1. Let \((X,d)\) be a metric space:

(a) Let \(y \in X\) be given. Define the function \(d_y : X \to \mathbb{R}\) by
\[
d_y(x) = d(x,y) \tag{1}
\]
Show that \(d_y\) is a continuous function on \(X\) for each \(y \in X\).

Pick the sequence of elements \(\{x_n\} \subset X\) such that \(x_n \to x\). To verify that \(d_y\) is continuous we only need to show \(d_y(x_n) \to d_y(x)\), that is \(d(x_n,y) \to d(x,y)\).

Because of triangle inequality:
\[
d(x_n,y) \leq d(x_n,x) + d(x,y) \Rightarrow d(x_n,y) - d(x,y) \leq d(x_n,x) \tag{2}
\]
That in turn implies the reverse triangle inequality:
\[
|d(x_n,y) - d(x,y)| \leq d(x_n,x) \tag{3}
\]
This verifies that \(\lim_{n \to \infty} |d(x_n,y) - d(x,y)| = 0\), because \(d(x_n,x) \to 0\). Hence \(d(x_n,y) \to d(x,y)\).

(b) Let \(A\) be a subset of \(X\) and \(x \in X\). Recall that the distance from the point \(x\) to the set \(A\) is defined as:
\[
\rho(x,A) = \inf \{d(x,a) : a \in A\} \tag{4}
\]
Show that the closure of set \(A\) is the set of all points with zero distance to \(A\), that is:
\[
\bar{A} = \{x \in X : \rho(x,A) = 0\} \tag{5}
\]

Let’s denote \(A' := \{x \in X : \rho(x,A) = 0\}\). Pick \(x \in A'\), then \(\rho(x,A) = 0\). Because of the infimum property for every \(n \in \mathbb{N}\) there exists \(a_n \in A\) such that \(d(x,a_n) \leq 1/n\), that in turn implies the sequence \(\{a_n\} \subset A\) converges to \(x\), hence \(x\) is in the closure of set \(A\), concluding that \(A' \subset \bar{A}\).

Conversely, take \(x \in \bar{A}\), which is to say there exists a sequence \(\{b_n\} \subset A\) such that \(b_n \to x\). Using the continuity of function \(d_x\) proved in part (a) we deduce that \(d_x(b_n) \to d_x(x) = 0\). This implies \(x \in A'\) (why?) and finishes the reverse direction, i.e \(\bar{A} \subset A'\).

(c) Now let \(A \subset X\) be a compact subset. Show that \(\rho(x,A) = d(x,a)\) for some \(a \in A\).

We have seen that the function \(d_x\) is continuous on \(X\), so is on \(A\) (why?). One can represent \(\rho(x,A)\) as
\[
\rho(x,A) = \inf \{d_x(a) : a \in A\}. \tag{6}
\]

\(^1\)In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu
It has been proved in the lecture notes that the continuous functions defined on the compact subsets assumes their extremums. Namely, there has to be some \( a \in A \) such that \( \rho(x, A) = d_x(a) \).

2. Let \( U \subseteq \mathbb{R}^d \) be an open set and \( f : [0, 1] \rightarrow U \) be continuous. For each \( n \in \mathbb{N} \), define the n-polygonal approximation of \( f \) to be the function \( \gamma_n : [0, 1] \rightarrow \mathbb{R}^d \) given by:

\[
\gamma_n(t) = f\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right)
\]

where \( i \in \{1, \ldots, n\} \) is such that \( t \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \).

(a) Show that \( \gamma_n \) is continuous for all \( n \in \mathbb{N} \).

