

# Econ 204 2022

## Lecture 10

### Outline

1. Diagonalization of Real Symmetric Matrices
2. Application to Quadratic Forms
3. Linear Maps Between Normed Spaces

## How Might This Matter

- Why does diagonalizability matter?

Consider a two-dimensional linear difference equation:

$$\begin{pmatrix} c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t = 0, 1, 2, 3, \dots$$

given an initial condition  $c_0, k_0$ , or, setting

$$y_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

we can rewrite this more compactly as

$$y_{t+1} = B y_t \quad \forall t$$

where  $b_{ij} \in \mathbf{R}$  each  $i, j$ .

We want to find a solution  $y_t$ ,  $t = 1, 2, 3, \dots$  given initial condition  $y_0$ . (Why?)

Such a dynamical system will arise for example as a characterization of the solution to a standard infinite-horizon optimal growth problem (202a, lecture 2).

If  $B$  is diagonalizable, this can be easily solved after a change of basis. If  $B$  is diagonalizable, choose an invertible  $2 \times 2$  real matrix  $P$  such that

$$P^{-1}BP = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Then

$$\begin{aligned} y_{t+1} = By_t \quad \forall t &\iff P^{-1}y_{t+1} = P^{-1}By_t \quad \forall t \\ &\iff P^{-1}y_{t+1} = P^{-1}BP P^{-1}y_t \quad \forall t \\ &\iff \bar{y}_{t+1} = D\bar{y}_t \quad \forall t \end{aligned}$$

where  $\bar{y}_t = P^{-1}y_t \ \forall t$ .

Since  $D$  is diagonal, after a change of basis to  $\bar{y}_t$ , we need to solve two **independent** linear univariate difference equations, which is easy:

$$\bar{y}_{it} = d_i^t \bar{y}_{i0} \quad \forall t$$

- Not all real  $n \times n$  matrices are diagonalizable (not even all invertible  $n \times n$  matrices are)...so can we identify some classes that are?
- Some types of matrices appear more frequently than others – especially real symmetric  $n \times n$  matrices (matrix representation of second derivatives of  $C^2$  functions, quadratic forms...).

- Recall that an  $n \times n$  real matrix  $A$  is *symmetric* if  $a_{ij} = a_{ji}$  for all  $i, j$ , where  $a_{ij}$  is the  $(i, j)^{\text{th}}$  entry of  $A$ .

# Orthonormal Bases

**Definition 1.** *Let*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

*A basis  $V = \{v_1, \dots, v_n\}$  of  $\mathbf{R}^n$  is orthonormal if  $v_i \cdot v_j = \delta_{ij}$ .*

In other words, a basis is orthonormal if each basis element has unit length ( $\|v_i\|^2 = v_i \cdot v_i = 1 \ \forall i$ ), and distinct basis elements are perpendicular ( $v_i \cdot v_j = 0$  for  $i \neq j$ ).

# Orthonormal Bases

**Remark:** Suppose that  $x = \sum_{j=1}^n \alpha_j v_j$  where  $\{v_1, \dots, v_n\}$  is an orthonormal basis of  $\mathbf{R}^n$ . Then

$$\begin{aligned} x \cdot v_k &= \left( \sum_{j=1}^n \alpha_j v_j \right) \cdot v_k \\ &= \sum_{j=1}^n \alpha_j (v_j \cdot v_k) \\ &= \sum_{j=1}^n \alpha_j \delta_{jk} \\ &= \alpha_k \end{aligned}$$

