Economics 204 Summer/Fall 2021
Final Exam – Suggested Solutions

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 7 questions for a total of 180 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. Use the points as a guide to allocating your time.

1. (15) Let D be an $n \times n$ matrix that is diagonal, so $d_{ij} = 0$ for all $i \neq j$, where d_{ij} is the ij^{th} entry of the matrix D (an $n \times n$ matrix M is *diagonal* if $m_{ij} = 0$ for all $i \neq j$, where m_{ij} denotes the ij^{th} entry of M). Show that for every $k \in \mathbb{N}$, D^k is also a diagonal matrix (where M^k denotes the product of k copies of the $n \times n$ matrix M).

   (**Hint:** use induction.)

   Solution: For the base case $k = 1$, the claim follows by definition: D is a diagonal matrix. For the induction hypothesis, assume that the claim is true for some $k \geq 1$, so D^k is a diagonal matrix. Then for $k + 1$,

$$D^{k+1} = D^k D$$

   Let $A = D^{k+1}$ and $B = D^k$, so $A = BD$. Then by the induction hypothesis, B is diagonal. Let b_i denote the ith row of B and d_j denote the jth column of D. Then $a_{ij} = b_i \cdot d_j$, where a_{ij} is the ij^{th} element of the matrix A. Since B and D are both diagonal matrices, $b_{ik} = 0$ for all $i \neq k$ and $d_{kj} = 0$ for all $k \neq j$. Then by definition

$$a_{ij} = b_i \cdot d_j = \sum_{k=1}^{n} b_{ik} d_{kj} = 0 \quad \forall i \neq j$$

   This implies $A = D^{k+1}$ is a diagonal matrix by definition. Thus by induction, D^k is a diagonal matrix for all $k \in \mathbb{N}$.

2. (15) Let (X, d) be a metric space and $f, g : X \to \mathbb{R}$ be continuous functions. Let $C = \{x \in X : f(x) \geq g(x)\}$. Show that C is a closed set.

   Solution: Let $h : X \to \mathbb{R}$ be given by $h = f - g$. Then note that h is continuous, because f and g are continuous, and

$$C = \{x \in X : f(x) \geq g(x)\} = \{x \in X : h(x) = f(x) - g(x) \geq 0\}$$

   Thus $C = h^{-1}([0, \infty))$. Since $[0, \infty) \subseteq \mathbb{R}$ is closed and h is continuous, $C = h^{-1}([0, \infty))$ is closed.

3. (30) Let X be a vector space over the field F, and let V be a proper subset of X, so $V \subseteq X$ and $V \neq X$. Suppose V is linearly independent. Show that V is a basis for X if and only if every proper superset of V is linearly dependent, that is, if and only if for every subset $W \subseteq X$ such that $V \subseteq W$ and $V \neq W$, W is linearly dependent.

Solution: First suppose V is a basis for X. Then let $W \subseteq X$ such that $V \subseteq W$ and $V \neq W$. Let $x \in W \setminus V$. Then since V is a basis for X, there exist $v_1, \ldots, v_n \in V$ and $\alpha_1, \ldots, \alpha_n \in F$ such that

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

Thus

$$0 = -x + \sum_{i=1}^{n} \alpha_i v_i$$

Since $V \subseteq W$ and $x \in W$, $\{x, v_1, \ldots, v_n\} \subseteq W$. The coefficients above are not all zero; in particular, $-1 \neq 0$. So W is linearly dependent.

For the converse, to show that V is a basis, let $x \in X \setminus V$. Then let $W = V \cup \{x\}$. By construction $V \subseteq W$ and $V \neq W$, so by assumption W is linearly dependent. Since V is linearly independent, there exists $\alpha_0, \alpha_1, \ldots, \alpha_n$ not all zero and $v_1, \ldots, v_n \in V$ such that

$$\alpha_0 x + \sum_{i=1}^{n} \alpha_i v_i = 0$$

and in addition, it must be that $\alpha_0 \neq 0$. Then this implies

$$-\alpha_0 x = \sum_{i=1}^{n} \alpha_i v_i$$

or

$$x = \sum_{i=1}^{n} -\frac{\alpha_i}{\alpha_0} v_i$$

Thus $x \in$ span V. Since $x \in X \setminus V$ was arbitrary, V spans X. Since V is linearly independent by assumption, V is a basis for X.

