Econ 204 – Problem Set 3[1]

Due Friday August 6, 2021

1. Let (X, d) be a metric space:

   (a) Let $y \in X$ be given. Define the function $d_y : X \to \mathbb{R}$ by

   $$d_y(x) = d(x, y) \tag{1}$$

   Show that d_y is a continuous function on X for each $y \in X$.

   (b) Let A be a subset of X and $x \in X$. Recall that the distance from the point x to the set A is defined as:

   $$\rho(x, A) = \inf \{d(x, a) : a \in A\} \tag{2}$$

   Show that the closure of set A is the set of all points with zero distance to A, that is:

   $$\bar{A} = \{x \in X : \rho(x, A) = 0\} \tag{3}$$

   (c) Now let $A \subset X$ be a compact subset. Show that $\rho(x, A) = d(x, a)$ for some $a \in A$.

2. Let $U \subseteq \mathbb{R}^d$ be an open set and $f : [0, 1] \to U$ be continuous. For each $n \in \mathbb{N}$, define the n-polygonal approximation of f to be the function $\gamma_n : [0, 1] \to \mathbb{R}^d$ given by:

   $$\gamma_n(t) = f\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right)$$

   where $i \in \{1, \ldots, n\}$ is such that $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$.

   (a) Show that γ_n is continuous for all $n \in \mathbb{N}$.

   (b) Show that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 \gamma_n(t) \in U$ for all $t \in [0, 1]$.

3. Let (X, d) be a metric space. Given $x \in X$, we define the connected component of x in X as the set

   $$C(x) = \bigcup_{\substack{U \subseteq X \text{ s.t } x \in U \\ U \text{ is connected}}} U$$

   Prove that:

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

(a) For every $x \in X$, $C(x)$ is a non-empty connected set.

(b) For every two elements $x, y \in X$, they either share a connected component $C(x) = C(y)$ or their connected components are disjoint $C(x) \cap C(y) = \varnothing$.

(c) Conclude that there exists a subset $\mathcal{A} \subseteq X$ such that $X = \dot{\bigcup}_{x \in \mathcal{A}} C(x)$, where $\dot{\cup}$ represents the disjoint union.

4. Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ [0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}. \tag{4}$$

Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

5. Let X be a metric space, and $I : X \to \mathbb{R}_+$ be a lower semi-continuous function [2].

(a) Prove that for every given $\varepsilon > 0$ there exists an open set U_ε containing $x \in X$ such that

$$\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon. \tag{5}$$

(b) Let $x \in X$. For each $n \in \mathbb{N}$ let

$$m_n = \inf\left\{I(y) : y \in B_{1/n}(x)\right\}. \tag{6}$$

Show that $\{m_n\}$ is an increasing sequence and that $m_n \to I(x)$.

6. Let x and y be moving objects in $\mathbb{R}$. Time is discrete, namely $t \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. In addition, $\beta > 1$ is a fixed parameter. For $a, b \in \mathbb{R}$, let $\rho(a,b) := |a - b| \wedge 1$ (as mentioned in the section, the symbol $\wedge$ is sometimes used to refer to the minimum of two elements). Then for any $x, y \in \mathbb{R}^\omega$ [3], let

$$d(x,y) = \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(x_t, y_t) \tag{7}$$

denotes the distance between $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$, where x_t is the position of x at time t on the real line.

(a) Show that d is a metric on $\mathbb{R}^\omega$.

(b) Show that $(\mathbb{R}^\omega, d)$ is a bounded metric space.

(c) Is $[0,1]^\omega$ an open or closed subset of $\mathbb{R}^\omega$? (in either case present a proof)

(d) Is $(\mathbb{R}^\omega, d)$ a complete metric space? (prove if yes, otherwise provide a counterexample)

[2] A function $I : X \to \mathbb{R}$ is called lower semi-continuous *iff* for every α the set $\{x : I(x) > \alpha\}$ is open in X.

[3] We define the infinite **cartesian product** of a set X with itself as $X^\omega := \prod_{i \in \mathbb{N}} X$.

(e) Is $[0, 1]^\omega$ a totally bounded subset under d? Is it a compact subset?