Economics 204 Summer/Fall 2021
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 7 questions for a total of 180 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. Use the points as a guide to allocating your time.

1. (15) Let D be an $n \times n$ matrix that is diagonal, so $d_{ij} = 0$ for all $i \neq j$, where d_{ij} is the ij^{th} entry of the matrix D (an $n \times n$ matrix M is *diagonal* if $m_{ij} = 0$ for all $i \neq j$, where m_{ij} denotes the ij^{th} entry of M). Show that for every $k \in \mathbb{N}$, D^k is also a diagonal matrix (where M^k denotes the product of k copies of the $n \times n$ matrix M).

   (**Hint:** use induction.)

2. (15) Let (X, d) be a metric space and $f, g : X \to \mathbb{R}$ be continuous functions. Let $C = \{x \in X : f(x) \geq g(x)\}$. Show that C is a closed set.

3. (30) Let X be a vector space over the field F, and let V be a proper subset of X, so $V \subseteq X$ and $V \neq X$. Suppose V is linearly independent. Show that V is a basis for X if and only if every proper superset of V is linearly dependent, that is, if and only if for every subset $W \subseteq X$ such that $V \subseteq W$ and $V \neq W$, W is linearly dependent.

4. (30) Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f'(x) \neq 0$ for all $x \in (a, b)$ then f is one-to-one.

5. (30) Let (X, d) be a metric space and $C \subseteq X$ be compact. Let $\{x_n\} \subseteq C$ be a sequence and let A be the set of cluster points of $\{x_n\}$.

   a. Show that A is closed and $A \subseteq C$.

   b. Show that $A \cup \{x_n : n \in \mathbb{N}\}$ is compact.

   (**Hint:** Use the open cover definition of compactness.)

6. (30) Let $a, b \in \mathbb{R}$ with $a \leq b$. Suppose $\varphi : [a, b] \to 2^{\mathbb{R}}$ is a continuous correspondence with nonempty, compact, convex values. Thus for every $x \in [a, b]$, $\varphi(x) \subseteq \mathbb{R}$ is nonempty, compact, and convex. Define the function $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}\left(\sup \varphi(x) + \inf \varphi(x)\right) \quad \text{for each } x \in [a, b]$$

   a. Show that $f(x) \in \varphi(x)$ for each $x \in [a, b]$.

   b. Show that f is continuous.

7. (30) Let (X, d) be a nonempty complete metric space, and let $f : X \to X$. Suppose there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha\left(d(x, f(x)) + d(y, f(y))\right)$$

   Show that f has a unique fixed point.