4. (30) Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f'(x) \neq 0$ for all $x \in (a, b)$ then f is one-to-one.

Solution: Let $x, y \in [a, b]$ such that $x \neq y$. Without loss of generality, take $x < y$. Then $[x, y] \subseteq [a, b]$, so f is continuous on $[x, y]$ and differentiable on (x, y). By the Mean Value Theorem, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x)$$

By assumption, $f'(z) \neq 0$, and $y - x \neq 0$, so $f(y) - f(x) \neq 0$, or $f(y) \neq f(x)$. Since $x, y \in [a, b]$ were arbitrary, this implies that f is one-to-one.

5. (30) Let (X, d) be a metric space and $C \subseteq X$ be compact. Let $\{x_n\} \subseteq C$ be a sequence and let A be the set of cluster points of $\{x_n\}$.

a. Show that A is closed and $A \subseteq C$.

Solution: To show that A is closed, let $\{y_k\} \subseteq A$ such that $y_k \to y$. It suffices to show $y \in A$, that is, that y is a cluster point of $\{x_n\}$. Then let $\varepsilon > 0$. Since $y_k \to y$, there exists K such that for all $k > K$, $y_k \in B_{\frac{\varepsilon}{2}}(y)$. Then fix $k > K$. Since $y_k \in A$, y_k is a cluster point of $\{x_n\}$. So by definition $\{n \in \mathbb{N} : x_n \in B_{\frac{\varepsilon}{2}}(y_k)\}$ is infinite. Then let $x_n \in B_{\frac{\varepsilon}{2}}(y_k)$.

$$\begin{aligned} d(x_n, y) &\leq d(x_n, y_k) + d(y_k, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus

$$\{n \in \mathbb{N} : x_n \in B_{\frac{\varepsilon}{2}}(y_k)\} \subseteq \{n \in \mathbb{N} : x_n \in B_\varepsilon(y)\}$$

This implies $\{n \in \mathbb{N} : x_n \in B_\varepsilon(y)\}$ is infinite. Since $\varepsilon > 0$ was arbitrary, this implies y is a cluster point of $\{x_n\}$ by definition, so $y \in A$.

To show that $A \subseteq C$, let $x \in A$. Since x is a cluster point of $\{x_n\}$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$. Then $\{x_n\} \subseteq C$ by assumption, so $\{x_{n_k}\} \subseteq C$. Since C is a compact subset of a metric space, C is closed. Thus $x \in C$. Thus $A \subseteq C$.

Here is an alternative argument to show A is closed using the subsequence characterization of cluster points. Let $\{y_k\} \subseteq A$ be a sequence such that $y_k \to y$. Then for each m there exists y_{k_m} such that $y_{k_m} \in B_{\frac{1}{2m}}(y)$. Now construct a subsequence of $\{x_n\}$ inductively as follows.

For $j = 1$, choose x_{n_1} such that $x_{n_1} \in B_{\frac{1}{2}}(y_{k_1})$. This is possible because y_{k_1} is a cluster point of $\{x_n\}$. Now suppose $n_j > n_{j-1} > \ldots > n_1$ have been chosen so that such that $x_{n_i} \in B_{\frac{1}{2i}}(y_{k_i})$ for each i. Then choose $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in B_{\frac{1}{2(j+1)}}(y_{k_{j+1}})$. Again this is possible because $y_{k_{j+1}}$ is a cluster point of $\{x_n\}$.

