# Econ 204 2021

## Lecture 13

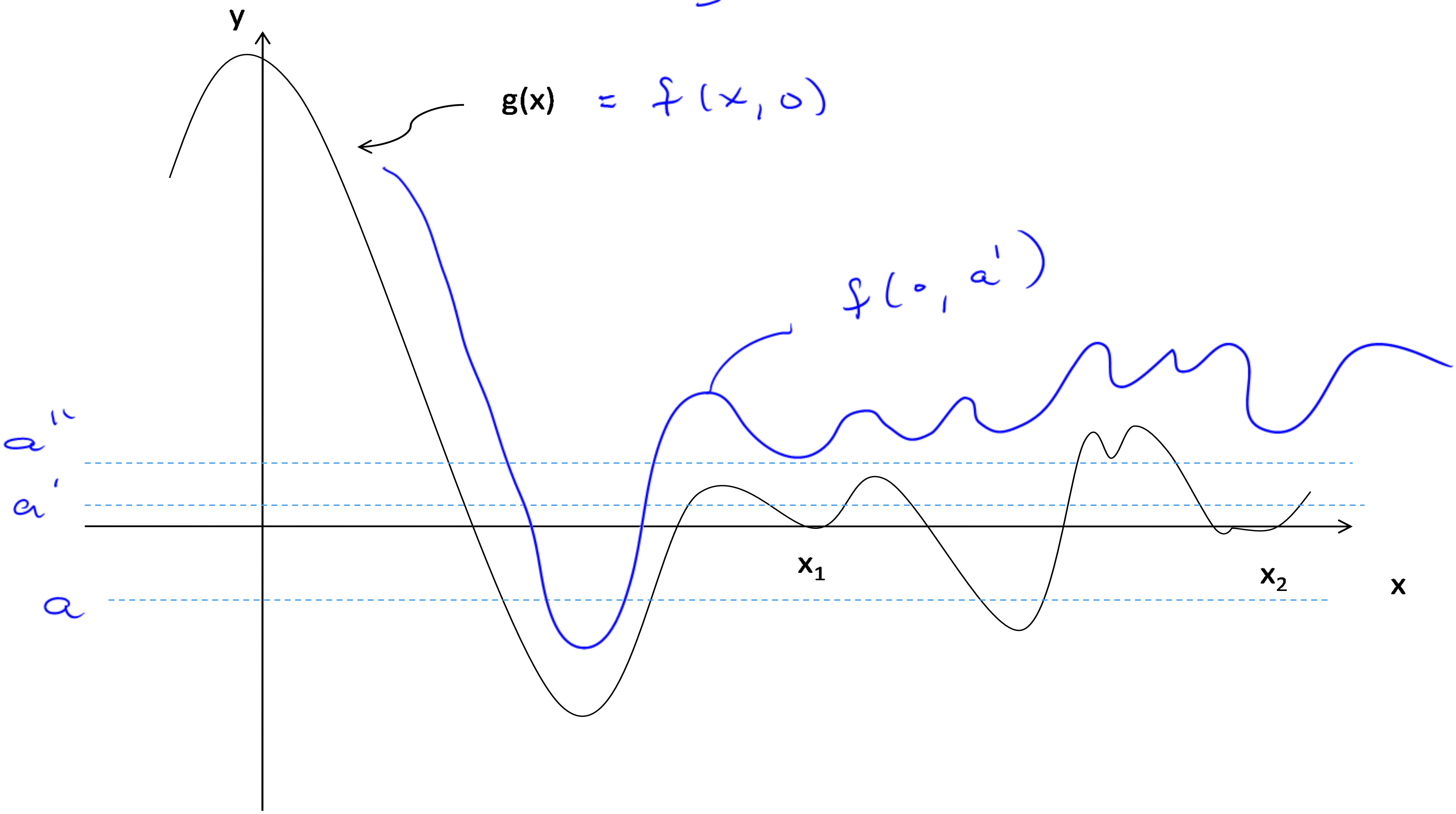
### Outline

0. Transversality Theorem

1. Fixed Points for Functions
2. Brouwer's Fixed Point Theorem
3. Fixed Points for Correspondences
4. Kakutani's Fixed Point Theorem
5. Separating Hyperplane Theorems

$g(x) = b$

$g(x) + a = 0$ vs. $f(x, a) = 0$



# Transversality

Suppose $f : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^m$. We care about the parameterized family of equations

$$f(x, a) = 0$$

where, as above, we interpret $a \in \mathbf{R}^p$ to be a vector of parameters that indexes the function $f(\cdot, a)$.

For a given $a$, we are interested in the set of solutions

$$\psi(a) = \{x \in X : f(x, a) = 0\}$$

and the way that this correspondence depends on $a$.

If $f$ is separable in $a$, that is, $f(x, a) = g(x) + a$, then we can use Sard's Theorem (PS6 2010).

$(x, a)$ regular point for $f$

but $x$ not a regular point for $f_a$

# Transversality Theorem

$g(x) + a$ vs. $f(x, a)$

Separability is strong, and not required: If $f$ depends on $a$ in a nonseparable fashion, it is enough that from any solution $f(x, a) = 0$, any directional change in $f$ can be achieved by arbitrarily small changes in $x$ and $a$.

**Theorem 4** (Thm. 2.5', Transversality Theorem)**.** *Let $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ be open, and $f : X \times A \to \mathbf{R}^m$ be $C^r$ with $r \geq 1 + \max\{0, n - m\}$. Suppose that 0 is a regular value of $f$. Then there is a set $A_0 \subseteq A$ such that $A \setminus A_0$ has Lebesgue measure zero and for all $a \in A_0$, 0 is a regular value of $f_a = f(\cdot, a)$.*

$D_x f(x_0, a_0)$

**Remark:** Notice the important difference between the statement that 0 is a regular value of $f$ (one of the assumptions of the Transversality Theorem), and the statement that 0 is a regular value of $f_a$ for a fixed $a \in A_0$ (part of the conclusion of the Transversality Theorem). 0 is a regular value of $f$ if and only if $Df(x, a)$ has full rank for every $(x, a)$ such that $f(x, a) = 0$. Instead, for fixed $a_0 \in A_0$, 0 is a regular value of $f_{a_0} = f(\cdot, a_0)$ if and only if $D_x f(x, a_0)$ has full rank for every $x$ such that $f_{a_0}(x) = f(x, a_0) = 0$.

**Remark:** Consider the important special case in which $n = m$, so we have as many equations ($m$) as endogenous variables ($n$). In this case, suppose $f$ is $C^1$ (note that $1 = 1 + \max\{0, n - n\}$). If 0 is a regular value of $f$, that is, $Df(x, a)$ has rank $n = m$ for every $(x, a)$ such that $f(x, a) = 0$, then by the Transversality Theorem there is a set $A_0 \subset A$ such that $A \setminus A_0$ has Lebesgue measure zero and for every $a_0 \in A_0$, $D_x f(x, a_0)$ has rank $n = m$ for all $x$ such that $f(x, a_0) = 0$.

Fix $a_0 \in A_0$ and $x_0$ such that $f(x_0, a_0) = 0$. By the Implicit Function Theorem, there exist open sets $A^*$ containing $a_0$ and $X^*$ containing $x_0$, and a $C^1$ function $x : A^* \to X^*$ such that

- $x(a_0) = x_0$

- $f(x(a), a) = 0$ for every $a \in A^*$

- if $(x, a) \in X^* \times A^*$ then

$$f(x, a) = 0 \iff x = x(a)$$

  that is, $x_0$ is locally unique, and $x(a)$ is locally unique for each $a \in A^*$

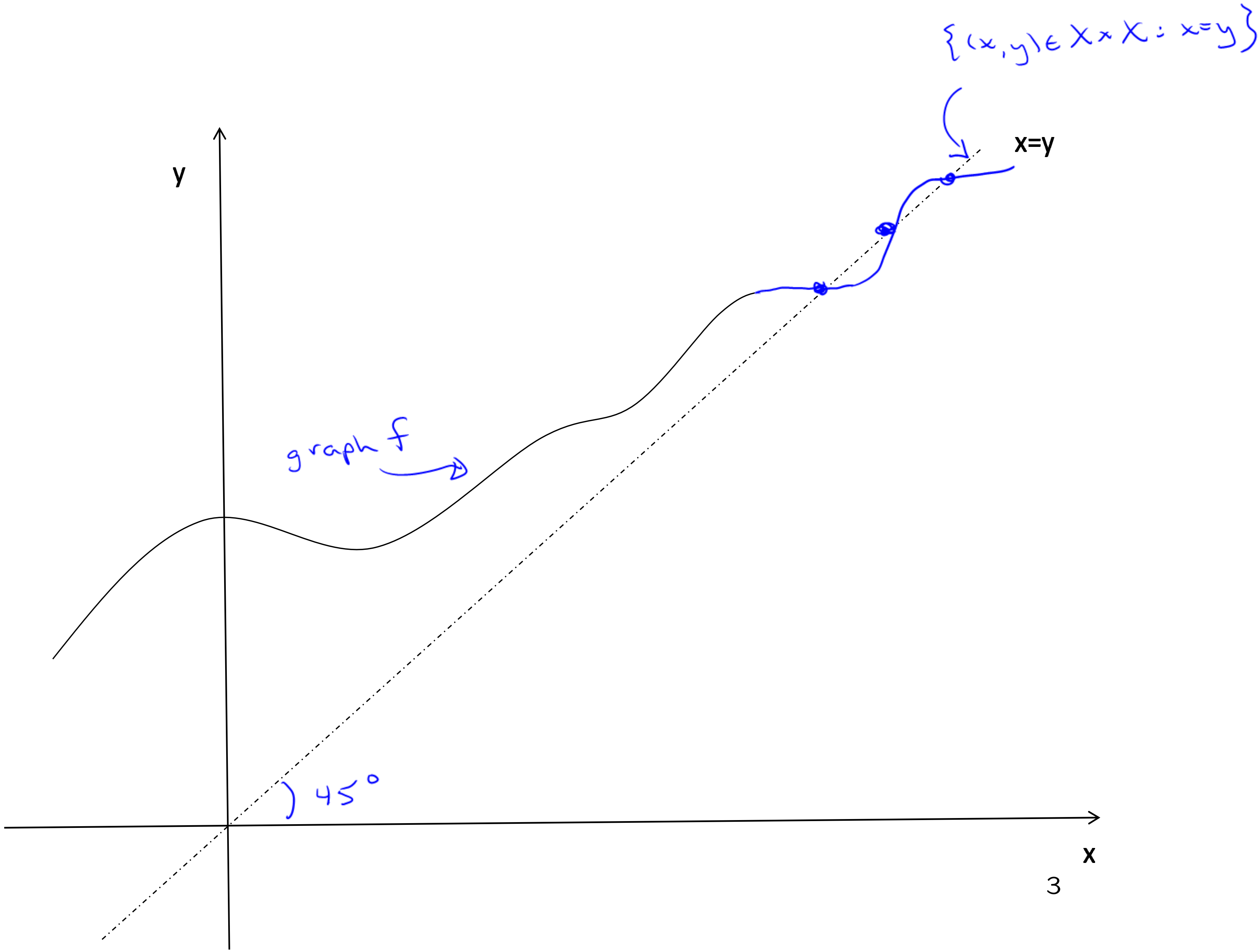
- $Dx(a_0) = -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0)$