so

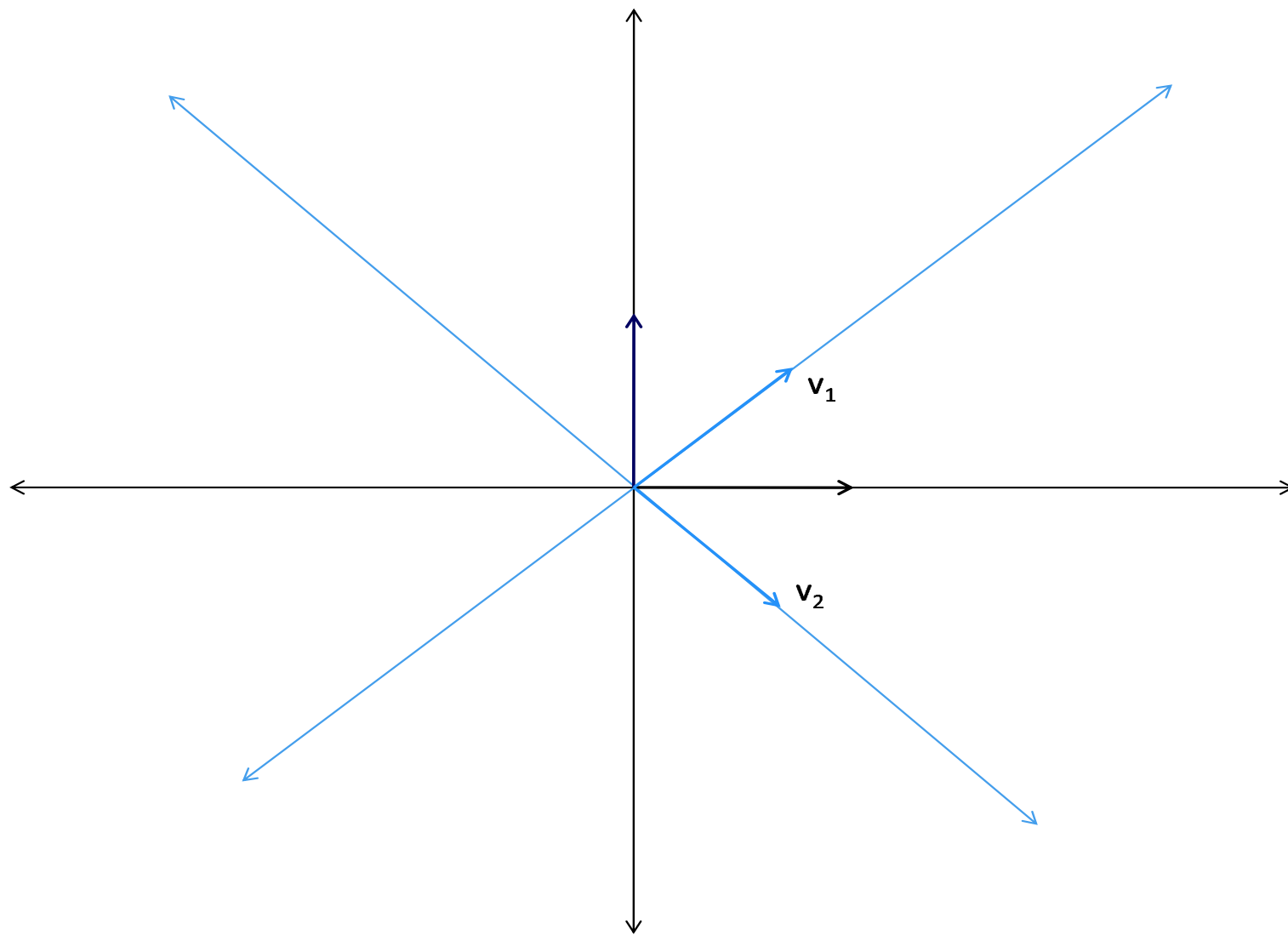
$$x = \sum_{j=1}^n (x \cdot v_j) v_j$$

# Orthonormal Bases

**Example:** The standard basis of  $\mathbf{R}^n$  is orthonormal.

(Why?)





# Unitary Matrices

Recall that for a real  $n \times m$  matrix  $A$ ,  $A^\top$  denotes the transpose of  $A$ : the  $(i, j)^{th}$  entry of  $A^\top$  is the  $(j, i)^{th}$  entry of  $A$ .