Then $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ and for each j, $x_{n_j} \in B_{\frac{1}{2j}}(y_{k_j})$. Using the triangle inequality and the choice of y_{k_j} above, this implies $x_{n_j} \in B_{\frac{1}{j}}(y)$ for each j. Then by construction, $x_{n_j} \to y$. Thus y is a cluster point of $\{x_n\}$, that is, $y \in A$.

b. Show that $A \cup \{x_n : n \in \mathbb{N}\}$ is compact.

(**Hint:** Use the open cover definition of compactness.)

Solution: Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A \cup \{x_n : n \in \mathbb{N}\}$. From (a), A is closed and $A \subseteq C$. Since C is compact, this implies A is compact. Then $\mathcal{U}$ is an open cover of A, so there exist $U_{\lambda_1}, \ldots, U_{\lambda_n} \in \mathcal{U}$ such that

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

Now claim $\{n \in \mathbb{N} : x_n \notin U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}\}$ is finite. To show this, suppose not. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \subseteq (U_{\lambda_1} \cup \cdots \cup U_{\lambda_n})^c$. Since $\{x_{n_k}\} \subseteq C$ and C is compact, there is a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ (and hence a subsequence of $\{x_n\}$) such that $x_{n_{k_j}} \to x \in C$. But then x is a cluster point of $\{x_n\}$, so $x \in A$. This implies $x \in U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$, and thus $x \in U_{\lambda_i}$ for some λ_i. Since U_{λ_i} is open, this implies there exists N such that $x_{n_{k_j}} \in U_{\lambda_i} \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$ for all $n_{k_j} > N$. This is a contradiction, since $x_{n_{k_j}} \notin U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$ for all n_{k_j} by construction.

Thus $\{n \in \mathbb{N} : x_n \notin U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}\}$ is finite. Then let $m_1, \ldots, m_r$ be these indexes, so

$$\{x_{m_1}, \ldots, x_{m_r}\} = \{x_n\} \setminus (U_{\lambda_1} \cup \cdots \cup U_{\lambda_n})$$

Since $\mathcal{U}$ is an open cover of $A \cup \{x_n : n \in \mathbb{N}\}$, for each m_i there exists $U_{m_i} \in \mathcal{U}$ such that $x_{m_i} \in U_{m_i}$. Thus

$$A \cup \{x_n : n \in \mathbb{N}\} \subseteq (U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}) \cup (U_{m_1} \cup \cdots \cup U_{m_r})$$

Since $\mathcal{U}$ was arbitrary, $A \cup \{x_n : n \in \mathbb{N}\}$ is compact.

Here is an alternative argument using sequential compactness. Let $\{y_k\}$ be a sequence such that $\{y_k\} \subseteq A \cup \{x_n : n \in \mathbb{N}\}$. Then either $\{y_k\}$ has a subsequence in A or a subsequence in $\{x_n : n \in \mathbb{N}\}$. We consider these cases in turn.

Case 1: Suppose $\{y_k\}$ has a subsequence $\{y_{k_j}\} \subseteq A$. Then from above, A is compact, so $\{y_{k_j}\}$ has a further subsequence $\{y_{k_{j_\ell}}\}$, which is also a subsequence of $\{y_k\}$, such that $y_{k_{j_\ell}} \to y \in A \subseteq A \cup \{x_n : n \in \mathbb{N}\}$.

Case 2: Suppose $\{y_k\}$ has a subsequence in $\{x_n : n \in \mathbb{N}\}$. Then there are two possible subcases.

Case 2a: $\{y_k\}$ has a constant subsequence $\{y_{k_j}\}$, so that $y_{k_j} = x_m$ for all k_j, for some fixed $x_m \in \{x_n : n \in \mathbb{N}\}$. In this case, $y_{k_j} \to x_m \in \{x_n : n \in \mathbb{N}\}$.

Case 2b: $\{y_k\}$ has a subsequence $\{y_{k_j}\}$ that is also a subsequence of $\{x_n\}$. In this case, since $\{y_{k_j}\} \subseteq C$ and C is compact, $\{y_{k_j}\}$ has a further subsequence $\{y_{k_{j_\ell}}\}$, which is also a subsequence of $\{y_k\}$ and of $\{x_n\}$, such that $y_{k_{j_\ell}} \to y \in C$. This implies y is a cluster point of $\{x_n\}$, so $y \in A$.

