b) = M$, so $f(c) = M$.
- $f(a) < M$ and $f(b) < M$.

Both cases imply that $M \in f((a, b))$. This means that $f((a, b))$ is not open, since open sets do not contain their supremum. This is a contradiction with $f$ being open.

A similar argument can be made in case (ii) using the infimum instead of the supremum. We conclude that $f$ is strictly monotonic.
Problem 5.

Let $X = [0, 1)$ and define

$$d(x, x') := \inf \{ |x - x' + k| : k \in \mathbb{Z} \}, \quad \forall x, x' \in X$$

Show that $d(x, x')$ is a metric over $X$. Hint: find the $k$ that yields the infimum for each pair $(x, x') \in X^2$.

Solution

Note that, since $(x, x') \in X^2$, $-1 < x - x' < 1$, which implies that $|x - x'| < 1$. Then $|x - x' + k| \geq |k| - |x - x'| > |k| - 1$. This implies that, if $|k| \geq 2$, then $|x - x' + k| > 1 > |x - x' + 0|$. Then, for any $(x, x') \in X^2$, the infimum is achieved when $k$ is $-1, 0, 1$. Let $k = k(x, x')$ be the value that achieves the infimum for the pair $(x, x')$. By inspection we can see that

$$k(x, x') = \begin{cases} 
-1 & \text{if } \frac{1}{2} \leq x - x' < 1 \\
0 & \text{if } -\frac{1}{2} \leq x - x' \leq \frac{1}{2} \\
1 & \text{if } -1 < x - x' \leq -\frac{1}{2}
\end{cases}$$

Note that there are “multiple solutions” at $x-x' \in \{-1/2, 1/2\}$, however that do not play any role in the derivation below. Given this definition, we can write $d(x, x') = |x - x' + k(x, x')|$. We can now proceed to check if $d$ is a metric.

Checking that $d(x, x') \geq 0$ is direct. Note that $d(x, x') = 0 \iff x - x' + k(x, x') = 0$. From the definition of $k(x, x')$, we see that the only way in which $x - x' = -k(x, x')$ is when $k = 0$ and $x - x' = 0$, which implies that $x = x'$. Then $d(x, x') = 0 \iff x = x'$.

To show that $d(x, x') = d(x', x)$, note that $k(x, x') = -k(x', x)$. Then, symmetry follows from the property of the absolute value that states that $|a| = | -a |$.

Finally, we have to check the triangle inequality. Let $x, x', x'' \in X$. Then

$$d(x, x'') = |x - x'' + k(x, x'')|$$

$$\leq |x - x'' + k(x, x') + k(x', x'')|$$

$$= |x - x' + k(x, x') + x' - x'' + k(x', x'')|$$

$$\leq |x - x' + k(x, x')| + |x' - x'' + k(x', x'')|$$

$$= d(x, x') + d(x', x''),$$

where the second line follows from $k(x, x') + k(x', x'') \in \mathbb{Z}$ and $k(x, x'')$ being, by definition, the integer that gives the minimum value.
Problem 6.

For some metric space \((X, d)\), take any two sets \(A, B \subseteq X\) such that \(\text{int} A = \text{int} B = \emptyset\), and \(A\) is closed. Prove that \(\text{int}(A \cup B) = \emptyset\).

Solution

Towards a contradiction, assume \(x \in \text{int}(A \cup B)\). By definition, this implies that there is some open ball \(B_\epsilon(x) \subset A \cup B\). Consider the set \(E = B_\epsilon(x) \setminus A = B_\epsilon(x) \cap A^c\). Since \(A\) is closed, \(A^c\) is open. Since \(E\) is the finite intersection of two open sets, then it is open. We have two cases:

- \(E = \emptyset\). This implies that \(B_\epsilon(x) \subset A\), which implies that \(x \in \text{int} A\), a contradiction.
- \(E \neq \emptyset\). Then, for any \(y \in E\), \(y \in B\), so \(E \subset B\). Since \(E\) is open, this implies that \(B\) has non-empty interior, a contradiction.

Since both cases lead to a contradiction, we conclude that \(\text{int}(A \cup B) = \emptyset\).