So the  $i^{th}$  row of  $A^\top$  is the  $i^{th}$  column of  $A$ .

**Definition 2.** A real  $n \times n$  matrix  $A$  is unitary if  $A^\top = A^{-1}$ .

Notice that by definition every unitary matrix is invertible.

# Unitary Matrices

**Theorem 1.** *A real  $n \times n$  matrix  $A$  is unitary if and only if the columns of  $A$  are orthonormal.*

*Proof.* Let  $v_j$  denote the  $j^{th}$  column of  $A$ .

$$\begin{aligned} A^\top &= A^{-1} && \iff A^\top A = I \\ &&& \iff v_i \cdot v_j = \delta_{ij} \quad \forall i, j \\ &&& \iff \{v_1, \dots, v_n\} \text{ is orthonormal} \end{aligned}$$



# Unitary Matrices

If  $A$  is unitary, let  $V$  be the set of columns of  $A$  and  $W$  be the standard basis of  $\mathbf{R}^n$ . Since  $A$  is unitary, it is invertible, so  $V$  is a basis of  $\mathbf{R}^n$ .

$$A^\top = A^{-1} = \text{Mtx}_{V,W}(\text{id})$$

Since  $V$  is orthonormal, the transformation between bases  $W$  and  $V$  preserves all geometry, including lengths and angles.

# Diagonalization of Real Symmetric Matrices

**Theorem 2.** *Let  $T \in L(\mathbf{R}^n, \mathbf{R}^n)$  and  $W$  be the standard basis of  $\mathbf{R}^n$ . Suppose that  $Mtx_W(T)$  is symmetric. Then the eigenvectors of  $T$  are all real, and there is an orthonormal basis  $V = \{v_1, \dots, v_n\}$  of  $\mathbf{R}^n$  consisting of eigenvectors of  $T$ , so that  $Mtx_W(T)$  is diagonalizable:*

$$Mtx_W(T) = Mtx_{W,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,W}(id)$$

*where  $Mtx_V T$  is diagonal and the change of basis matrices  $Mtx_{V,W}(id)$  and  $Mtx_{W,V}(id)$  are unitary.*

The proof of the theorem requires a lengthy digression into the linear algebra of *complex* vector spaces. A brief outline is in the notes.

# Quadratic Forms

**Example:** Let

$$f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

Let

$$A = \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix}$$

so  $A$  is symmetric and

$$\begin{aligned}x^\top Ax &= (x_1, x_2) \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\&= (x_1, x_2) \begin{pmatrix} \alpha x_1 + \frac{\beta}{2} x_2 \\ \frac{\beta}{2} x_1 + \gamma x_2 \end{pmatrix} \\&= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 \\&= f(x)\end{aligned}$$

# Quadratic Forms

Consider a quadratic form

$$f(x_1, \dots, x_n) = \sum_{i=1}^n \alpha_{ii} x_i^2 + \sum_{i < j} \beta_{ij} x_i x_j \quad (1)$$

Let

$$\alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ji}}{2} & \text{if } i > j \end{cases}$$

Let

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \text{ so } f(x) = x^\top A x$$



# Quadratic Forms

$A$  is symmetric, so let  $V = \{v_1, \dots, v_n\}$  be an orthonormal basis of eigenvectors of  $A$  with corresponding eigenvalues  $\lambda_1, \dots, \lambda_n$ .

$$\begin{aligned} \text{Then } A &= U^\top D U \\ \text{where } D &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \\ \text{and } U &= \text{Mtx}_{V,W}(\text{id}) \text{ is unitary} \end{aligned}$$