Recall :

# Fixed Points for Functions

**Definition 1.** *Let $X$ be a nonempty set and $f : X \to X$. A point $x^* \in X$ is a* fixed point *of $f$ if $f(x^*) = x^*$.*

$x^*$ is a fixed point of $f$ if it is "fixed" by the map $f$.


{(x,y) ∈ X × X : x=y}
x=y
y
graph f
45°
x

# Fixed Points for Functions

**Examples:**

1. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x$. Then $x = 0$ is a fixed point of $f$ (and is the unique fixed point of $f$).

   $f(x) = 2x = x \iff x = 0$

2. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x$. Then every point in $\mathbf{R}$ is a fixed point of $f$ (in particular, fixed points need not be unique).

3. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x + 1$. Then $f$ has no fixed points.

   $f(x) = x+1 \neq x \quad \forall x \in \mathbb{R}$

4. Let $X = [0, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}(x + 1)$. Then

$$\begin{aligned} f(x) &= \frac{1}{2}(x+1) = x \\ &\iff x + 1 = 2x \\ &\iff x = 1 \end{aligned}$$

So $x = 1$ is the unique fixed point of $f$. Notice that $f$ is a contraction (why?), so we already knew that $f$ must have a unique fixed point on $\mathbf{R}$ from the Contraction Mapping Theorem.

5. Let $X = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $f : X \to X$ be given by $f(x) = 1 - x$. Then $f$ has no fixed points.

$f(x) = 1 - x = x \iff x = \frac{1}{2} \notin X$

6. Let $X = [-2, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}x^2$. Then $f$ has two fixed points, $x = 0$ and $x = 2$. If instead $X' = (0, 2)$, then $f : X' \to X'$ but $f$ has no fixed points on $X'$.

7. Let $X = \{1, 2, 3\}$ and $f : X \to X$ be given by $f(1) = 2, f(2) = 3, f(3) = 1$ (so $f$ is a permutation of $X$). Then $f$ has no fixed points.

8. Let $X = [0, 2]$ and $f : X \to X$ be given by

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

Then $f$ has no fixed points.

# A Simple Fixed Point Theorem

**Theorem 1.** *Let $X = [a, b]$ for $a, b \in \mathbf{R}$ with $a < b$ and let $f : X \to X$ be continuous. Then $f$ has a fixed point.*

*Proof.* Let $g : [a, b] \to \mathbf{R}$ be given by

$$g(x) = f(x) - x$$

$g(x) = 0 \iff x$ is a fixed point of $f$

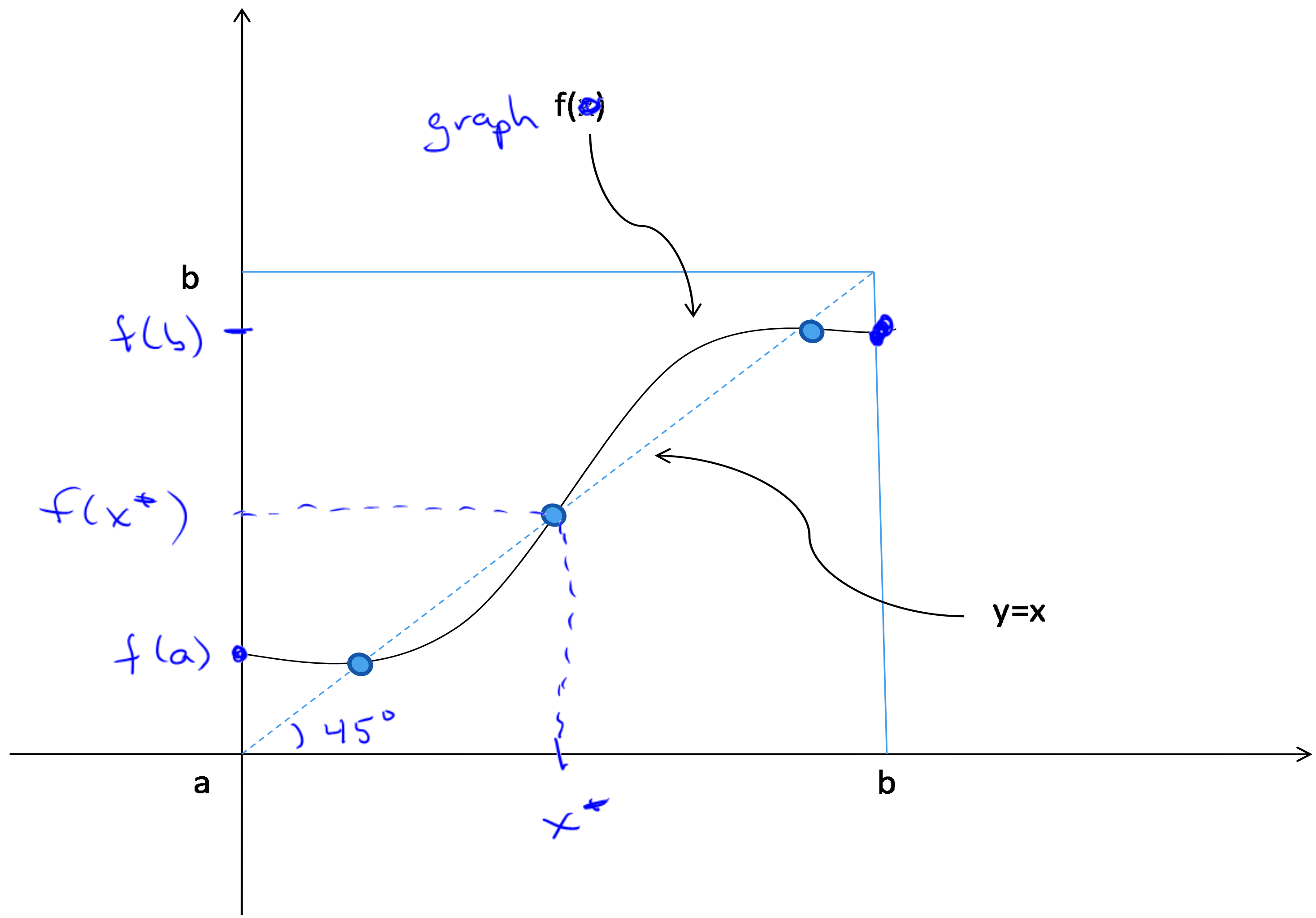
If either $f(a) = a$ or $f(b) = b$, we're done. So assume $f(a) > a$ and $f(b) < b$. Then

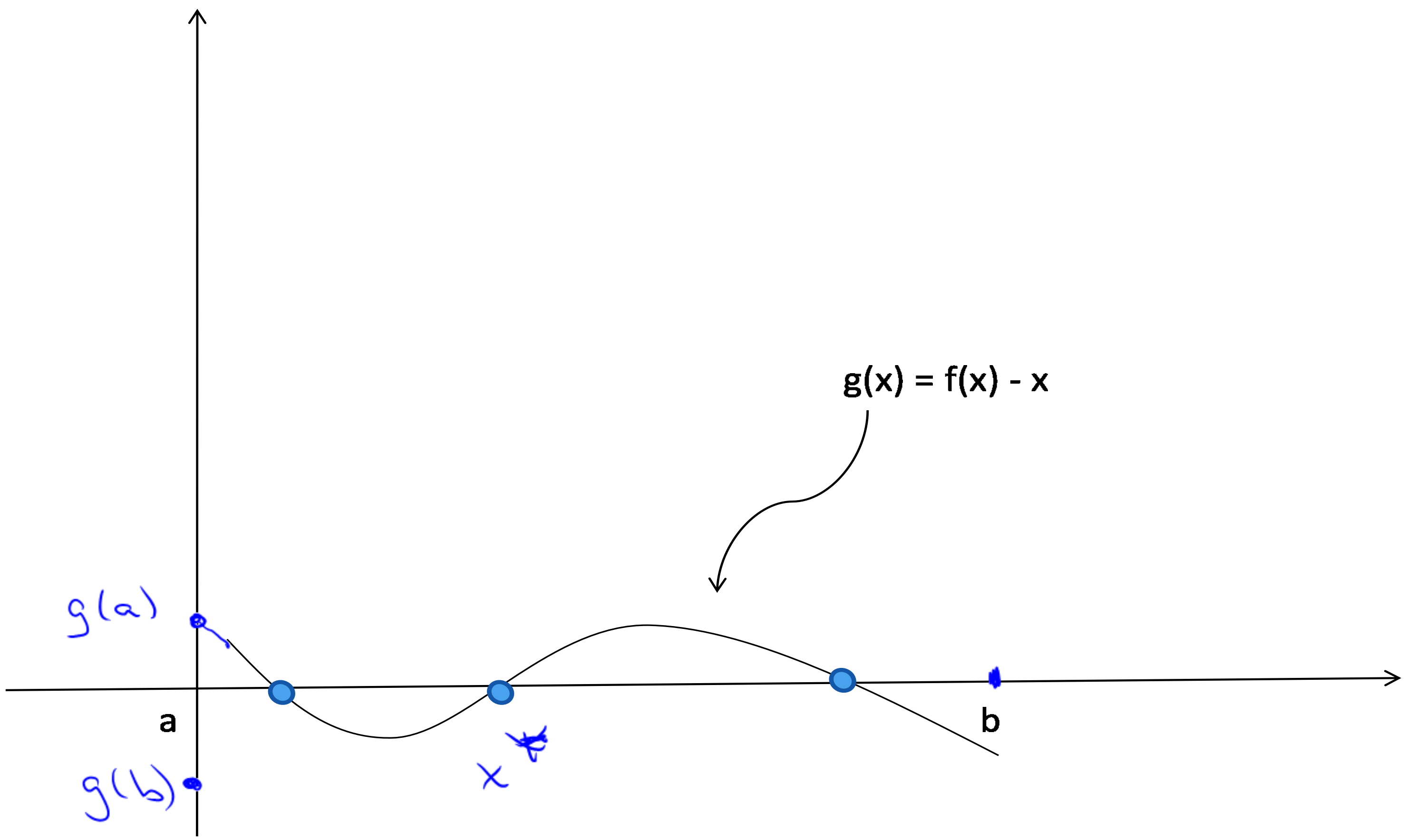
($f(a) \in [a,b]$)

