

# Econ 204 2022

## Lecture 1

### Outline

1. Introductions
2. About the Course and Other Administrative Details
3. Methods of Proof
4. Equivalence Relations
5. Cardinality

# Introductions

Welcome

- 204
- Berkeley Economics
- UC Berkeley
- Berkeley
- California

- US...

# Introductions

- Chris Shannon
- Bruno Smaniotto
- Damian Vergara

# About the Course

- **Schedule:** Lectures MTWThF 9:00 - 11:30 am (Berkeley time), often going over so don't schedule anything before 12:00

**Discussion Sections:** MTWThF 1:00 - 3:00 pm (first section today)

**Office hours:** Chris Shannon MTWThF 11:30 - 12:30 (end of lecture + 1 hour), also by appointment

Bruno + Dami MTWThF 3:00 - 5:00

- **Final Exam:** Tuesday August 16 9am - 12pm

- **Prerequisites:** Math 53-54 at Berkeley or equivalent
  - 4 semesters college mathematics
  - linear algebra
  - multivariable calculus
  - rigorous approach - theorems stated carefully and some proofs given
  - stream for engineers and scientists

## Course requirements:

- problems sets: 6 total  
(no late problem sets...no exceptions)
- exam
- reading/working on your own

**Grade:** 10% problem sets (5 highest scores out of 6), 90% final exam

## Grading in First Year Economics Courses:

- median grade = B+ : solid command of material
- A and A- are very good grades, A+ for truly exceptional work
- B : ready to go on to further work...a B in 204 means you are ready to go on to 201a/b, 202a/b, 240a/b
- B- : very marginal, but we won't make you take the class again. B- in 204 means you will have a very hard time in 201a/b. Recommend you take Math 53 and 54 this year, maybe Math 104, come back next year to retake 204 and



take 201a/b. B- is a passing grade, but you must maintain a B average

- C: not passing. Definitely not ready for 201a/b, 202a/b, 240a/b. Take Math 53-54 this year, maybe Math 104, retake 204 next year
- 204 with at least a B- (or a waiver from 204 requirement) is a strictly enforced prerequisite for enrollment in 201a/b
- F: means you didn't take the final exam. Be sure to withdraw if you don't or can't take the final.

## Resources:

Book: de la Fuente, *Mathematical Methods and Models for Economists*

Lecture notes: for every lecture + supplements for several topics

*Be sure to read Corrections Handout with dIF*

Seek out other references

This class is not normal...

- lectures
- expectations

## Goals for 204

- reduce heterogeneity of math backgrounds for students in Econ graduate classes
- advance everyone's math skills and knowledge
- present some particular concepts and results used in first-year economics courses 201a/b, 202a/b, 240a/b
- challenge everyone - so not everyone will understand everything

- develop basic math skills and knowledge needed to work as a professional economist and read academic economics
- develop ability to read and evaluate purported proofs...essential for reading and working in all branches of economics - theoretical, empirical, experimental
- develop ability to compose simple proofs...essential to working in all branches of economics - theoretical, empirical, experimental
- cover selected material from real analysis and linear algebra at moderate level of abstraction (considerably more advanced and abstract than Math 53 + 54)

- **not** to review Math 53 + 54. If you are weak on this material, take Math 53-54 this year, and take 204 next year.

## Learning by Doing

- to learn this sort of mathematics you need to do more than just read the book and notes and listen to lectures
- active reading: work through each line, be sure you know how to get from one line to the next
- active listening: follow each step as we work through arguments in class
- working problems: the most valuable part of the class

- working in groups strongly encouraged...
- but, always try to work through all of the problems before talking to others
- everyone must write up their own solutions
- best test of understanding: can you explain it to others



# Methods of Proof

**What is a proof?** The million dollar question...

**Main Methods of Proof:**

- deduction
- contraposition
- induction
- contradiction

We'll examine each of these in turn.

# Proof by Deduction

**Proof by Deduction:** A list of statements, the last of which is the statement to be proven. Each statement in the list is either

- an axiom: a fundamental assumption about mathematics, or part of definition of the object under study; or
- a previously established theorem; or
- follows from previous statements in the list by a valid rule of inference

## Proof by Deduction

**Example:** Prove that the function  $f(x) = x^2$  is continuous at  $x = 5$ .

Recall from one-variable calculus that  $f(x) = x^2$  is continuous at  $x = 5$  means

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$$

That is, “for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that whenever  $x$  is within  $\delta$  of 5,  $f(x)$  is within  $\varepsilon$  of  $f(5)$ .”

