Econ 204 2022

Lecture 3

Outline

0. Intermediate Value Theorem
1. Metric Spaces and Normed Spaces
2. Convergence of Sequences in Metric Spaces
3. Sequences in $\mathbf{R}$ and $\mathbf{R}^n$

Intermediate Value Theorem

$\subseteq \mathbb{R}$

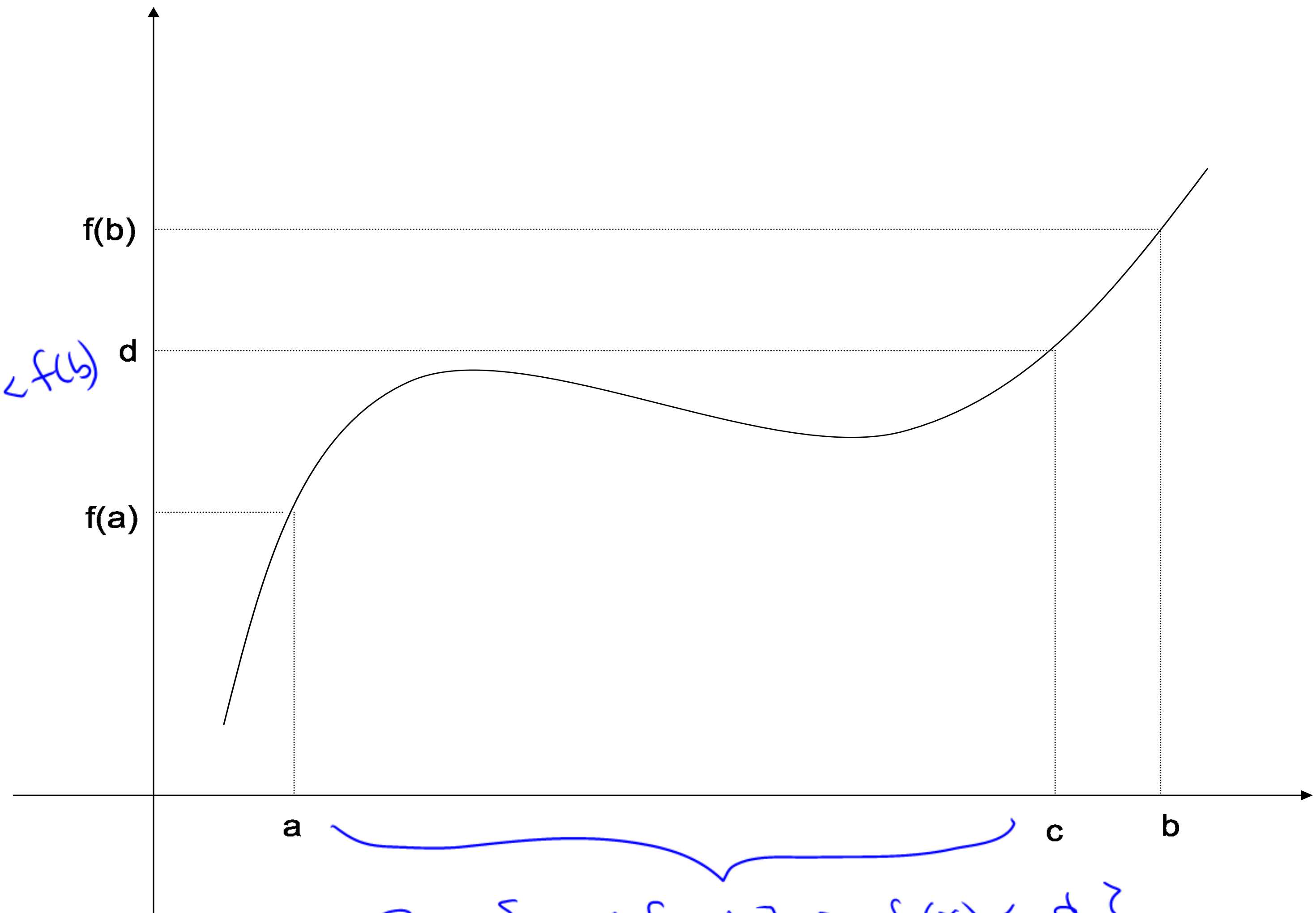
Theorem 4 (Intermediate Value Theorem)**.** *Suppose* $f : [a, b] \to \mathbf{R}$ *is continuous, and* $f(a) < d < f(b)$. *Then there exists* $c \in (a, b)$ *such that* $f(c) = d$.