($f(b) \in [a,b]$)

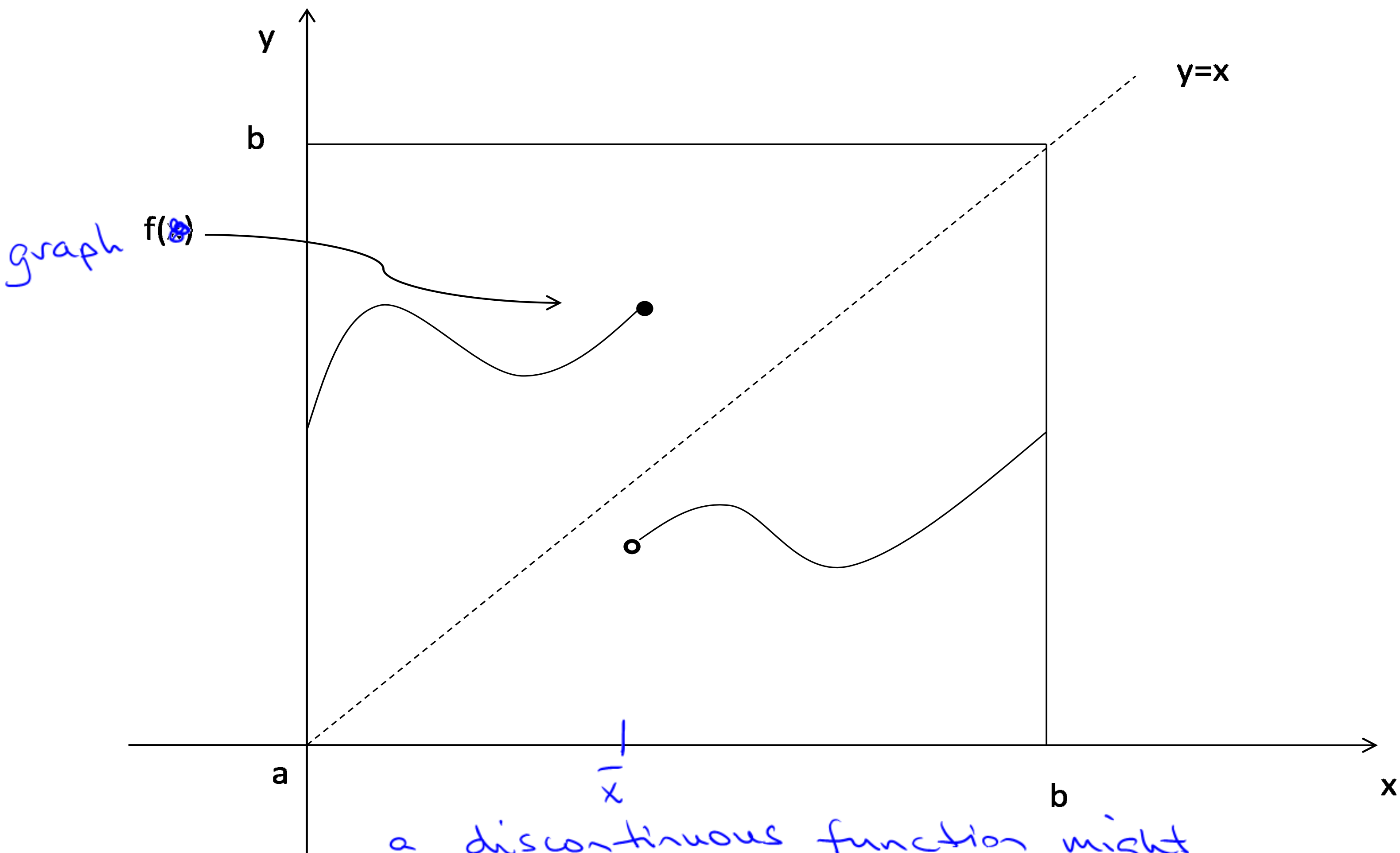
$$\begin{aligned} g(a) &= f(a) - a > 0 \\ g(b) &= f(b) - b < 0 \end{aligned}$$

$g$ is continuous, so by the Intermediate Value Theorem, $\exists x^* \in (a, b)$ such that $g(x^*) = 0$, that is, such that $f(x^*) = x^*$. $\square$


graph f(x)
b
f(b)
f(x*)
f(a)
45°
a
x*
b
y=x




$\exists x^* \in (a, b)$ s.t. $g(x^*) = 0$




a discontinuous function might have no fixed points

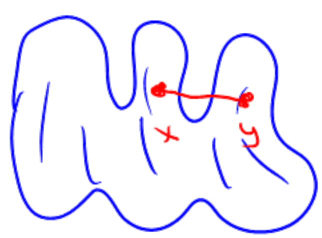
# Brouwer's Fixed Point Theorem

**Theorem 2** (Thm. 3.2. Brouwer's Fixed Point Theorem)**.** *Let $X \subseteq \mathbf{R}^n$ be nonempty, compact, and convex, and let $f : X \to X$ be continuous. Then $f$ has a fixed point.*

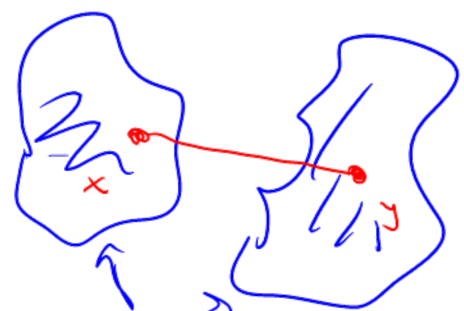

$X \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall x, y \in X$ $\forall \alpha \in [0,1]$

$\alpha x + (1-\alpha) y \in X$


X convex




D <u>not</u> convex




B not convex

$X \subseteq \mathbb{R}^n$ nonempty, compact, convex homeomorphic to $B^m$ in $\mathbb{R}^m$ for some $m$

