Econ 204 – Problem Set 4

Due Tuesday, August 10

1. Let $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ be defined by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

2. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(av) = af(v)$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but f is not linear.

3. Prove that if $(v_1, ..., v_n)$ spans V and $T \in L(V, W)$ is onto, then $(Tv_1, ..., Tv_n)$ spans W.

4. Let V be a finite-dimensional vector space and $W \subset V$ be a vector subspace. Prove that W has a complement in V, i.e., there exists a vector subspace $W' \subset V$ such that $W \cap W' = \{0\}$ and $W + W' = V$.

5. Let U and V be vector spaces. Suppose $T : U \to V$ is a linear transformation and $v \in V$. Prove that, if the preimage $T^{-1}(v)$ is non-empty, and $u \in T^{-1}(v)$, then $T^{-1}(v) = \{u + z | z \in \ker T\} = u + \ker T$.

6. Let V be a finite dimensional vector space and $T, S \in L(V, V)$. Prove that TS is invertible if and only if T and S are invertible.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (4x - 2y, x + y)$. Let V be the standard basis and $W = \{(5, 3), (1, 1)\}$ be another basis of $\mathbb{R}^2$.

   (a) Find $Mtx_V(T)$.

   (b) Find $Mtx_W(T)$.

   (c) Compute $T(4, 3)$ using the matrix representation of W.