Econ 204 – Problem Set 5[1]

Due Friday August 13, 2021

1. Let $C = C([a, b])$ be the set of continuous functions from $[a, b]$ to $\mathbb{R}$ equipped with the sup norm.

   (a) Define, for any $t \in [a, b]$, the function $\mathcal{A}_t : C \to \mathbb{R}$ as

   $$\mathcal{A}_t(f) = f(t)$$

   Prove that $\mathcal{A}_t$ is 1-Lipschitz, but not L-Lipschitz for any $L < 1$. [2].

   (b) Define $\mathcal{I} : C \to \mathbb{R}$ as

   $$\mathcal{I}(f) = \int_a^b f(t)dt$$

   Find the constant K such that $\mathcal{I}$ is K-Lipschitz, but not L-Lipschitz for any $L < K$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 (twice continuously differentiable) function. The function and its second derivative are bounded, namely there exist $M, N > 0$ such that $\sup_{x\in\mathbb{R}} \left|f(x)\right| \leq M$ and $\sup_{x\in\mathbb{R}} \left|f''(x)\right| \leq N$. Show that $\sup_{x\in\mathbb{R}} \left|f'(x)\right| \leq 2\sqrt{MN}$.

3. Let E be a Banach space. Define E^* as the set of continuous linear transformations from E to $\mathbb{R}$. Prove that [3]

   (a) E^* is a vector space with the operations

   $$(T + L)(x) = T(x) + L(x), \text{ for } T, L \in E^*, x \in E$$
   $$(\alpha T)(x) = \alpha T(x) \text{ for } T \in E^*, x \in E, \alpha \in \mathbb{R}$$

   (b) The function $||T|| = \sup_{||x||=1} |T(x)|$ is a norm on E^*.

   (c) The space E^* with the norm above is a Banach space.

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

[2] We say that a function $f : X \to Y$ is L-Lipschitz, or Lipschitz with constant L, if $||f(x) - f(y)||_Y \leq L||x - y||_X \forall x, y \in X$

[3] These are special cases of general results we discussed in class. The goal of the exercise is to prove these results directly in this simpler setting.

4. The goal of this exercise is to verify the **Banach-Steinhaus** theorem. Let $\{T_n\}$ be a sequence of bounded linear functions $T_n : X \to Y$ from a Banach (complete normed vector) space X into a normed vector space Y, such that $\{T_n(x)\}$ is bounded for every $x \in X$, that is for all $x \in X$ there exists $c_x \in \mathbb{R}_+$ such that:

$$\left\|T_n(x)\right\| \leq c_x \quad \forall n \in \mathbb{N} \tag{1}$$

Then, we want to show that the sequence of norms $\left\{\|T_n\|\right\}$ is bounded, that is there exists $c > 0$ such that $\|T_n\| \leq c$ for all $n \in \mathbb{N}$.

(a) For every $k \in \mathbb{N}$ let $A_k \subseteq X$ be the set of all $x \in X$ such that $\left\|T_n(x)\right\| \leq k$ for all n. Show that A_k is closed under the X-norm.

(b) Use equation (1) to show that $X = \bigcup_{k \in \mathbb{N}} A_k$.

(c) The **Baire's** theorem states that in this case since X is complete, there exists some A_{k_0} that contains an open ball, say $B_\varepsilon(x_0) \subseteq A_{k_0}$. Take this result as given, and prove there exists some constant $c > 0$ such that

$$\|T_n\| \leq c \quad \forall n \in \mathbb{N}. \tag{2}$$

Hint: For every nonzero $x \in X$ there exists $\gamma > 0$ such that $x = \frac{1}{\gamma}(z - x_0)$, where $x_0, z \in B_\varepsilon(x_0)$ and $\gamma > 0$.

5. Suppose $\Psi : X \to 2^X$ is a non-empty and compact-valued upper-hemicontinuous correspondence. The metric space X is compact. Show that there exists a non-empty compact set $C \subset X$ such that $\Psi(C) = C$ (you can use the exercises that are proved in the sections).