The columns of  $U^\top$  (the rows of  $U$ ) are the coordinates of  $v_1, \dots, v_n$ , expressed in terms of the standard basis  $W$ . Given  $x \in \mathbf{R}^n$ , recall

$$x = \sum_{i=1}^n \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$

# Quadratic Forms

So

$$\begin{aligned} f(x) &= f\left(\sum \gamma_i v_i\right) \\ &= \left(\sum \gamma_i v_i\right)^\top A \left(\sum \gamma_i v_i\right) \\ &= \left(\sum \gamma_i v_i\right)^\top U^\top D U \left(\sum \gamma_i v_i\right) \\ &= \left(U \sum \gamma_i v_i\right)^\top D \left(U \sum \gamma_i v_i\right) \\ &= \left(\sum \gamma_i U v_i\right)^\top D \left(\sum \gamma_i U v_i\right) \\ &= (\gamma_1, \dots, \gamma_n) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \\ &= \sum \lambda_i \gamma_i^2 \end{aligned}$$

# Quadratic Forms

The equation for a level set of  $f$  is

$$\left\{ \gamma \in \mathbf{R}^n : \sum_{i=1}^n \lambda_i \gamma_i^2 = C \right\}$$

- If  $\lambda_i \geq 0$  for all  $i$ , the level set is an ellipsoid, with principal axes in the directions  $v_1, \dots, v_n$ . The length of the principal axis along  $v_i$  is  $\sqrt{C/\lambda_i}$  if  $C \geq 0$  (if  $\lambda_i = 0$ , the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if  $C < 0$ .
- If  $\lambda_i \leq 0$  for all  $i$ , the level set is an ellipsoid, with principal axes in the directions  $v_1, \dots, v_n$ . The length of the principal

axis along  $v_i$  is  $\sqrt{C/\lambda_i}$  if  $C \leq 0$  (if  $\lambda_i = 0$ , the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if  $C > 0$ .

- If  $\lambda_i > 0$  for some  $i$  and  $\lambda_j < 0$  for some  $j$ , the level set is a hyperboloid. For example, suppose  $n = 2$ ,  $\lambda_1 > 0$ ,  $\lambda_2 < 0$ . The equation is

$$\begin{aligned} C &= \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2 \\ &= \left( \sqrt{\lambda_1} \gamma_1 + \sqrt{|\lambda_2|} \gamma_2 \right) \left( \sqrt{\lambda_1} \gamma_1 - \sqrt{|\lambda_2|} \gamma_2 \right) \end{aligned}$$