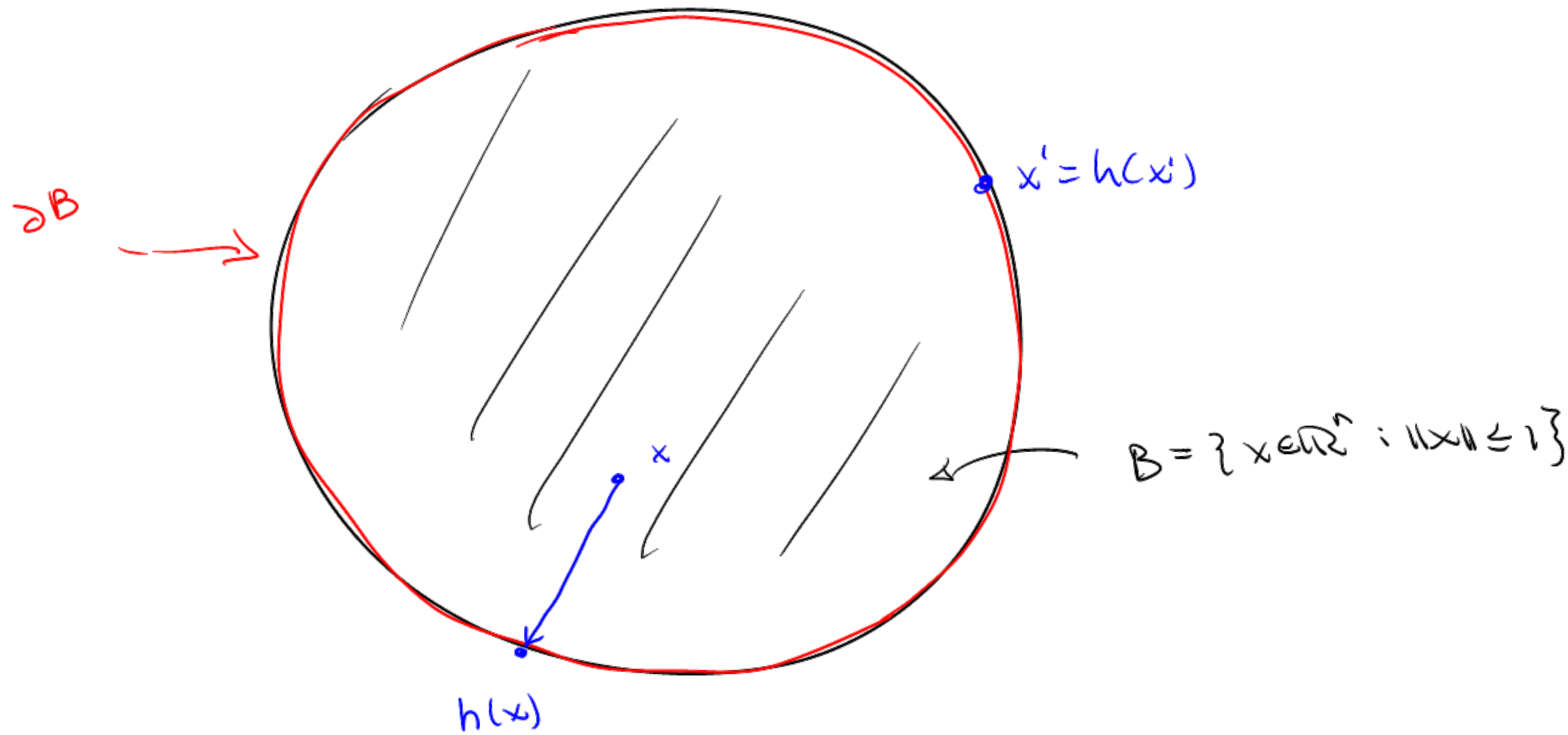
# Sketch of Proof of Brouwer

Consider the case when the set $X$ is the closed unit ball in $\mathbf{R}^n$, i.e. $X = B_1[0] = B = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$. Let $f : B \to B$ be a continuous function. Recall that $\partial B$ denotes the boundary of $B$, so $\partial B = \{x \in \mathbf{R}^n : \|x\| = 1\}$.

**Fact:** Let $B$ be the unit ball in $\mathbf{R}^n$. Then there is no continuous function $h : B \to \partial B$ such that $h(x') = x'$ for every $x' \in \partial B$.

See J. Franklin, Methods of Mathematical Economics, for an elementary (but long) proof.

(also Y. Kannai, Am. Math. Monthly, April 1981, pp. 264-268.)




$\nexists\ h : B \to \partial B$ continuous such that

$$x' = h(x') \quad \forall x' \in \partial B$$

$f: B \to B$ cont.

Now to establish Brouwer's theorem, suppose, by way of contradiction, that $f$ has no fixed points in $B$. Thus for every $x \in B$, $x \neq f(x)$.

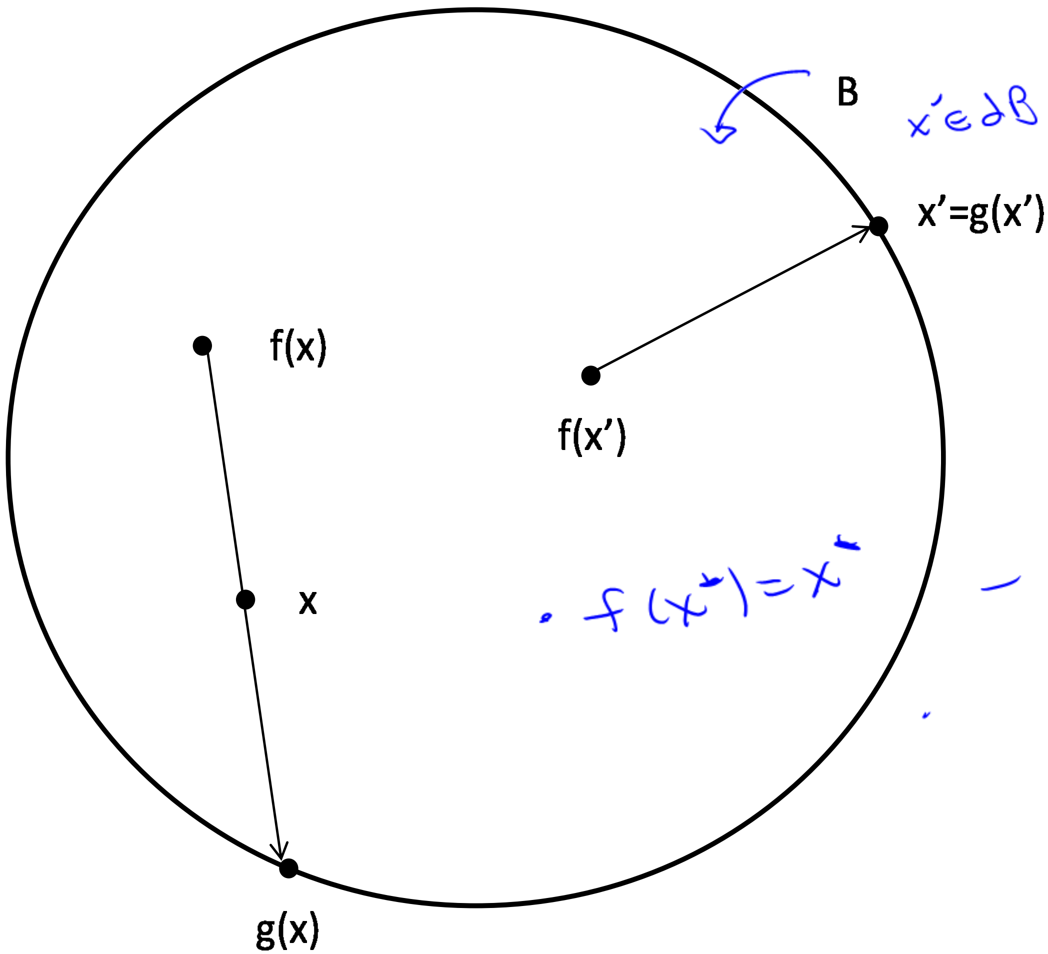
Since $x \neq f(x)$ for every $x$, we can carry out the following construction. For each $x \in B$, construct the line segment originating at $f(x)$ and going through $x$. Let $g(x)$ denote the intersection of this line segment with $\partial B$.

This construction is well-defined, and gives a continuous function $g : B \to \partial B$. Furthermore, if $x' \in \partial B$, then $x' = g(x')$. That is, $g|_{\partial B} = \mathrm{id}_{\partial B}$. Since there are no such functions by the fact above, we have a contradiction. Therefore there exists $x^* \in B$ such that $f(x^*) = x^*$, that is, $f$ has a fixed point in $B$.

$$g(x) = x + tu$$

where $u = \dfrac{x - f(x)}{\| x - f(x) \|}$

$$t = -x \cdot u + \sqrt{1 - x \cdot x + (x \cdot u)^2}$$



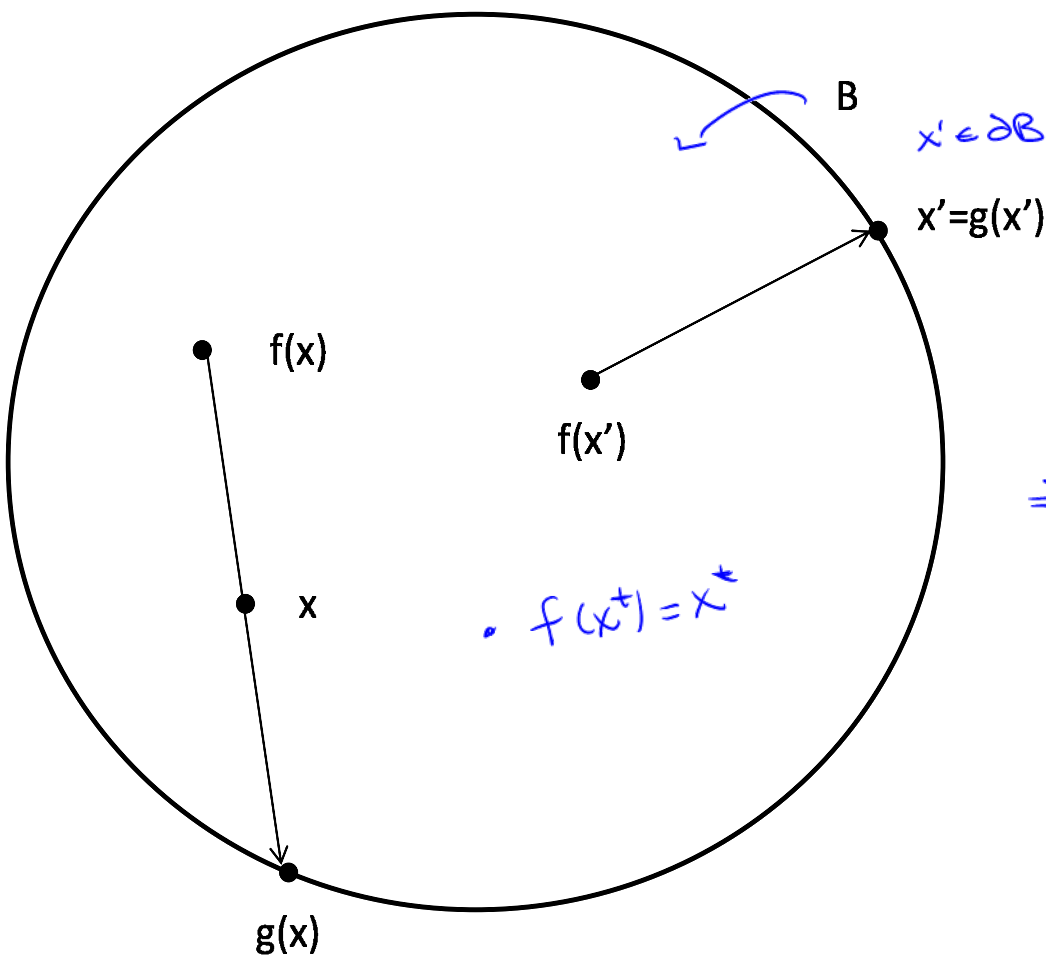

$x' \in \partial B \Rightarrow$

- $f(x^*) = x^*$
- what is $g(x^*)$?