Fix \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). Let \( a \in [0, 1] \) and \( j \in \mathbb{N} \) be such that \( a \in \left[\frac{j-1}{n}, \frac{j}{n}\right] \). Note that

\[
\gamma_n(a) - f\left(\frac{j-1}{n}\right) = f\left(\frac{j-1}{n}\right) + n\left(a - \frac{j-1}{n}\right)\left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right) - f\left(\frac{j-1}{n}\right) = n\left(a - \frac{j-1}{n}\right)\left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right)
\]

A similar inequality holds for the distance between the images of \( a \) and \( j/n \). This will be useful to find out the \( \delta \) in the definition of continuity. I encourage you to think about what we have just done graphically. Let now \( M = \max_{x \in [0, 1]} |f(x)| \) and define

\[
\delta = \min\left(\frac{\varepsilon}{2Mn}, \frac{1}{n}\right)
\]

and let \( a, b \in [0, 1] \) be such that \( |a - b| < \delta \). Without loss of generality, assume that \( a < b \). We then have two cases:

**Case 1:** \( \exists j \in \{1, \ldots, n\} \) such that \( a, b \in \left[\frac{j-1}{n}, \frac{j}{n}\right] \).

We then have that:
\[ |\gamma_n(b) - \gamma_n(a)| = \left| f\left(\frac{j - 1}{n}\right) + n\left(b - \frac{j - 1}{n}\right)\left(f\left(\frac{j}{n}\right)\right) - f\left(\frac{j - 1}{n}\right) - n\left(a - \frac{j - 1}{n}\right)\left(f\left(\frac{j}{n}\right)\right) \right| \]

\[ = \left| n(b - a) f\left(\frac{j}{n}\right) \right| \]

\[ < n\frac{\varepsilon}{2Mn} \left| f\left(\frac{j}{n}\right) \right| \]

\[ < \varepsilon \]

**Case 2:** There exists \( j \in \{1, \ldots, n\} \) such that \( \max(|b - \frac{j}{n}|, |a - \frac{j}{n}|) < \delta \). By the inequality we proved above, we have that

\[ |\gamma_n(b) - \gamma_n(a)| = \left| f(b) - f\left(\frac{j}{n}\right) + f\left(\frac{j}{n}\right) - f(a) \right| \]

\[ \leq \left| n\left(a - \frac{j}{n}\right)\left(f\left(\frac{j}{n}\right) - f\left(\frac{j - 1}{n}\right)\right) \right| + \left| n\left(b - \frac{j}{n}\right)\left(f\left(\frac{j + 1}{n}\right) - f\left(\frac{j - 1}{n}\right)\right) \right| \]

\[ < n\delta \left| f\left(\frac{j}{n}\right) - f\left(\frac{j - 1}{n}\right) \right| + n\delta \left| f\left(\frac{j + 1}{n}\right) - f\left(\frac{j - 1}{n}\right) \right| \]

\[ < n\frac{\varepsilon}{2Mn} \left| f\left(\frac{j}{n}\right) - f\left(\frac{j - 1}{n}\right) \right| + n\frac{\varepsilon}{2Mn} \left| f\left(\frac{j + 1}{n}\right) - f\left(\frac{j - 1}{n}\right) \right| \]

\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \]

\[ = \varepsilon \]

where we have used the fact that, for any \( j \in \{1, \ldots, n\} \),

\[ \left| f\left(\frac{j}{n}\right) - f\left(\frac{j - 1}{n}\right) \right| \leq \left| f\left(\frac{j}{n}\right) \right| + \left| f\left(\frac{j - 1}{n}\right) \right| \]

\[ \leq 2M \]

(b) Show that there exists \( n_0 \in \mathbb{N} \) such that \( \forall n \geq n_0 \gamma_n(t) \in U \) for all \( t \in [0, 1] \).

First note that, for any \( t \in [0, 1] \) and \( n \in \mathbb{N} \), we have that
\[ |f(t) - \gamma_n(t)| = |f(t) - f\left(\frac{i-1}{n}\right) + n \left(t - \frac{i-1}{n}\right) \left( f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right)| \]
\[ \leq |f(t) - f\left(\frac{i-1}{n}\right)| + n \left|t - \frac{i-1}{n}\right| |f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)| \]
\[ \leq |f(t) - f\left(\frac{i-1}{n}\right)| + n \left|\frac{i}{n} - \frac{i-1}{n}\right| |f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)| \]
\[ = |f(t) - f\left(\frac{i-1}{n}\right)| + n \left|\frac{i}{n} - \frac{i-1}{n}\right| |f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)| \]