This is a hyperbola with asymptotes

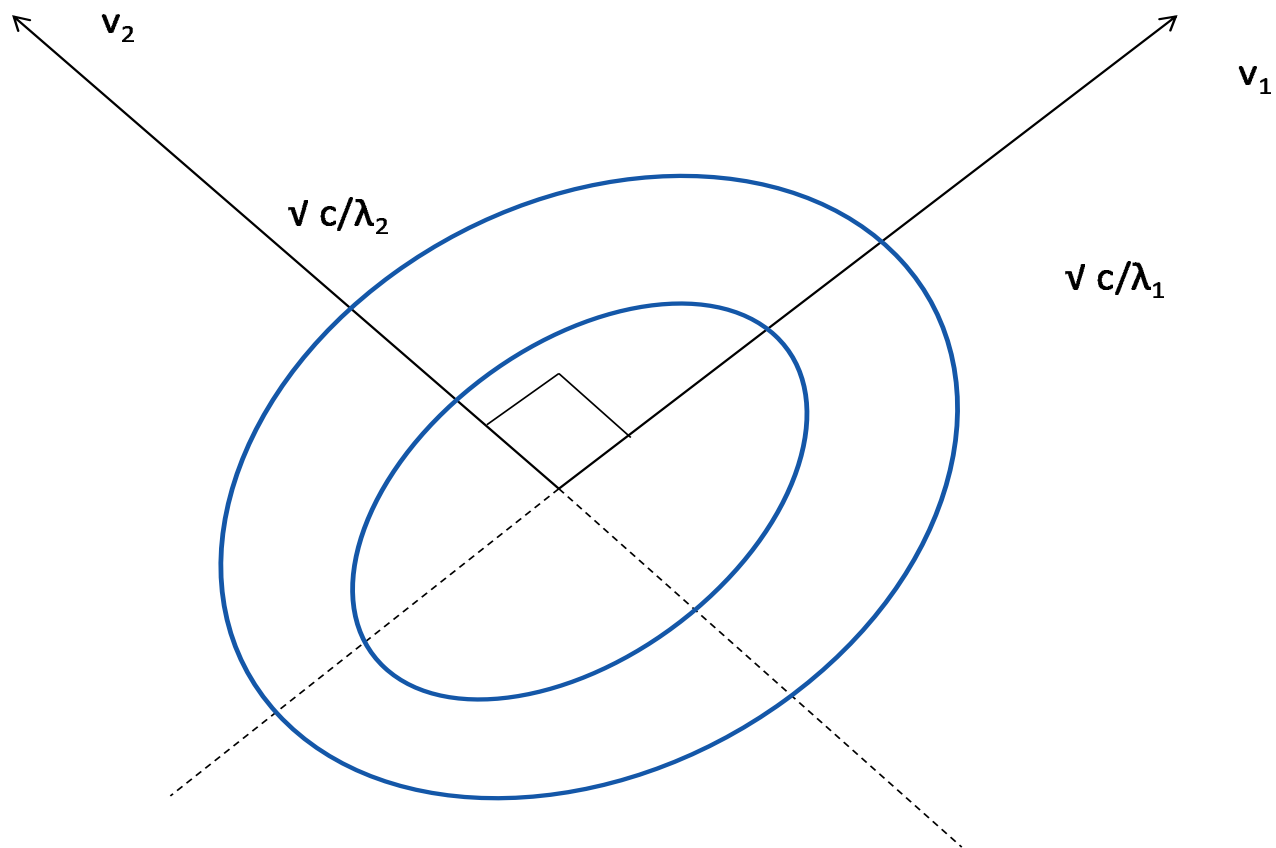
$$0 = \sqrt{\lambda_1}\gamma_1 + \sqrt{|\lambda_2|}\gamma_2$$

