Econ 204 – Problem Set 2

Due Tuesday, August 3

1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Let (E, d) be a metric space and $S \subset E$ a subset. Show that $A \subset S$ is open relative to S if and only if $A = S \cap U$ for some $U \subset E$ open.[1]

3. Let (X, d) be a metric space. Assume $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous on (X, d) and $(\mathbb{R}, |\cdot|)$, with $|\cdot|$ the absolute-value norm.
   (a) Show that $f + g : X \to \mathbb{R}$ is uniformly continuous, where $(f + g)(x) = f(x) + g(x)$.
   (b) Show that $\max\{f, g\} : X \to \mathbb{R}$ is uniformly continuous, where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.
   (c) Give a counterexample to the following statement: $f \cdot g : X \to \mathbb{R}$ *is uniformly continuous on* (X, d) *and* $(\mathbb{R}, |\cdot|)$, where $f \cdot g = f(x) \cdot g(x)$.

4. A function $f : X \to Y$ is open if $\forall A \subset X$ such that A is open, $f(A)$ is open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotonic.

5. Let $X = [0, 1)$ and define
$$d(x, x') := \inf \{|x - x' + k| : k \in \mathbb{Z}\}, \quad \forall x, x' \in X$$
   Show that $d(x, x')$ is a metric over X. *Hint*: find the k that yields the infimum for each pair $(x, x') \in X^2$.

6. For some metric space (X, d), take any two sets $A, B \subset X$ such that $\operatorname{int} A = \operatorname{int} B = \emptyset$, and A is closed. Prove that $\operatorname{int}(A \cup B) = \emptyset$.

[1] $A \subset S$ is open relative to S if $\forall x \in A\ \exists r_x > 0$ such that $B_{r_x}(x) \cap S \subset A$.