Since \( U \) is open, this means that \( \forall t \in [0, 1] \exists \varepsilon_t > 0 \) such that \( B(f(t), \varepsilon_t) \subseteq U \). Since \( f \) is continuous and \([0, 1]\) is compact, we have that \( f([0, 1]) \) is a compact set. Notice that the collection \( B(f(t), \varepsilon_t/3) \) is an open covering of \( f([0, 1]) \). By definition (or a characterization) of compactness, this means that there exists \( t_1, \ldots, t_N \) such that

\[ f([0, 1]) \subseteq B(f(t_1), \varepsilon_{t_1}/3) \cup \cdots \cup B(f(t_N), \varepsilon_{t_N}/3) \subseteq U \]

Let now \( \varepsilon = \min (\varepsilon_{t_1}/3, \ldots, \varepsilon_{t_N}/3)/2 \). Since \( f \) is a continuous function and \([0, 1]\) is a compact set, \( f \) is uniformly continuous, thus there exists \( \delta > 0 \) such that

\[ |a - b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon \]

Let \( n_0 \) be such that \( 1/n_0 < \delta \), and let \( t \in [0, 1], n > n_0 \) be arbitrary. Let \( k \) be such that \( f(t) \in B(f(t_k), \varepsilon_{t_k}/3) \). We then have that

\[ |\gamma_n(t) - f(t_k)| \leq |\gamma_n(t) - f(t)| + |f(t) - f(t_k)| \]
\[ \leq |f(t) - f\left(\frac{i-1}{n}\right)| + |f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)| + |f(t) - f(t_k)| \]
\[ < \varepsilon + \varepsilon + \varepsilon_{t_k}/3 \]
\[ < 3 \ast \varepsilon_{t_k}/3 \]
\[ = \varepsilon_{t_k} \]

Thus \( \gamma(t) \in B(f(t_k), \varepsilon_{t_k}) \subseteq U \), which finishes the proof.
3. Let \((X, d)\) be a metric space. Given \(x \in X\), we define the connected component of \(x\) in \(X\) as the set

\[
C(x) = \bigcup_{U \subseteq X \text{ is connected} \atop U \ni x} U
\]

Prove that:

(a) For every \(x \in X\), \(C(x)\) is a non-empty connected set.

Note that \(\{x\}\) is a connected set, thus the union is taken over a non-empty collection, thus \(x \in C(x)\) and \(C(x)\) is non-empty.

Suppose now, by contradiction, that \(C(x)\) is not connected. We then have that there exists \(A, B\) open, disjoint, non-empty sets such that \(C(x) \cap A \neq \emptyset\), \(C(x) \cap B \neq \emptyset\) and \(C(x) \subseteq A \cup B\).

Since \(x \in C(x)\), we have that \(x \in A\) or \(x \in B\), but not both, since they are disjoint. Without loss of generality, let \(x \in A\).

Let \(y \in C(x) \cap B\), which exists because \(C(x) \cap B \neq \emptyset\). Since \(y \in C(x)\), we have that there exists \(U_y\) connected such that \(x \in U_y, y \in U_y, U_y \subseteq X\) and \(U_y\) is connected.

Let now

\[
E_1 = \{z \in U_y \mid z \in A\} \\
E_2 = \{z \in U_y \mid z \in B\}
\]

We know that \(x \in E_1\) and \(y \in E_2\). We also have that \(U_y \subseteq C(x) \subseteq A \cup B\), so that \(E_1 \cup E_2 = U_y\). This implies that \(U_y\) is not a connected set, which is a contradiction. Thus \(C(x)\) is a connected set.

(b) For every two elements \(x, y \in X\), they either share a connected component \(C(x) = C(y)\) or their connected components are disjoint \(C(x) \cap C(y) = \emptyset\).