$g(x) = x + tu$

where $u = \dfrac{x - f(x)}{\|x - f(x)\|}$

$t = -x\cdot u + \sqrt{1 - x\cdot x + (x\cdot u)^2}$




$x' \in \partial B \Rightarrow$

$\cdot\ f(x^*) = x^*$

$x \neq f(x)\ \forall x$

$\Rightarrow\ g: B \to \partial B$ is well-defined and continuous

$\Rightarrow\!\Leftarrow$

# Fixed Points for Correspondences

**Definition 2.** *Let* $X$ *be nonempty and* $\Psi : X \to 2^X$ *be a correspondence. A point* $x^* \in X$ *is a* fixed point *of* $\Psi$ *if* $x^* \in \Psi(x^*)$*.*

Note here that we do *not* require $\Psi(x^*) = \{x^*\}$, that is $\Psi$ need not be single-valued at $x^*$. So $x^*$ can be a fixed point of $\Psi$ but there may be other elements of $\Psi(x^*)$ different from $x^*$.

**Examples:**

1. Let $X = [0,4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} [x+1, x+2] & \text{if } x < 2 \\ [0,4] & \text{if } x = 2 \\ [x-2, x-1] & \text{if } x > 2 \end{cases}$$

Then $x = 2$ is the unique fixed point of $\Psi$.

$2 \in \Psi(2) = [0,4]$

$x \notin [x+1, x+2]$

$x \notin [x-2, x-1]$

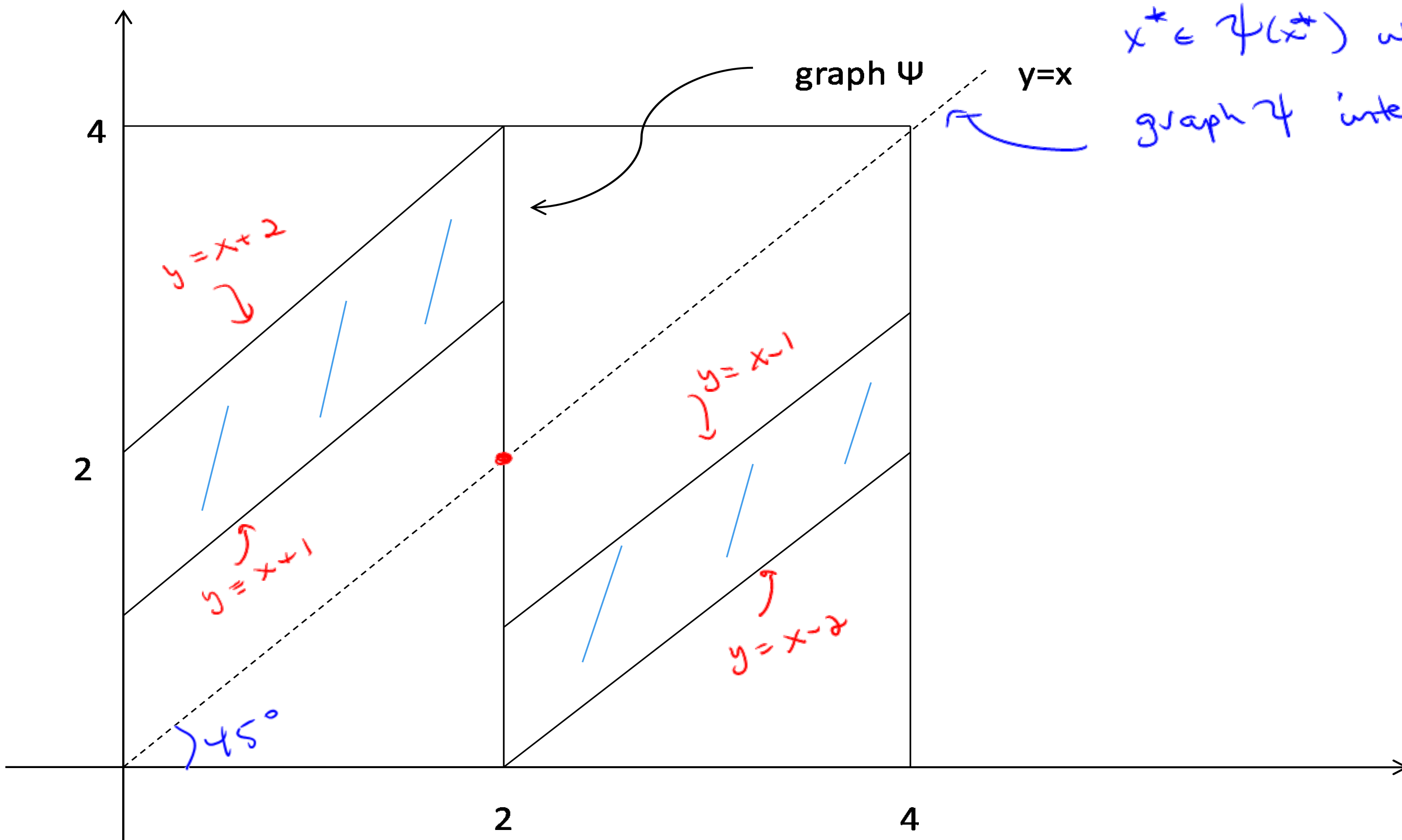
$\Rightarrow x \notin \Psi(x) \quad \forall x \neq 2$

2. Let $X = [0,4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} [x+1, x+2] & \text{if } x < 2 \\ [0,1] \cup [3,4] & \text{if } x = 2 \\ [x-2, x-1] & \text{if } x > 2 \end{cases}$$

Then $\Psi$ has no fixed points.