$$\Rightarrow \gamma_1 = -\sqrt{\frac{|\lambda_2|}{\lambda_1}}\gamma_2$$

$$0 = \left( \sqrt{\lambda_1}\gamma_1 - \sqrt{|\lambda_2|}\gamma_2 \right)$$

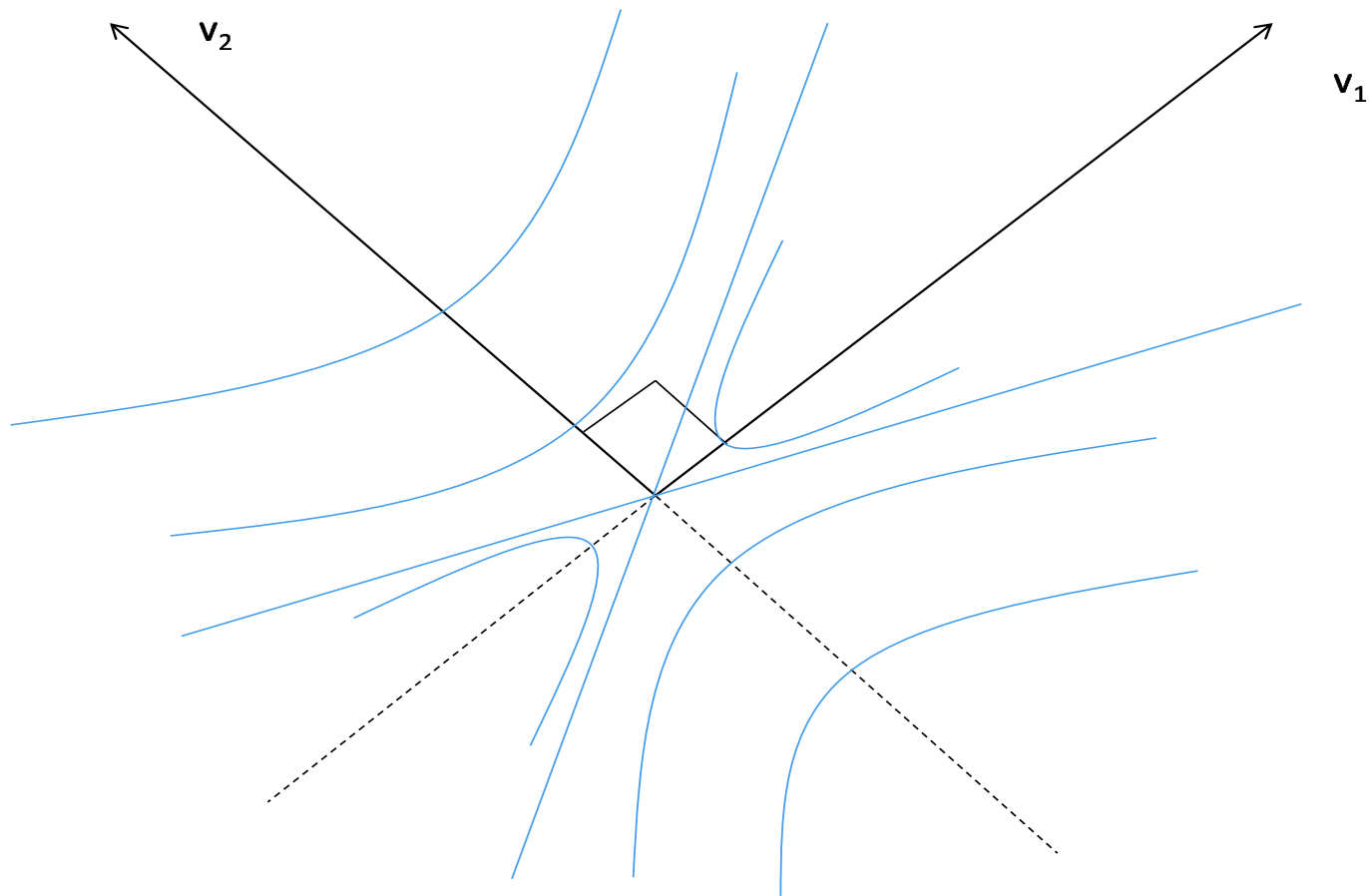
$$\Rightarrow \gamma_1 = \sqrt{\frac{|\lambda_2|}{\lambda_1}}\gamma_2$$

$$\lambda_1 > 0, \lambda_2 > 0$$



$$\lambda_1 > 0, \lambda_2 < 0$$

$$v_1 = v |\lambda_2| / \lambda_1$$



# Quadratic Forms

This proves the following corollary of Theorem 2.

**Corollary 1.** *Consider the quadratic form (1).*

- 1.  $f$  has a global minimum at 0 if and only if  $\lambda_i \geq 0$  for all  $i$ ; the level sets of  $f$  are ellipsoids with principal axes aligned with the orthonormal eigenvectors  $v_1, \dots, v_n$ .*
- 2.  $f$  has a global maximum at 0 if and only if  $\lambda_i \leq 0$  for all  $i$ ; the level sets of  $f$  are ellipsoids with principal axes aligned with the orthonormal eigenvectors  $v_1, \dots, v_n$ .*



3. *If  $\lambda_i < 0$  for some  $i$  and  $\lambda_j > 0$  for some  $j$ , then  $f$  has a saddle point at  $0$ ; the level sets of  $f$  are hyperboloids with principal axes aligned with the orthonormal eigenvectors  $v_1, \dots, v_n$ .*

## Bounded Linear Maps

**Definition 3.** Suppose  $X, Y$  are normed vector spaces and  $T \in L(X, Y)$ . We say  $T$  is bounded if

$$\exists \beta \in \mathbf{R} \text{ s.t. } \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X$$

Note this implies that  $T$  is Lipschitz with constant  $\beta$ .