If \(C(x) \cap C(y) = \emptyset\), there is nothing to be done. If this is not the case, then there exists \(z \in C(x) \cap C(y)\).

We will prove now that \(C(x) \cup C(y)\) is connected. Assume, by contradiction, that this is not the case. Then there exists \(A, B\) open, non-empty disjoint sets such that \((C(x) \cup C(y)) \cap A \neq \emptyset\), \((C(x) \cup C(y)) \cap B \neq \emptyset\) and \(C(x) \cup C(y) \subseteq A \cup B\).

Since \(z \in C(x) \cup C(y)\), we have that \(z \in A \cup B\), without loss of generality assume that \(z \in A\). Since \((C(x) \cup C(y)) \cap B \neq \emptyset\), there exists \(w \in (C(x) \cup C(y)) \cap B\), without loss of generality assume that \(w \in C(x) \cap B\). Note also that \(z \in C(x) \cap A\).

This contradicts the fact that \(C(x)\) is connected, so it must be the case that \(C(x) \cup C(y)\) is connected.

By definition, this means that \(C(y) \subseteq C(x) \cup C(y) \subseteq C(x)\) and \(C(x) \subseteq C(x) \cup C(y) \subseteq C(y)\), and thus \(C(x) = C(y)\).
(c) Conclude that there exists a subset $\mathcal{A} \subseteq X$ such that $X = \bigcup_{x \in \mathcal{A}} C(x)$, where $\bigcup$ represents the disjoint union.

Define the following equivalence relation on $X$: $x \sim y \iff C(x) = C(y)$, and consider the partition $X/\sim$. Let $\mathcal{A}$ be a set formed by taking one element of each set of $X/\sim$. By construction, $X = \bigcup_{x \in \mathcal{A}} C(x)$, and the union is disjoint by b).

4. Define the correspondence $\Gamma: [0,1] \rightarrow 2^{[0,1]}$ by:

$$
\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\
[0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} 
\end{cases}
$$

(7)

Show that $\Gamma$ is not continuous, but it is lower-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

Consider the open set $V = (0,1)$ which contains $\Gamma(q) = [0,1] \setminus \mathbb{Q}$ for every $q \in [0,1] \cap \mathbb{Q}$. Then any open set containing $q$ will also contain an irrational number $x \in [0,1] \setminus \mathbb{Q}$, and $\Gamma(x) = [0,1] \cap \mathbb{Q} \not\subset V$. Hence $\Gamma$ is not upper-hemicontinuous at any rational number.

Now fix some $y \in [0,1] \setminus \mathbb{Q}$ and consider the open set $V = [0,y) \cup (y,1]$ in $[0,1]$. For any $x \in [0,1] \setminus \mathbb{Q}$ we have $\Gamma(x) \subset V$, but every open set containing $x$ will also contain a rational number $q \in [0,1] \cap \mathbb{Q}$ and $\Gamma(q) = [0,1] \setminus \mathbb{Q} \not\subset V$. Thus $\Gamma$ is nowhere upper-hemicontinuous and hence nowhere continuous.

Next, let $V$ be any open set satisfying $V \cap [0,1] \neq \emptyset$. Then we have $V \cap ([0,1] \cap \mathbb{Q}) \neq \emptyset$ and $V \cap ([0,1] \setminus \mathbb{Q}) \neq \emptyset$, since every $\varepsilon$-ball in the reals contains both rational and irrational numbers. But then $\Gamma(x) \cap V \neq \emptyset$ for every $x$ in the domain of $\Gamma$. This proves that $\Gamma$ is lower-hemicontinuous.

The correspondence does not have a closed graph. Remember that $\text{gr}(\Gamma)$ is a subset of $[0,1] \times [0,1]$. Fix some $y \in [0,1] \setminus \mathbb{Q}$ and take any sequence $\{q_n\} \subset [0,1] \cap \mathbb{Q}$ such that $q_n \rightarrow y$. Then the sequence $(q_n, y) \in \text{gr}(\Gamma)$ but $(y, y) \notin \text{gr}(\Gamma)$. Hence the graph is not closed.