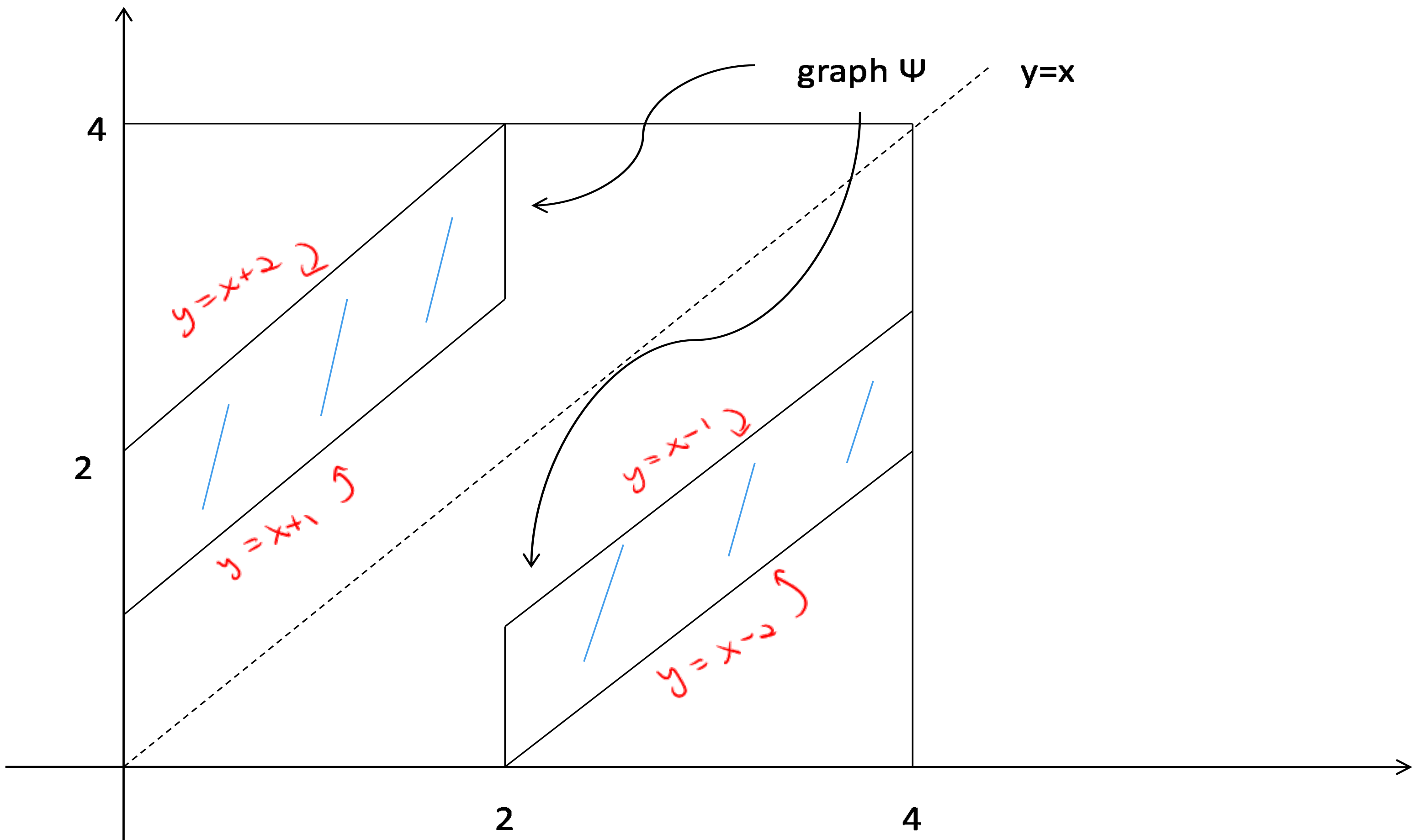
$2 \notin \Psi(2) = [0,1] \cup [3,4]$




$x^* \in \Psi(x^*)$ where graph $\Psi$ intersects

$2 \in \Psi(2) = [0,4]$

$\Psi(2)$ is $\underline{\text{not}}$ convex




$2 \notin \Psi(2) = [0,1] \cup [3,4]$



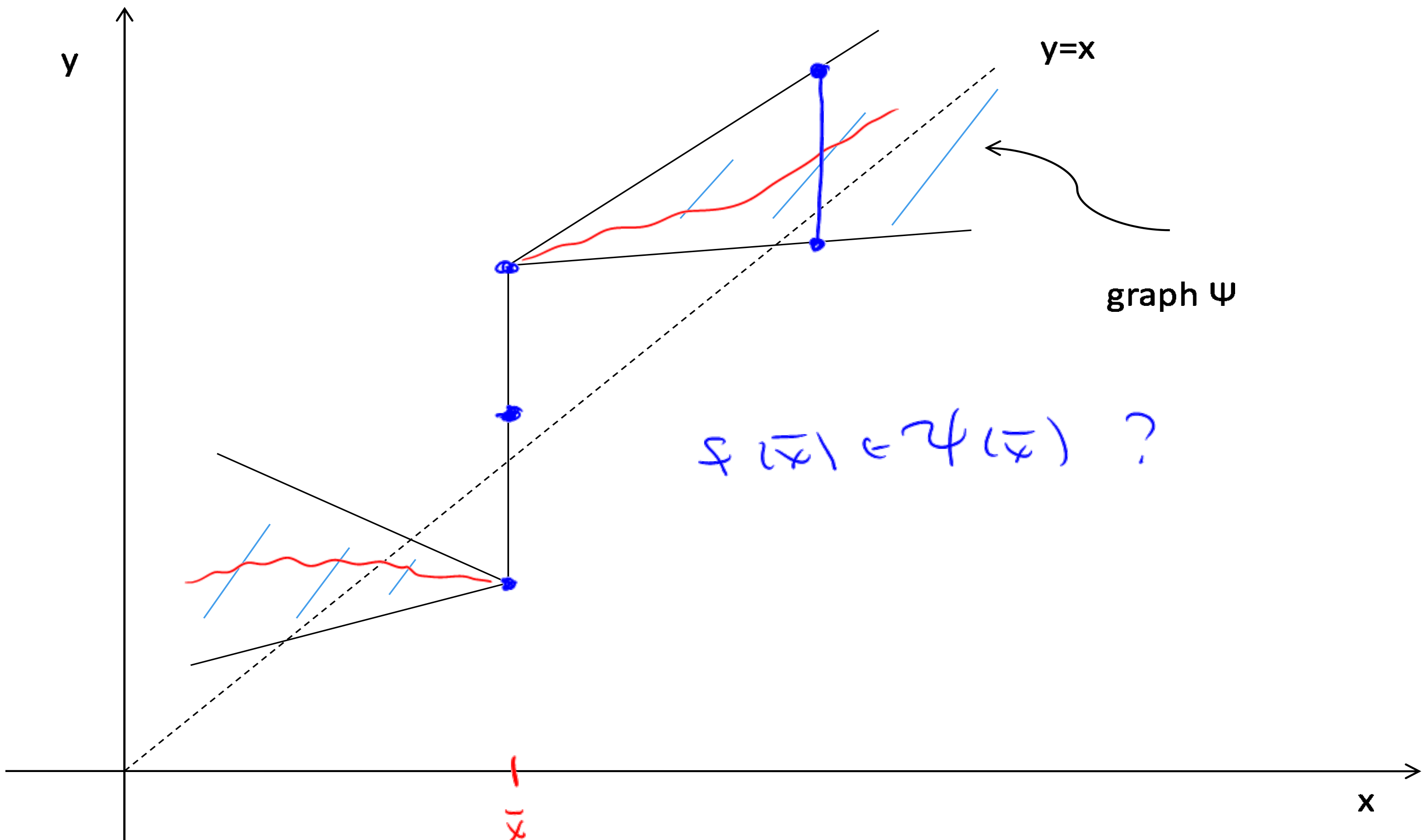
Note: $\Psi$ is uhc in both cases

$\forall x$: $\Psi(x)$ nonempty, compact, convex

# Kakutani's Fixed Point Theorem

**Theorem 3. (Thm. 3.4'. Kakutani's Fixed Point Theorem)** *Let $X \subseteq \mathbf{R}^n$ be a non-empty, compact, convex set and $\Psi : X \to 2^X$ be an upper hemi-continuous correspondence with non-empty, convex, compact values. Then $\Psi$ has a fixed point in $X$.*

*Proof.* **(sketch)** Here, the idea is to use Brouwer's theorem after appropriately approximating the correspondence with a function. The catch is that there won't necessarily exist a continuous selection from $\Psi$, that is, a continuous function $f : X \to X$ such that $f(x) \in \Psi(x)$ for every $x \in X$. If such a function existed, then by applying Brouwer to $f$ we would have a fixed point of $\Psi$ (because if $\exists x^* \in X$ such that $x^* = f(x^*)$, then $x^* = f(x^*) \in \Psi(x^*)$).




$f(\bar{x}) \in \psi(\bar{x})$ ?

? $\exists\, f: X \to X$ such that $f(x) \in \psi(x)\ \forall x$ and $f$ continuous

$\psi(x)$ convex $\forall x \in X$, $\psi$ uhc

Instead, we look for a weaker type of approximation. Let $X \subset \mathbf{R}^n$ be a non-empty, compact, convex set, and let $\Psi : X \to 2^X$ be an uhc correspondence with non-empty, compact, convex values. For every $\varepsilon > 0$, define the $\varepsilon$ ball about graph $\Psi$ to be

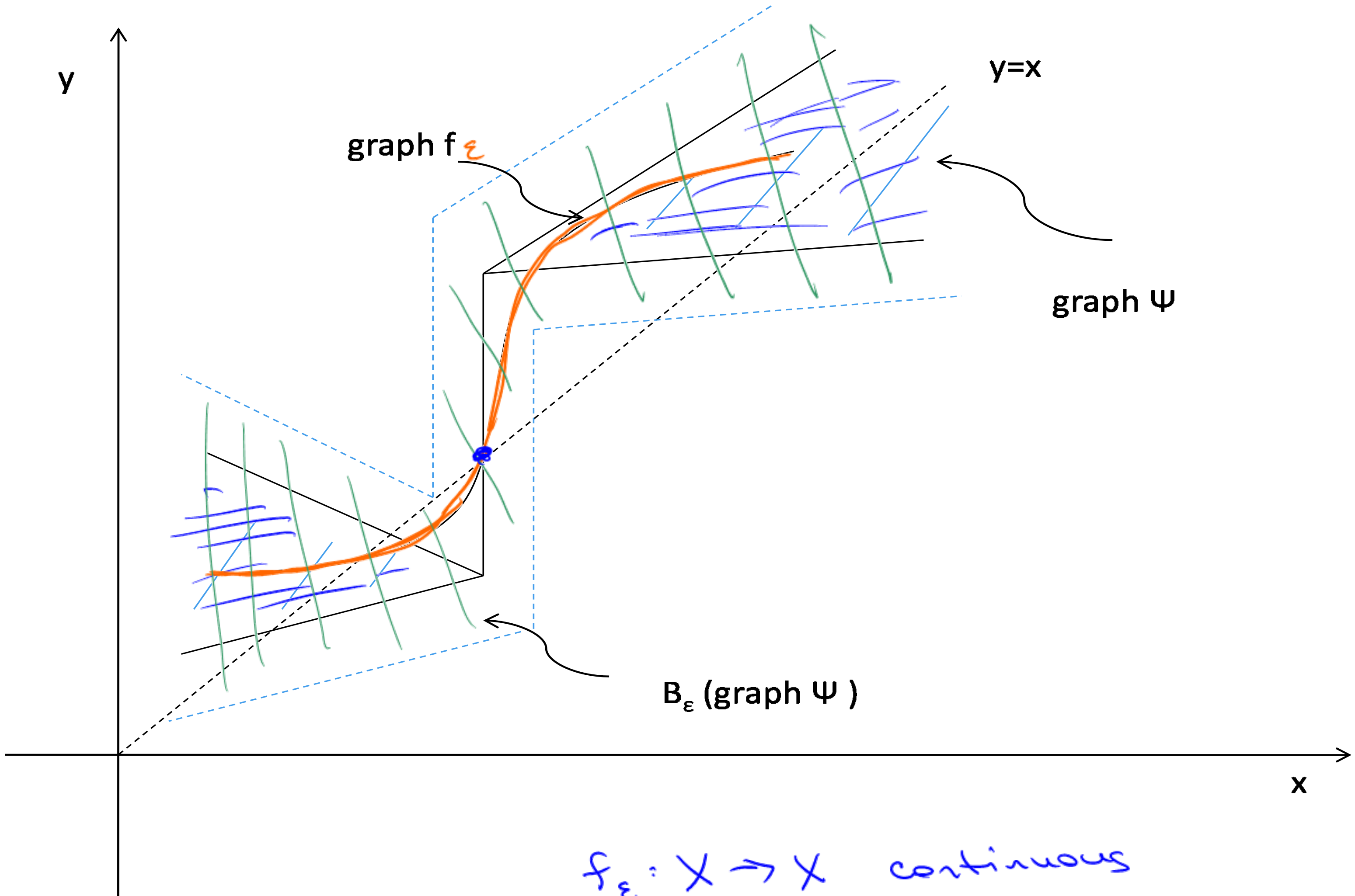
$$B_\varepsilon(\text{ graph } \Psi) = \left\{ z \in X \times X : d(z, \text{ graph } \Psi) = \inf_{(x,y) \in \text{ graph } \Psi} d(z,(x,y)) < \varepsilon \right\}$$

Here $d$ denotes the ordinary Euclidean distance. Since $\Psi$ is uhc and convex-valued, for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : X \to X$ such that graph $f_\varepsilon \subseteq B_\varepsilon($ graph $\Psi)$.

$\exists\, f_n : X \to X$ graph $f_{\frac{1}{n}} \subseteq B_{\frac{1}{n}}(\text{graph } \Psi)$ th

cont. s.t.