To prove the claim, we must systematically verify that this definition is satisfied.

*Proof.* Let  $\varepsilon > 0$  be given. Let

$$\delta = \min \left\{ 1, \frac{\varepsilon}{11} \right\} > 0$$

Where did that come from ? Suppose  $|x - 5| < \delta$ . Since  $\delta \leq 1$ ,  $4 < x < 6$ , so  $9 < x + 5 < 11$  and  $|x + 5| < 11$ . Then

$$\begin{aligned} |f(x) - f(5)| &= |x^2 - 25| \\ &= |(x + 5)(x - 5)| \\ &= |x + 5||x - 5| \\ &< 11 \cdot \delta \\ &\leq 11 \cdot \frac{\varepsilon}{11} \\ &= \varepsilon \end{aligned}$$

Thus, we have shown that for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$ , so  $f$  is continuous at  $x = 5$ .  $\square$

# Proof by Contraposition

Recall some basics of logic.

$\neg P$  means “ $P$  is false.”

$P \wedge Q$  means “ $P$  is true *and*  $Q$  is true.”

$P \vee Q$  means “ $P$  is true *or*  $Q$  is true (or possibly both).”

$\neg P \wedge Q$  means  $(\neg P) \wedge Q$ ;  $\neg P \vee Q$  means  $(\neg P) \vee Q$ .

$P \Rightarrow Q$  means “whenever  $P$  is satisfied,  $Q$  is also satisfied.”

Formally,  $P \Rightarrow Q$  is equivalent to  $\neg P \vee Q$ .

## Proof by Contraposition

The *contrapositive* of the statement  $P \Rightarrow Q$  is the statement  $\neg Q \Rightarrow \neg P$ .

**Theorem 1.**  $P \Rightarrow Q$  is true if and only if  $\neg Q \Rightarrow \neg P$  is true.

*Proof.* Suppose  $P \Rightarrow Q$  is true. Then either  $P$  is false, or  $Q$  is true (or possibly both). Therefore, either  $\neg P$  is true, or  $\neg Q$  is false (or possibly both), so  $\neg(\neg Q) \vee (\neg P)$  is true, that is,  $\neg Q \Rightarrow \neg P$  is true.

Conversely, suppose  $\neg Q \Rightarrow \neg P$  is true. Then either  $\neg Q$  is false, or  $\neg P$  is true (or possibly both), so either  $Q$  is true, or  $P$  is false (or possibly both), so  $\neg P \vee Q$  is true, so  $P \Rightarrow Q$  is true.  $\square$

# Proof by Induction

We illustrate with an example:

**Theorem 2.** *For every  $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ ,*

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

*i.e.*  $1 + 2 + \dots + n = \frac{n(n+1)}{2}.$

*Proof.* **Base step**  $n = 0$ : LHS =  $\sum_{k=1}^0 k =$  the empty sum = 0. RHS =  $\frac{0 \cdot 1}{2} = 0$

So the claim is true for  $n = 0$ .



**Induction step:** Suppose

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \text{ for some } n \geq 0$$

We must show that

$$\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}$$

$$\begin{aligned}
\text{LHS} &= \sum_{k=1}^{n+1} k \\
&= \sum_{k=1}^n k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \text{ by the Induction hypothesis} \\
&= (n+1) \left( \frac{n}{2} + 1 \right) \\
&= \frac{(n+1)(n+2)}{2} \\
\text{RHS} &= \frac{(n+1)((n+1)+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} = \text{LHS}
\end{aligned}$$

So by mathematical induction,  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$  for all  $n \in \mathbf{N}_0$ .  $\square$

## Proof by Contradiction

Assume the negation of what is claimed, and work toward a contradiction.