5. Let $X$ be a metric space, and $I : X \rightarrow \mathbb{R}_+$ be a lower semi-continuous function.²

(a) Prove that for every given $\varepsilon > 0$ there exists an open set $U_\varepsilon$ containing $x \in X$ such that

$$
\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon.
$$

(8)

Because of lower semi-continuity the set $U_\varepsilon := \{y : I(y) > I(x) - \varepsilon\}$ is open in $X$, that trivially contains $x$, and hence is non-empty. Now, it should be clear that

$$
\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon.
$$

(9)

²A function $I : X \rightarrow \mathbb{R}$ is called lower semi-continuous iff for every $\alpha$ the set $\{x : I(x) > \alpha\}$ is open in $X$. 

6
6. Let \( x \) and \( y \) be moving objects in \( \mathbb{R} \). Time is discrete, namely \( t \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N} \). In addition, \( \beta > 1 \) is a fixed parameter. For \( a, b \in \mathbb{R} \), let \( \rho(a, b) := |a - b| \land 1 \) (as mentioned in the section, the symbol \( \land \) is sometimes used to refer to the minimum of two elements). Then for any \( x, y \in \mathbb{R}^\omega \), let
\[
d(x, y) = \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(x_t, y_t)
\] denotes the distance between \( x = (x_0, x_1, \ldots) \) and \( y = (y_0, y_1, \ldots) \), where \( x_t \) is the position of \( x \) at time \( t \) on the real line.

(a) Show that \( d \) is a metric on \( \mathbb{R}^\omega \).

The first two metric properties are easy to validate, and we only verify the triangle inequality. We have seen in the section that \( \rho \) is a bounded metric on \( \mathbb{R} \), hence \( \rho(x_t, z_t) \leq \rho(x_t, y_t) + \rho(y_t, z_t) \). Multiply both sides by \( \beta^{-t} \) and sum over \( t \); since the right hand side is bounded by \( \sum_{t \in \mathbb{Z}_+} \beta^{-t} \), this sum is well-defined. Therefore
\[
\sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(x_t, z_t) \leq \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(x_t, y_t) + \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(y_t, z_t)
\] holds, and the triangle inequality falls out.

(b) Show that \( (\mathbb{R}^\omega, d) \) is a bounded metric space.

It is a bounded metric because one can pick \( x = (0, 0, \ldots) \). Then for every \( y \in \mathbb{R}^\omega \)
\[
\rho(x, y) = \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(0, y_t) \leq \sum_{t \in \mathbb{Z}_+} \beta^{-t} = \frac{\beta}{\beta - 1} < \infty
\]

(c) Is \([0, 1]^\omega\) an open or closed subset of \( \mathbb{R}^\omega \)? (in either case present a proof)

It is a closed subset of \( \mathbb{R}^\omega \). To see this, take the sequence \( \{x^{(n)}\} \subseteq [0, 1]^\omega \) and suppose \( d(x^{(n)}, x) \to 0 \) for some \( x \in \mathbb{R}^\omega \). Since \( \beta^{-t} \rho(x^{(n)}_t, x_t) \leq d(x^{(n)}, x) \),

---

3We define the infinite cartesian product of a set \( X \) with itself as \( X^\omega := \prod_{i \in \mathbb{N}} X \).
then \( |x_t^{(n)} - x_t| \to 0 \) for all \( t \in \mathbb{Z}_+ \). Since \([0, 1]\) is closed and \( \{x_t^{(n)}\} \subset [0, 1] \) then it has to be the case that \( x_t \in [0, 1] \) too. This in turn implies that \( x \in [0, 1]^\omega \), thereby \([0, 1]^\omega\) is a closed subset.