$\varepsilon > 0$




$f_\varepsilon : X \to X$ continuous

graph $f_\varepsilon \subseteq B_\varepsilon(\text{graph } \Psi)$

$\varepsilon_n = \frac{1}{n}\ \forall n$ $\quad f_n : X \to X\ \forall n$

Now by letting $\varepsilon \to 0$, this means that we can find a sequence of continuous functions $\{f_n\}$ such that graph $f_n \subseteq B_{\frac{1}{n}}($ graph $\Psi)$ for each $n$. By Brouwer's Fixed Point Theorem, each function $f_n$ has a fixed point $\hat{x}_n \in X$, and

$$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \text{ graph } f_n \subseteq B_{\frac{1}{n}}(\text{ graph } \Psi) \text{ for each } n$$

So for each $n$ there exists $(x_n, y_n) \in$ graph $\Psi$ such that $d((\hat{x}_n, \hat{x}_n), (x_n, y_n)) < \frac{1}{n}$

$$d(\hat{x}_n, x_n) < \frac{1}{n} \text{ and } d(\hat{x}_n, y_n) < \frac{1}{n}$$

Since $X$ is compact, $\{\hat{x}_n\}$ has a convergent subsequence $\{\hat{x}_{n_k}\}$, with $\hat{x}_{n_k} \to \hat{x} \in X$. Then $x_{n_k} \to \hat{x}$ and $y_{n_k} \to \hat{x}$. Since $\Psi$ is uhc and closed-valued, it has closed graph, so $(\hat{x}, \hat{x}) \in$ graph $\Psi$. Thus $\hat{x} \in \Psi(\hat{x})$, that is, $\hat{x}$ is a fixed point of $\Psi$. $\square$

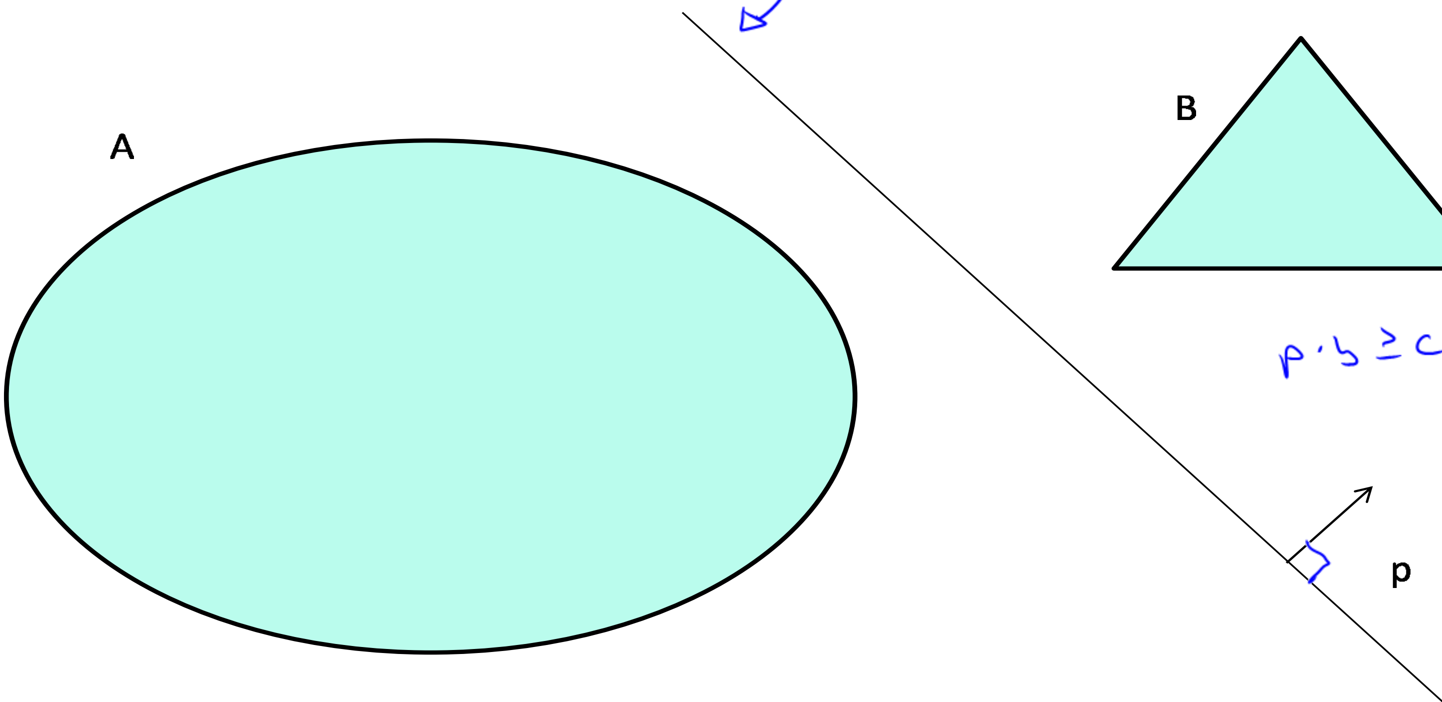
# Separating Hyperplane Theorems

**Theorem 4** (1.26, Separating Hyperplane Theorem)**.** *Let* $A, B \subseteq \mathbf{R}^n$ *be nonempty, disjoint convex sets. Then there exists a nonzero vector* $p \in \mathbf{R}^n$ *such that*

$$p \cdot a \leq p \cdot b \quad \forall a \in A, b \in B$$

hyperplane: $\{ z \in \mathbb{R}^n : v \cdot z = c \}$ for some $v \in \mathbb{R}^n$, $v \neq 0$,
and some $c \in \mathbb{R}$

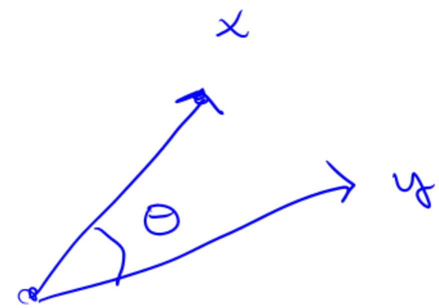
$H = \{ z \in \mathbb{R}^n ; p \cdot z = c \}$




$p \cdot b \geq c \quad \forall b \in B$

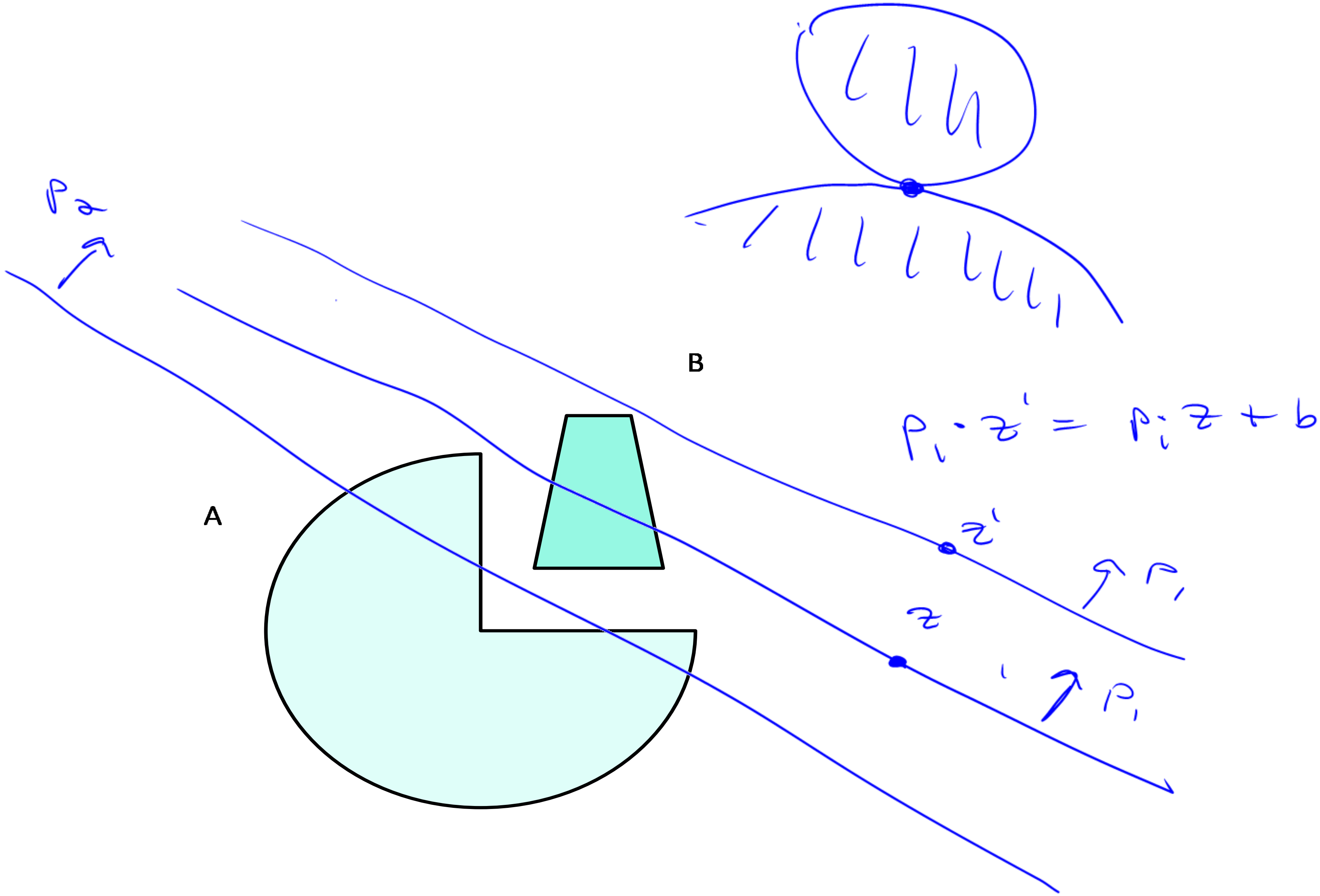
$p \cdot a \leq c \quad \forall a \in A$