Proof. Later, we will give a slick proof. Here, we give a bare-hands proof using the Supremum Property. Let

$$B = \{x \in [a, b] : f(x) < d\}$$

$a \in B$, so $B \neq \emptyset$; $B \subseteq [a, b]$, so B is bounded above. By the Supremum Property, $\sup B$ exists and is real so let $c = \sup B$. Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$.

$f(a) < d < f(b)$



$B = \{ x \in [a,b] : f(x) < d \}$

$c = \sup B$

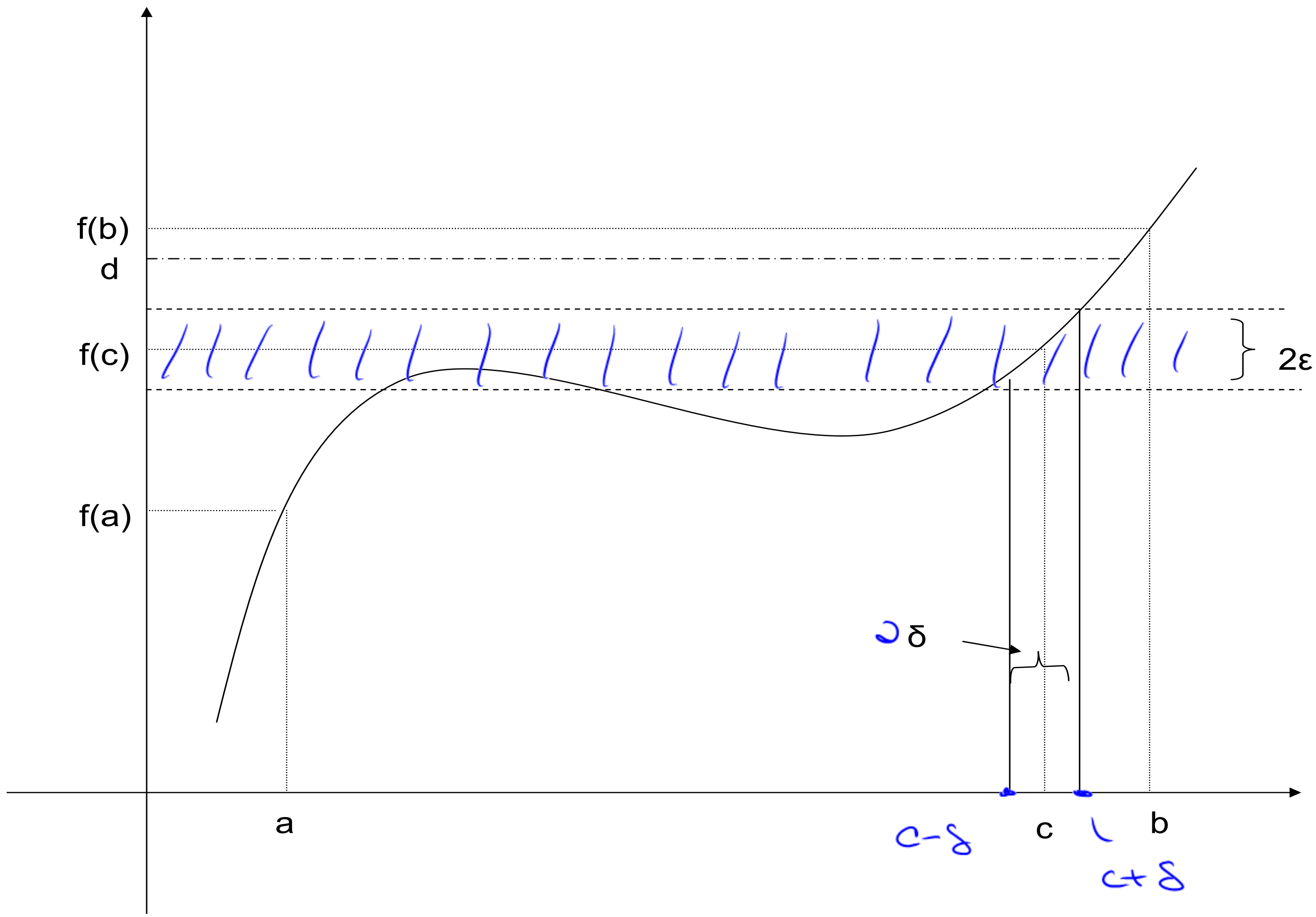
Claim: $f(c) = d$

$f(x) \in (f(c) - \varepsilon, \ f(c) + \varepsilon)$

We claim that $f(c) = d$. If not, suppose $f(c) < d$. Then since $f(b) > d$, $c \neq b$, so $c < b$. Let $\varepsilon = \frac{d - f(c)}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$
\begin{aligned}
|x - c| < \delta \ \Rightarrow \ |f(x) - f(c)| &< \varepsilon \\
\Rightarrow \ f(x) &< f(c) + \varepsilon \\
&= f(c) + \frac{d - f(c)}{2} \\
&= \frac{f(c) + d}{2} \\
&< \frac{d + d}{2} \\
&= d
\end{aligned}
$$

so $(c, c + \delta) \subseteq B$, so $c \neq \sup B$, contradiction.



$f(c) < d \Rightarrow \exists\, \delta > 0$ s.t. for $x \in (c-\delta, c+\delta)$, $f(x) < d$