In each case, $\{y_k\}$ has a convergent subsequence that converges to an element of $A \cup \{x_n : n \in \mathbb{N}\}$. Thus $A \cup \{x_n : n \in \mathbb{N}\}$ is sequentially compact, and hence compact.

Finally, another solution is to show that $A \cup \{x_n : n \in \mathbb{N}\}$ is closed. Since C is compact and $A \cup \{x_n : n \in \mathbb{N}\} \subseteq C$, using (a) and the assumption that $\{x_n\} \subseteq C$, this will imply that $A \cup \{x_n : n \in \mathbb{N}\}$ is compact. To show that $A \cup \{x_n : n \in \mathbb{N}\}$ is closed, let $\{y_k\}$ be a sequence such that $\{y_k\} \subseteq A \cup \{x_n : n \in \mathbb{N}\}$ and $y_k \to y$. Then we must show $y \in A \cup \{x_n : n \in \mathbb{N}\}$. The argument is similar to the previous argument to show sequential compactness, considering the possible cases above.

6. (30) Let $a, b \in \mathbb{R}$ with $a \leq b$. Suppose $\varphi : [a, b] \to 2^{\mathbb{R}}$ is a continuous correspondence with nonempty, compact, convex values. Thus for every $x \in [a, b]$, $\varphi(x) \subseteq \mathbb{R}$ is nonempty, compact, and convex. Define the function $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}(\sup \varphi(x) + \inf \varphi(x)) \quad \text{for each } x \in [a, b]$$

a. Show that $f(x) \in \varphi(x)$ for each $x \in [a, b]$.

Solution: Fix $x \in [a, b]$. It suffices to show that $\sup \varphi(x) \in \varphi(x)$ and $\inf \varphi(x) \in \varphi(x)$ since $\varphi(x)$ is convex. Then note that $\varphi(x) \subseteq \mathbb{R}$ is nonempty and compact, hence bounded, so $\sup \varphi(x)$ and $\inf \varphi(x)$ are both finite. Then for each $n \in \mathbb{N}$ there exists $y_n \in \varphi(x)$ such that

$$\sup \varphi(x) - \frac{1}{n} \leq y_n \leq \varphi(x)$$

So $y_n \to \sup \varphi(x)$ by construction. Then $\varphi(x)$ is compact, hence closed, and $\{y_n\} \subseteq \varphi(x)$, so $\sup \varphi(x) \in \varphi(x)$. The argument for $\inf \varphi(x)$ is similar. Therefore $f(x) = \frac{1}{2}(\sup \varphi(x) + \inf \varphi(x)) \in \varphi(x)$. Since $x \in [a, b]$ was arbitrary, this establishes the claim.

b. Show that f is continuous.

Solution: Let $g, h : [a, b] \to \mathbb{R}$ be given by

$$g(x) = \sup \varphi(x) \quad \text{and } h(x) = \inf \varphi(x) \quad \text{for each } x \in [a, b]$$

Then $f = \frac{1}{2}(g+h)$, so it suffices to show that g and h are continuous. To that end, let $x \in [a, b]$ and let $\varepsilon > 0$. Then $(g(x) - \varepsilon, g(x) + \varepsilon) = (\sup \varphi(x) - \varepsilon, \sup \varphi(x) + \varepsilon)$ is open and $(\sup \varphi(x) - \varepsilon, \sup \varphi(x) + \varepsilon) \cap \varphi(x) \neq \emptyset$. Since φ is lhc, there exists an open set U_1 with $x \in U_1$ such that for all $y \in U_1 \cap [a, b]$,

$$\varphi(y) \cap (\sup \varphi(x) - \varepsilon, \sup \varphi(x) + \varepsilon) \neq \emptyset$$