$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}$$

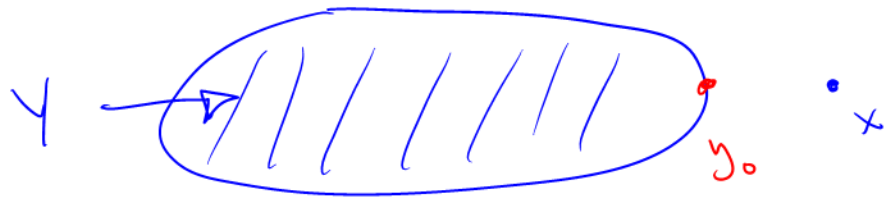



$$\cos\theta \geq 0 \iff \theta \leq 90^\circ$$

$$\cos\theta \leq 0 \iff \theta \geq 90^\circ$$




Convexity important: no hyperplane separates A and B




# Separating a Point from a Set

**Theorem 5.** *Let $Y \subseteq \mathbf{R}^n$ be a nonempty convex set and $x \notin Y$. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that*

$$p \cdot x \leq p \cdot y \quad \forall y \in Y$$

*Proof.* We sketch the proof in the special case that $Y$ is compact. We will see that in this case we actually get a stronger conclusion:

$$\exists p \in \mathbf{R}^n,\ p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

Choose $y_0 \in Y$ such that $\|y_0 - x\| = \inf\{\|y - x\| : y \in Y\}$; such a point exists because $Y$ is compact, so the distance function $g(y) = \|y - x\|$ assumes its minimum on $Y$. Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$. Let $p = y_0 - x$. The set

g is continuous

$$H = \{z \in \mathbf{R}^n : p \cdot z = p \cdot y_0\}$$

is the hyperplane perpendicular to $p$ through $y_0$. See Figure 12. Then

$$\begin{aligned}
p \cdot y_0 &= (y_0 - x) \cdot y_0 \\
&= (y_0 - x) \cdot (y_0 - x + x) \\
&= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x \\
&= \|y_0 - x\|^2 + p \cdot x \\
&> p \cdot x
\end{aligned}$$

We claim that

$$y \in Y \Rightarrow p \cdot y \geq p \cdot y_0 \quad > p \cdot x$$

If not, suppose there exists $y \in Y$ such that $p \cdot y < p \cdot y_0$. Given $\alpha \in (0,1)$, let

$$w_\alpha = \alpha y + (1-\alpha) y_0$$

Since $Y$ is convex, $w_\alpha \in Y$. Then for $\alpha$ sufficiently close to zero,

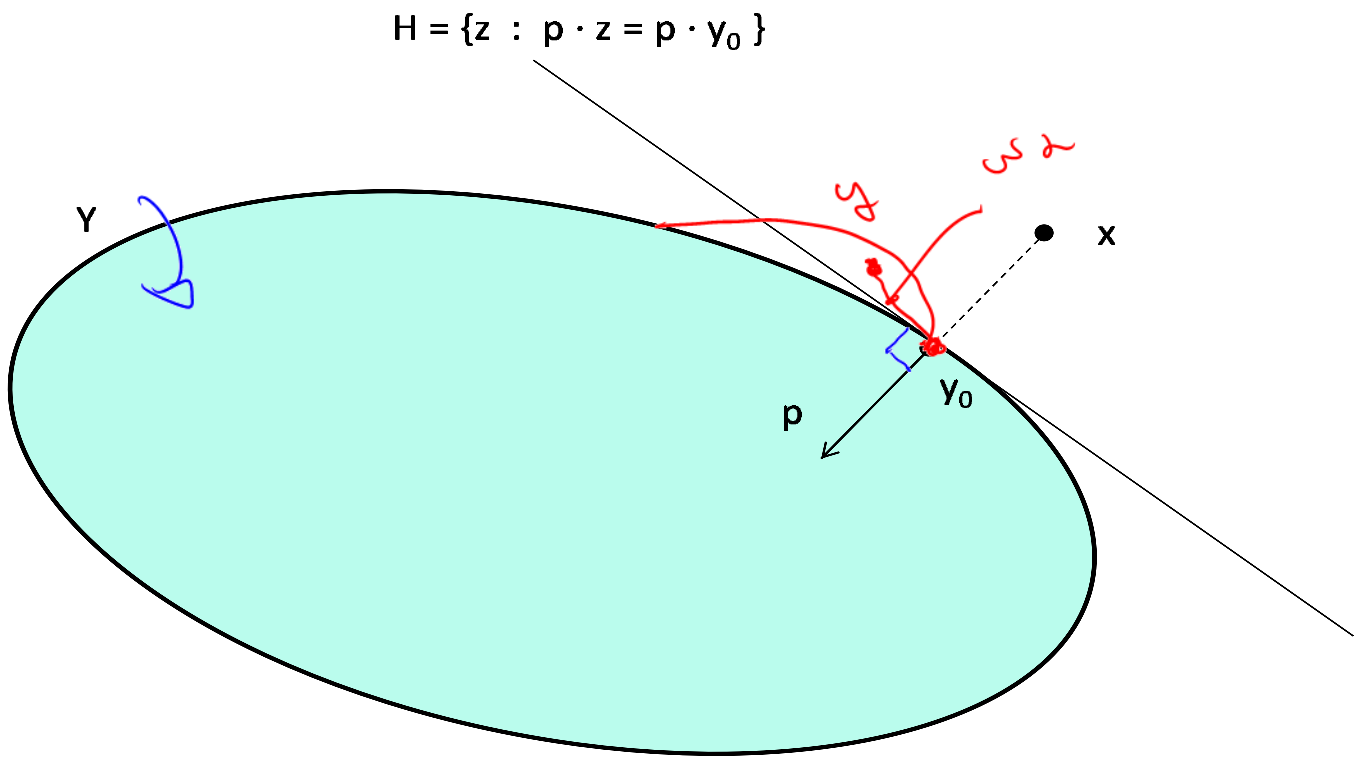
$$
\begin{aligned}
\|x - w_\alpha\|^2 &= \|x - \alpha y - (1-\alpha)y_0\|^2 && \text{defn of } w_\alpha \\
&= |x - y_0 + \alpha(y_0 - y)|^2 && \text{algebra} \\
&= |-p + \alpha(y_0 - y)|^2 && \text{defn of } p \\
&= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2 && \text{more algebra} \\
&= |p|^2 + \alpha\left(\underbrace{-2p \cdot (y_0 - y)}_{\text{negative}} + \underbrace{\alpha |y_0 - y|^2}_{\to 0 \text{ as } \alpha \to 0}\right) && \text{''} \\
&< |p|^2 \quad \text{for } \alpha \text{ close to } 0, \text{ as } p \cdot y_0 > p \cdot y \\
&= \|y_0 - x\|^2
\end{aligned}
$$

Thus for $\alpha$ sufficiently close to zero,

$$\|w_\alpha - x\| < \|y_0 - x\|$$

which implies $y_0$ is not the closest point in $Y$ to $x$, contradiction.

$\square$


H = {z : p · z = p · y0 }
Y
x
y0
p

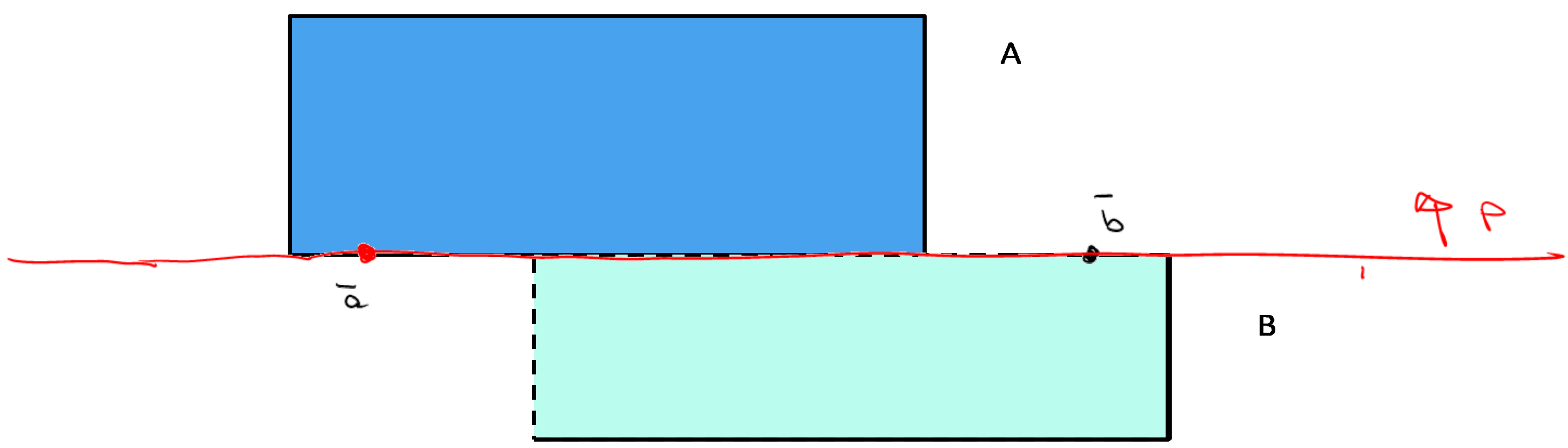
The general version of the Separating Hyperplane Theorem can be derived from this special case by noting that if $A \cap B = \emptyset$, then $0 \notin \underbrace{A - B}_{\text{convex}} = \{a - b : a \in A, b \in B\}$.

# Strict Separation

For the special case of $Y$ compact and $X = \{x\}$, we actually could *strictly separate* $Y$ and $X$:

$$x \notin Y \Rightarrow \quad \exists p \in \mathbf{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

When can we do this in general? Will require additional assumptions...




A, B nonempty, disjoint, convex $\Rightarrow$

$\exists p \in \mathbb{R}^n$, $p \neq 0$ s.t. $p \cdot a \leq p \cdot b$ $\forall a \in A$, $\forall b \in B$

But

$p \cdot \bar{a} = p \cdot \bar{b}$ for some $\bar{a} \in A$ and $\bar{b} \in B$

(for any such $p$)

# Strict Separation

**Theorem 6. (Strict Separating Hyperplane Theorem)** *Let $A, B \subseteq \mathbf{R}^n$ be nonempty, disjoint, closed, convex sets. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that*

$$p \cdot a < p \cdot b \quad \forall a \in A, b \in B$$