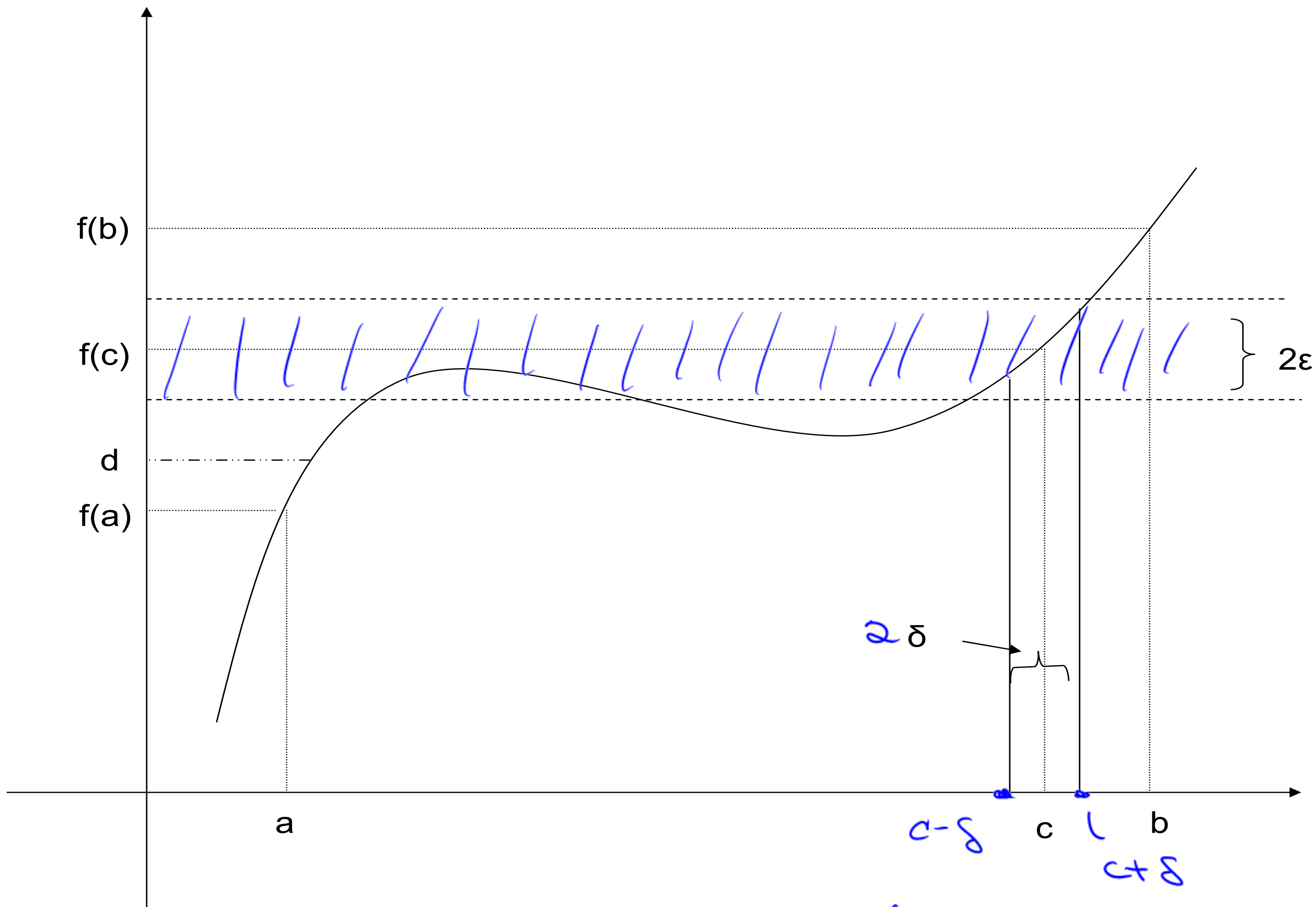
$\Rightarrow c \neq \sup B$

Suppose $f(c) > d$. Then since $f(a) < d$, $a \neq c$, so $c > a$. Let $\varepsilon = \frac{f(c)-d}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$\begin{array}{rcrcl} |x-c| < \delta & \Rightarrow & |f(x)-f(c)| & < & \varepsilon \\ & \Rightarrow & f(x) & > & f(c) - \varepsilon \\ & & & = & f(c) - \frac{f(c)-d}{2} \\ & & & = & \frac{f(c)+d}{2} \\ & & & > & \frac{d+d}{2} \\ & & & = & d \end{array}$$

so $(c-\delta, c+\delta) \cap B = \emptyset$. So either there exists $x \in B$ with $x \geq c+\delta$ (in which case c is not an upper bound for B) or $c-\delta$ is an upper bound for B (in which case c is not the least upper bound for B); in either case, $c \neq \sup B$, contradiction.

$f(c) > d \Rightarrow \exists\, \delta > 0$ s.t. $f(x) > d \;\; \forall x \in (c-\delta, c+\delta)$



$(c-\delta, c+\delta) \cap B = \emptyset \Rightarrow$ either $\exists\, y \in [c+\delta, b] \cap B$

or $B \subseteq [a, c-\delta]$

in either case, $c \neq \sup B$