Thus for all $y \in U_1 \cap [a, b]$, $\sup \varphi(y) > \sup \varphi(x) - \varepsilon$, that is, $g(y) > g(x) - \varepsilon$. Similarly, $V = (\inf \varphi(x) - \varepsilon, \sup \varphi(x) + \varepsilon)$ is an open set and $\varphi(x) \subseteq V$. Since φ is uhc, there exists an open set U_2 with $x \in U_2$ such that for all $y \in U_2 \cap [a, b]$,

$$\varphi(y) \subseteq V = (\inf \varphi(x) - \varepsilon, \sup \varphi(x) + \varepsilon)$$

Thus for all $y \in U_2 \cap [a, b]$, $\sup \varphi(y) < \sup \varphi(x) + \varepsilon$, that is, $g(y) < g(x) + \varepsilon$. Let $U = U_1 \cap U_2$. Then U is open, $x \in U$, and for all $y \in U \cap [a, b]$, $g(y) \in (g(x) - \varepsilon, g(x) + \varepsilon)$. Since U is open and $x \in U$, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \cap [a, b] \subseteq U$. For all $y \in (x - \delta, x + \delta) \cap [a, b]$, $g(y) \in (g(x) - \varepsilon, g(x) + \varepsilon)$ by the previous argument. Since $\varepsilon > 0$ and $x \in [a, b]$ were arbitrary, this implies g is continuous. The argument for h is similar. Therefore $f = \frac{1}{2}(g+h)$ is continuous.

Here is an alternative argument using the sequential characterizations of uhc and lhc. First note that φ is compact-valued, so the sequential characterization of

uhc is valid. Then let $x \in [a,b]$ and let $x_n \to x$. Fix $\varepsilon > 0$. Since φ is lhc and $\sup \varphi(x) \in \varphi(x)$, for every n there exists $z_n \in \varphi(x_n)$ such that $z_n \to \sup \varphi(x)$. Then there exists N_1 such that for all $n > N_1$,

$$z_n > \sup \varphi(x) - \varepsilon$$

Since $z_n \in \varphi(x_n)$ for each n, this implies

$$\sup \varphi(x_n) \geq z_n > \sup \varphi(x) - \varepsilon \quad \forall n > N_1$$

Then claim there exists N_2 such that for all $n > N_2$,

$$\sup \varphi(x_n) < \sup \varphi(x) + \varepsilon$$

To see this, suppose not. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\sup \varphi(x_{n_k}) \geq \sup \varphi(x) + \varepsilon \quad \forall n_k$$

Then $x_{n_k} \to x$ and $\sup \varphi(x_{n_k}) \in \varphi(x_{n_k})$ for each n_k. Since φ is uhc and compact-valued, this implies there must be a subsequence $\{\sup \varphi(x_{n_{k_j}})\}$ of $\{\sup \varphi(x_{n_k})\}$ that converges to an element $y \in \varphi(x)$. But $\sup \varphi(x_{n_{k_j}}) \geq \sup \varphi(x) + \varepsilon$ for all n_{k_j}, so if $\sup \varphi(x_{n_{k_j}}) \to y$, then $y \geq \sup \varphi(x) + \varepsilon$. This implies $y \notin \varphi(x)$. This is a contradiction.

So there exists N_2 such that for all $n > N_2$, $\sup \varphi(x_n) < \sup \varphi(x) + \varepsilon$. Then let $N = \max(N_1, N_2)$. For all $n > N$,

$$\sup \varphi(x_n) \in (\sup \varphi(x) - \varepsilon, \sup \varphi(x) + \varepsilon)$$

Since $\varepsilon > 0$ was arbitrary, this implies $\sup \varphi(x_n) \to \sup \varphi(x)$. Since $x \in [a,b]$ was arbitrary, this shows $g = \sup \varphi$ is continuous. The argument for $h = \inf \varphi$ is similar.