**Theorem 3.** *There is no rational number  $q$  such that  $q^2 = 2$ .*

*Proof.* Suppose  $q^2 = 2$  where  $q \in \mathbf{Q}$ . Then we can write  $q = \frac{m}{n}$  for some integers  $m, n \in \mathbf{Z}$ . Moreover, we can assume that  $m$  and  $n$  have no common factor; if they did, we could divide it out.

$$2 = q^2 = \frac{m^2}{n^2}$$

Therefore,  $m^2 = 2n^2$ , so  $m^2$  is even.

We claim that  $m$  is even. If not, then  $m$  is odd, so  $m = 2p + 1$  for some  $p \in \mathbf{Z}$ . Then

$$\begin{aligned} m^2 &= (2p + 1)^2 \\ &= 4p^2 + 4p + 1 \\ &= 2(2p^2 + 2p) + 1 \end{aligned}$$

which is odd, contradiction. Therefore,  $m$  is even, so  $m = 2r$  for some  $r \in \mathbf{Z}$ .

$$\begin{aligned} 4r^2 &= (2r)^2 \\ &= m^2 \\ &= 2n^2 \\ n^2 &= 2r^2 \end{aligned}$$

So  $n^2$  is even, which implies (by the argument given above) that  $n$  is even. Therefore,  $n = 2s$  for some  $s \in \mathbf{Z}$ , so  $m$  and  $n$  have a

common factor, namely 2, contradiction. Therefore, there is no rational number  $q$  such that  $q^2 = 2$ .  $\square$

# Binary Relations

**Definition 1.** A binary relation  $R$  from  $X$  to  $Y$  is a subset  $R \subseteq X \times Y$ . We write  $xRy$  if  $(x, y) \in R$  and “not  $xRy$ ” if  $(x, y) \notin R$ .  $R \subseteq X \times X$  is a binary relation on  $X$ .

**Example:** Suppose  $f : X \rightarrow Y$  is a function from  $X$  to  $Y$ . The binary relation  $R \subseteq X \times Y$  defined by

$$xRy \iff f(x) = y$$

is exactly the graph of the function  $f$ . A function can be considered a binary relation  $R$  from  $X$  to  $Y$  such that for each  $x \in X$  there exists exactly one  $y \in Y$  such that  $(x, y) \in R$ .

**Example:** Suppose  $X = \{1, 2, 3\}$  and  $R$  is the binary relation on  $X$  given by  $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$ . This is the binary relation “is weakly greater than,” or  $\geq$ .

## Equivalence Relations

**Definition 2.** A binary relation  $R$  on  $X$  is

(i) reflexive if  $\forall x \in X, xRx$

(ii) symmetric if  $\forall x, y \in X, xRy \Leftrightarrow yRx$

(iii) transitive if  $\forall x, y, z \in X, (xRy \wedge yRz) \Rightarrow xRz$

**Definition 3.** A binary relation  $R$  on  $X$  is an equivalence relation if it is reflexive, symmetric and transitive.

# Equivalence Relations

**Definition 4.** Given an equivalence relation  $R$  on  $X$ , write

$$[x] = \{y \in X : xRy\}$$

$[x]$  is called the equivalence class containing  $x$ .

The set of equivalence classes is the quotient of  $X$  with respect to  $R$ , denoted  $X/R$ .

**Example:** The binary relation  $\geq$  on  $\mathbf{R}$  is not an equivalence relation because it is not symmetric.

**Example:** Let  $X = \{a, b, c, d\}$  and

$$R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$$

$R$  is an equivalence relation (why?) and the equivalence classes of  $R$  are  $\{a, b\}$  and  $\{c, d\}$ .  $X/R = \{\{a, b\}, \{c, d\}\}$



# Equivalence Relations

The equivalence classes of an equivalence relation form a *partition* of  $X$ : every element of  $X$  belongs to exactly one equivalence class.

**Theorem 4.** *Let  $R$  be an equivalence relation on  $X$ . Then  $\forall x \in X, x \in [x]$ . Given  $x, y \in X$ , either  $[x] = [y]$  or  $[x] \cap [y] = \emptyset$ .*

*Proof.* If  $x \in X$ , then  $xRx$  because  $R$  is reflexive, so  $x \in [x]$ .

Suppose  $x, y \in X$ . If  $[x] \cap [y] = \emptyset$ , we're done. So suppose  $[x] \cap [y] \neq \emptyset$ . We must show that  $[x] = [y]$ , i.e. that the elements of  $[x]$  are exactly the same as the elements of  $[y]$ .