# Bounded Linear Maps

Much more is true:

**Theorem 3** (Thms. 4.1, 4.3). *Let  $X$  and  $Y$  be normed vector spaces and  $T \in L(X, Y)$ . Then*

- $T$  is continuous at some point  $x_0 \in X$*
- $\iff$   *$T$  is continuous at every  $x \in X$*
- $\iff$   *$T$  is uniformly continuous on  $X$*
- $\iff$   *$T$  is Lipschitz*
- $\iff$   *$T$  is bounded*

*Proof.* Suppose  $T$  is continuous at  $x_0$ . Fix  $\varepsilon > 0$ . Then there exists  $\delta > 0$  such that

$$\|z - x_0\| < \delta \Rightarrow \|T(z) - T(x_0)\| < \varepsilon$$

Now suppose  $x$  is any element of  $X$ . If  $\|y - x\| < \delta$ , let  $z = y - x + x_0$ , so  $\|z - x_0\| = \|y - x\| < \delta$ .

$$\begin{aligned}\|T(y) - T(x)\| &= \|T(y - x)\| \\ &= \|T(y - x + x_0 - x_0)\| \\ &= \|T(z) - T(x_0)\| \\ &< \varepsilon\end{aligned}$$

which proves that  $T$  is continuous at every  $x$ , and uniformly continuous.

We claim that  $T$  is bounded if and only if  $T$  is continuous at 0. Suppose  $T$  is not bounded. Then

$$\exists \{x_n\} \text{ s.t. } \|T(x_n)\| > n\|x_n\| \quad \forall n$$

Note that  $x_n \neq 0$ . Let  $\varepsilon = 1$ . Fix  $\delta > 0$  and choose  $n$  such that  $\frac{1}{n} < \delta$ . Let

$$\begin{aligned}
 x'_n &= \frac{x_n}{n\|x_n\|} \\
 \|x'_n\| &= \frac{\|x_n\|}{n\|x_n\|} \\
 &= \frac{1}{n} \\
 &< \delta \\
 \|T(x'_n) - T(0)\| &= \|T(x'_n)\| \\
 &= \frac{1}{n\|x_n\|} \|T(x_n)\| \\
 &> \frac{n\|x_n\|}{n\|x_n\|} \\
 &= 1 \\
 &= \varepsilon
 \end{aligned}$$

Since this is true for every  $\delta$ ,  $T$  is not continuous at 0. Therefore,  $T$  continuous at 0 implies  $T$  is bounded. Now, suppose  $T$  is bounded, so find  $M$  such that  $\|T(x)\| \leq M\|x\|$  for every  $x \in X$ . Given  $\varepsilon > 0$ , let  $\delta = \varepsilon/M$ . Then

$$\begin{aligned}\|x - 0\| < \delta &\Rightarrow \|x\| < \delta \\ &\Rightarrow \|T(x) - T(0)\| = \|T(x)\| < M\delta \\ &\Rightarrow \|T(x) - T(0)\| < \varepsilon\end{aligned}$$

so  $T$  is continuous at 0.

Thus, we have shown that continuity at some point  $x_0$  implies uniform continuity, which implies continuity at every point, which implies  $T$  is continuous at 0, which implies that  $T$  is bounded, which implies that  $T$  is continuous at 0, which implies that  $T$  is

continuous at some  $x_0$ , so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose  $T$  is bounded, with constant  $M$ . Then

$$\begin{aligned}\|T(x) - T(y)\| &= \|T(x - y)\| \\ &\leq M\|x - y\|\end{aligned}$$

so  $T$  is Lipschitz with constant  $M$ ; conversely, if  $T$  is Lipschitz with constant  $M$ , then  $T$  is bounded with constant  $M$ . So all the statements are equivalent.  $\square$

# Bounded Linear Maps

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

**Theorem 4** (Thm. 4.5). *Let  $X$  and  $Y$  be normed vector spaces, with  $\dim X = n$ . Every  $T \in L(X, Y)$  is bounded.*

*Proof.* See de la Fuente.