7. (30) Let (X,d) be a nonempty complete metric space, and let $f : X \to X$. Suppose there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha \left(d(x, f(x)) + d(y, f(y))\right)$$

Show that f has a unique fixed point.

Solution: Let $x_0 \in X$. Define $\{x_n\}$ by

$$x_n = f(x_{n-1}) \quad \text{for each } n \in \mathbb{N}$$

Now claim $\{x_n\}$ is a Cauchy sequence. To see this, first note that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\ &\leq \alpha \left(d(x_n, f(x_n)) + d(x_{n-1}, f(x_{n-1}))\right) \\ &= \alpha \left(d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right) \end{aligned}$$

This implies

$$(1-\alpha)d(x_{n+1},x_n) \leq \alpha d(x_n,x_{n-1})$$

or

$$d(x_{n+1},x_n) \leq \frac{\alpha}{1-\alpha} d(x_n,x_{n-1})$$

Then let $\beta = \frac{\alpha}{1-\alpha}$. Since $\alpha \in (0,\frac{1}{2})$, $\beta \in (0,1)$, and from the above argument, $d(x_{n+1},x_n) \leq \beta d(x_n,x_{n-1})$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, repeating yields

$$\begin{aligned} d(x_{n+1},x_n) &\leq \beta d(x_n,x_{n-1}) \\ &\leq \beta^2 d(x_{n-1},x_{n-2}) \\ &\vdots \\ &\leq \beta^n d(x_1,x_0) \end{aligned}$$

Then fix $n,m \in \mathbb{N}$ with $n \geq m$.

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m) \\ &\leq (\beta^n + \beta^{n-1} + \cdots + \beta^m) d(x_1,x_0) \\ &= \sum_{k=m}^{n} \beta^k d(x_1,x_0) \\ &< \sum_{k=m}^{\infty} \beta^k d(x_1,x_0) \\ &= \frac{\beta^m}{1-\beta} d(x_1,x_0) \end{aligned}$$

Then let $\varepsilon > 0$. Since $\frac{\beta^m}{1-\beta} d(x_1,x_0) \to 0$ as $m \to \infty$, choose N such that for all $m > N$, $\frac{\beta^m}{1-\beta} d(x_1,x_0) < \varepsilon$. Then if $n,m > N$ with $n \geq m$,

$$d(x_n,x_m) \leq \frac{\beta^m}{1-\beta} d(x_1,x_0) < \varepsilon$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$.

Now claim that $f(x^*) = x^*$. To see this, note that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(f(x_n),f(x^*)) &\leq \alpha\left(d(x_n,f(x_n)) + d(x^*,f(x^*))\right) \\ &= \alpha\left(d(x_n,x_{n+1}) + d(x^*,f(x^*))\right) \end{aligned}$$

Thus

$$\begin{aligned} d(x_{n+1},f(x^*)) = d(f(x_n),f(x^*)) &\leq \alpha\left(d(x_n,x_{n+1}) + d(x^*,f(x^*))\right) \\ &\leq \alpha\left(\beta^n d(x_1,x_0) + d(x^*,f(x^*))\right) \end{aligned}$$

Then note that $\beta^n d(x_1, x_0) \to 0$ as $n \to \infty$, and since $x_n \to x^*$ and the metric d is continuous, $d(x_{n+1}, f(x^*)) \to d(x^*, f(x^*))$. Putting these together with the previous inequality implies

$$d(x^*, f(x^*)) \leq \alpha d(x^*, f(x^*))$$

Since $\alpha \in (0, \frac{1}{2})$, this implies $d(x^*, f(x^*)) = 0$. Thus $x^* = f(x^*)$, that is, x^* is a fixed point of f.

Finally, to show f has a unique fixed point, suppose $y^* \in X$ and $f(y^*) = y^*$. Then

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \alpha \left(d(x^*, f(x^*)) + d(y^*, f(y^*)) \right) = 0$$

Since $\alpha > 0$, this implies $d(x^*, y^*) = 0$, thus $x^* = y^*$.