Choose  $z \in [x] \cap [y]$ . Then  $z \in [x]$ , so  $xRz$ . By symmetry,  $zRx$ . Also  $z \in [y]$ , so  $yRz$ . By symmetry again,  $zRy$ . Now choose  $w \in [x]$ . By definition,  $xRw$ . Since  $zRx$  and  $R$  is transitive,  $zRw$ . By symmetry,  $wRz$ . Since  $zRy$ ,  $wRy$  by transitivity again. By symmetry,  $yRw$ , so  $w \in [y]$ , which shows that  $[x] \subseteq [y]$ . Similarly,  $[y] \subseteq [x]$ , so  $[x] = [y]$ . □

## Cardinality

**Definition 5.** *Two sets  $A, B$  are numerically equivalent ( or have the same cardinality) if there is a bijection  $f : A \rightarrow B$ , that is, a function  $f : A \rightarrow B$  that is 1-1 ( $a \neq a' \Rightarrow f(a) \neq f(a')$ ), and onto ( $\forall b \in B \exists a \in A$  s.t.  $f(a) = b$ ).*

**Example:**  $A = \{2, 4, 6, \dots, 50\}$  is numerically equivalent to the set  $\{1, 2, \dots, 25\}$  under the function  $f(n) = 2n$ .

$B = \{1, 4, 9, 16, 25, 36, 49 \dots\} = \{n^2 : n \in \mathbb{N}\}$  is numerically equivalent to  $\mathbb{N}$ .

# Cardinality

A set is either finite or infinite. A set is *finite* if it is numerically equivalent to  $\{1, \dots, n\}$  for some  $n$ . A set that is not finite is *infinite*.

In particular,  $A = \{2, 4, 6, \dots, 50\}$  is finite,  $B = \{1, 4, 9, 16, 25, 36, 49 \dots\}$  is infinite.

A set is *countable* if it is numerically equivalent to the set of natural numbers  $\mathbb{N} = \{1, 2, 3, \dots\}$ . An infinite set that is not countable is called *uncountable*.

# Cardinality

**Example:** The set of integers  $\mathbf{Z}$  is countable.

$$\mathbf{Z} = \{0, 1, -1, 2, -2, \dots\}$$

Define  $f : \mathbf{N} \rightarrow \mathbf{Z}$  by

$$\begin{aligned} f(1) &= 0 \\ f(2) &= 1 \\ f(3) &= -1 \\ &\vdots \\ f(n) &= (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

where  $\lfloor x \rfloor$  is the greatest integer less than or equal to  $x$ . It is straightforward to verify that  $f$  is one-to-one and onto.

# Cardinality

**Theorem 5.** *The set of rational numbers  $\mathbb{Q}$  is countable.*

**“Picture Proof”:**

$$\begin{aligned}\mathbb{Q} &= \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\} \\ &= \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}\end{aligned}$$

		$m$								
		0		1		$-1$		2		$-2$
$n$	1	0	$\rightarrow$	1		$-1$	$\rightarrow$	2		$-2$
			$\swarrow$		$\nearrow$		$\swarrow$		$\nearrow$	
	2	0		$\frac{1}{2}$		$-\frac{1}{2}$		1		$-1$
		$\downarrow$	$\nearrow$		$\swarrow$		$\nearrow$			
	3	0		$\frac{1}{3}$		$-\frac{1}{3}$		$\frac{2}{3}$		$-\frac{2}{3}$
		$\swarrow$		$\nearrow$						
	4	0		$\frac{1}{4}$		$-\frac{1}{4}$		$\frac{1}{2}$		$-\frac{1}{2}$
		$\downarrow$	$\nearrow$							
	5	0		$\frac{1}{5}$		$-\frac{1}{5}$		$\frac{2}{5}$		$-\frac{2}{5}$

Go back and forth on upward-sloping diagonals, omitting the

repeats:

$$f(1) = 0$$

$$f(2) = 1$$

$$f(3) = \frac{1}{2}$$

$$f(4) = -1$$

$$\vdots$$

$f : \mathbf{N} \rightarrow \mathbf{Q}$ ,  $f$  is one-to-one and onto.