(d) Is \((\mathbb{R}^\omega, d)\) a complete metric space? (prove if yes, otherwise provide a counterexample)

Let’s take the Cauchy sequence \( \{x^{(n)}\} \subset [0, 1]^\omega \). By definition of Cauchy sequence, for every \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n, m \geq N \):

\[
d(x^{(n)}, x^{(m)}) < \varepsilon.
\]

This in particular implies that \( \beta^{-t}\rho(x^{(n)}, x^{(m)}) < \varepsilon \), and hence for every \( t \in \mathbb{Z}_+ \), \( \{x_t^{(n)}\} \) is a Cauchy sequence in \( \mathbb{R} \), thereby convergent to a point in \( \mathbb{R} \) say \( x_t \) (remember that \( \mathbb{R} \) is complete). This constructs a tuple \( x \in \mathbb{R}^\omega \). It is only left to show \( x^{(n)} \to x \) under the metric \( d \). For given \( \varepsilon > 0 \), pick \( T \in \mathbb{Z}_+ \) large enough such that \( \sum_{t>T} \beta^{-t} < \varepsilon / 2 \); this is possible because the infinite sum is bounded. Then

\[
d(x^{(n)}, x) = \sum_{t \leq T} \beta^{-t}\rho(x_t^{(n)}, x_t) + \sum_{t > T} \beta^{-t}\rho(x_t^{(n)}, x_t) < \sum_{t \leq T} \beta^{-t}\rho(x_t^{(n)}, x_t) + \varepsilon / 2
\]

The first term on the RHS of the last inequality is a sum over finitely many numbers which are all converging to 0. Therefore, for large enough \( n \) this sum can be made smaller than \( \varepsilon / 2 \), and hence \( d(x^{(n)}, x) < \varepsilon \); concluding the proof of \( x^{(n)} \to x \). In part (c), we have seen that \([0, 1]^\omega\) is a closed subset, therefore \( x \) as the limit point of \( \{x^{(n)}\} \) is in \([0, 1]^\omega\) as well. This verifies that \([0, 1]^\omega\) is a complete subset.

We could have also proved that \( \mathbb{R}^\omega \) is a complete metric space, and then conclude that \([0, 1]^\omega\) must be complete because it is a closed subset of \( \mathbb{R}^\omega \).

(e) Is \([0, 1]^\omega\) a totally bounded subset under \( d \)? Is it a compact subset?

It is a totally bounded subset. Again for a given \( \varepsilon > 0 \) pick \( T \) large enough such that \( \sum_{t>T} \beta^{-t} < \varepsilon / 2 \). Since \([0, 1] \subset \mathbb{R}\) is totally bounded, for each \( t \in \{0, 1, \ldots, T\} \) there exists a collection of points \( \mathcal{A}_t^\varepsilon := \{a_1^t, a_2^t, \ldots, a_m^t\} \) such that:

\[
[0, 1] \subset \bigcup_{j=1}^{m_t} (a_j^t - \varepsilon \beta^j / 2T, a_j^t + \varepsilon \beta^j / 2T)
\]

Now consider the following set of points:

\( \mathcal{A}^\varepsilon := \{(a_1, a_2, \ldots) \in [0, 1]^\omega : a_t \in \mathcal{A}_t^\varepsilon \text{ for } t \leq T \text{ and } a_t = 0 \text{ for } t > T \} \)

This is a finite set (why?) and can be considered as the centers of open \( \varepsilon \)-balls that further on will cover the \([0, 1]^\omega\). Now for any point \( x \in [0, 1]^\omega \) one can find
a point $a \in \mathcal{A}^\varepsilon$ such that:

$$
\begin{align*}
\rho(x, a) & = \sum_{t \leq T} \beta^{-t} \rho(x_t, a_t) + \sum_{t > T} \beta^{-t} \rho(x_t, a_t) \\
& < \varepsilon / 2 + \varepsilon / 2 = \varepsilon
\end{align*}
$$

Therefore, $\{ B_\varepsilon(a) : a \in \mathcal{A}^\varepsilon \}$ is a collection of finitely many open sets that covers $[0, 1]$. Hence, $[0, 1]$ is totally bounded. This lets us to invoke theorem 9 of lecture note 6, and deduce the compactness of set $[0, 1]$ (since this subset is complete and totally bounded).