# Topological Isomorphism

**Definition 4.** A topological isomorphism *between normed vector spaces  $X$  and  $Y$*  is a linear transformation  $T \in L(X, Y)$  that is invertible (one-to-one, onto), continuous, and has a continuous inverse.

*Two normed vector spaces  $X$  and  $Y$  are topologically isomorphic if there is a topological isomorphism  $T : X \rightarrow Y$ .*

## The Space $B(X, Y)$

Suppose  $X$  and  $Y$  are normed vector spaces. We define

$$\begin{aligned} B(X, Y) &= \{T \in L(X, Y) : T \text{ is bounded}\} \\ \|T\|_{B(X, Y)} &= \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X}, x \in X, x \neq 0 \right\} \\ &= \sup \{\|T(x)\|_Y : \|x\|_X = 1\} \end{aligned}$$

We skip the proofs of the rest of these results – read dIF.

## The Space $B(X, Y)$

**Theorem 5** (Thm. 4.8). *Let  $X, Y$  be normed vector spaces. Then*

$$\left( B(X, Y), \|\cdot\|_{B(X, Y)} \right)$$

*is a normed vector space.*

## The Space $B(\mathbf{R}^n, \mathbf{R}^m)$

**Theorem 6** (Thm. 4.9). *Let  $T \in L(\mathbf{R}^n, \mathbf{R}^m)$  ( $= B(\mathbf{R}^n, \mathbf{R}^m)$ ) with matrix  $A = (a_{ij})$  with respect to the standard bases. Let*

$$M = \max\{|a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$$

*Then*

$$M \leq \|T\| \leq M\sqrt{mn}$$

.

# Compositions

**Theorem 7** (Thm. 4.10). *Let  $R \in L(\mathbf{R}^m, \mathbf{R}^n)$  and  $S \in L(\mathbf{R}^n, \mathbf{R}^p)$ .  
Then*

$$\|S \circ R\| \leq \|S\| \|R\|$$

# Invertibility

Define  $\Omega(\mathbf{R}^n) = \{T \in L(\mathbf{R}^n, \mathbf{R}^n) : T \text{ is invertible}\}$

**Theorem 8** (Thm. 4.11'). *Suppose  $T \in L(\mathbf{R}^n, \mathbf{R}^n)$  and  $E$  is the standard basis of  $\mathbf{R}^n$ . Then*

*$T$  is invertible*

$$\iff \ker T = \{0\}$$

$$\iff \det(Mtx_E(T)) \neq 0$$

$$\iff \det(Mtx_{V,V}(T)) \neq 0 \text{ for every basis } V$$

$$\iff \det(Mtx_{V,W}(T)) \neq 0 \text{ for every pair of bases } V, W$$

## Invertibility

**Theorem 9** (Thm. 4.12). *If  $S, T \in \Omega(\mathbf{R}^n)$ , then  $S \circ T \in \Omega(\mathbf{R}^n)$  and*

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}$$

## Invertibility

**Theorem 10** (Thm. 4.14). *Let  $S, T \in L(\mathbf{R}^n, \mathbf{R}^n)$ . If  $T$  is invertible and*

$$\|T - S\| < \frac{1}{\|T^{-1}\|}$$

*then  $S$  is invertible. In particular,  $\Omega(\mathbf{R}^n)$  is open in  $L(\mathbf{R}^n, \mathbf{R}^n) = B(\mathbf{R}^n, \mathbf{R}^n)$ .*

**Theorem 11** (Thm. 4.15). *The function  $(\cdot)^{-1} : \Omega(\mathbf{R}^n) \rightarrow \Omega(\mathbf{R}^n)$  that assigns  $T^{-1}$  to each  $T \in \Omega(\mathbf{R}^n)$  is continuous.*