Econ 204 – Problem Set 6

Due Monday, August 16

1. Consider the following equations:

$$\begin{aligned} x^2 - yu &= 0, \\ xy + uv &= 0, \end{aligned}$$

where $(x, y, u, v) \in \mathbb{R}^4$. Using the implicit function theorem, describe under what condition these equations can be solved for u and v. Then solve the equations directly and check these conditions.

2. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$. Then, show that exactly one of the following two conditions hold:

   - $\exists x \in \mathbb{R}^n$ such that $Ax = b$, with $x \geq 0$;
   - $\exists y \in \mathbb{R}^{1\times m}$ such that $A'y \geq 0$, and $y'b < 0$.

   Hint: you may want to use the following definition and its properties. If $v_1, v_2, ..., v_n$ are the columns of A, define

$$Q = \text{cone}(A) \equiv \left\{ s \in \mathbb{R}^m : s = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \forall i \right\},$$

   i.e., Q is the set of all conic combinations of the columns of A. Note that Q is non-empty ($0 \in Q$), and assume it is closed and convex (you should be able to prove this!).

3. Call a vector $\pi \in \mathbb{R}^n$ a *probability vector* if

$$\sum_{i}^{n} \pi_i = 1 \text{ and } \pi_i \geq 0 \ \forall i$$

   We say there are n states of the world, and π_i is the probability that state i occurs. Suppose there are two traders (trader 1 and trader 2) who each have a set of prior probability distributions (Π_1 and Π_2) which are nonempty, convex, and compact. Call a *trade* a vector $f \in \mathbb{R}^n$, which denotes the net transfer trader 1 receives in each state of the world (and thus $-f$ is the net transfer trader 2 receives in each state of the world). A trade is *agreeable* if

$$\inf_{\pi\in\Pi_1} \sum_{i=1}^{n} \pi_i f_i > 0 \quad \text{and} \quad \inf_{\pi\in\Pi_2} \sum_{i=1}^{n} \pi_i(-f_i) > 0$$

   Prove that there exists an agreeable trade if and only if there is no common prior (that is, $\Pi_1 \cap \Pi_2 = \varnothing$).

4. (a) Let $A \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, \lambda_2, ..., \lambda_p \geq 0$, with $\sum_{i=1}^{p} \lambda_i = 1$. Prove that, if $x_1, x_2, ..., x_p \in A$, then $\sum_{i=1}^{p} \lambda_i x_i \in A$.

   (b) The sum $\sum_{i=1}^{p} \lambda_i x_i$ defined in (a) is called a convex combination. The convex hull of a set S, denoted by $co(S)$, is the intersection of all convex sets which contain S. Prove that the set of all convex combinations of the elements of S is exactly $co(S)$.

5. Consider a symmetric game with m players indexed by i. Each player strategy space is given by a finite set $S \subset \mathbb{R}$ with n distinct elements. The strategy for each player i consists on a vector $x^i \in \Delta = \{x \in \mathbb{R}^n : \sum_{j=1}^n x_j = 1, x_j \geq 0 \quad \forall j = 1, ..., n\}$ that assigns probabilities of implementing each of the elements of S. The utility function for each player i is given by $u(x^1, ..., x^m) \equiv u(x^i, x^{-i})$. Define a Nash equilibrium as a vector $x^* = (x^{1*}, ..., x^{m*}) \in \Delta^m \subset \mathbb{R}^{n \times m}$ such that $u(x^{i*}, x^{-i*}) \geq u(x^{i'}, x^{-i*})$, for all $x^{i'} \in S$ and i, where $\Delta^m = \Delta \times \Delta \times ... \times \Delta$ (m times).

   Define $\phi^i : \Delta^{m-1} \to \Delta$ as $\phi^i(x^{-i}) = z^i$, where $z^i = \arg\max_{z^{i'}} u(z^{i'}, x^{-i})$. Assume that ϕ^i is continuous and single-valued (i.e., that ϕ^i is a continuous function).[1] Show that there exists a Nash equilibrium.

6. Solve the following differential equation: $y'' - 5y' + 4y = e^{4x}$. Concretely, provide (i) the general solution of the homogeneous differential equation, and (ii) the particular and general solutions of the inhomogeneous differential equation. Solve explicitly for the constants using the following initial conditions: $y(0) = 3$, $y(0)' = \frac{19}{3}$.

[1] Assuming u is continuous and strictly quasi-concave would yield this result (see Berge's maximum theorem).