Since $f(c) \not< d$, $f(c) \not> d$, and the order is complete, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$. $\square$

Corollary 1. *There exists* $x \in \mathbf{R}$ *such that* $x^2 = 2$.

Proof. Let $f(x) = x^2$, for $x \in [0, 2]$. f is continuous (Why?). $f(0) = 0 < 2$ and $f(2) = 4 > 2$, so by the Intermediate Value Theorem, there exists $c \in (0, 2)$ such that $f(c) = 2$, i.e. such that $c^2 = 2$. □

Metric Spaces and Metrics

Generalize distance and length notions in $\mathbf{R}^n$

Definition 1. *A* metric space *is a pair* (X, d)*, where* X *is a set and* $d : X \times X \to \mathbf{R}_+$ *a function satisfying*

$\mathbb{R}_+ = \{ r \in \mathbb{R} : r \geq 0 \}$

1. $d(x,y) \geq 0,\ d(x,y) = 0 \Leftrightarrow x = y\ \forall x, y \in X$

2. $d(x,y) = d(y,x)\ \forall x, y \in X$

3. triangle inequality*:*

$$d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in X$$

$$
\begin{tikzcd}
& y \arrow[dr, "d(y,z)"] & \\
x \arrow[ur, "d(x,y)"] \arrow[rr, "d(x,z)"'] & & z
\end{tikzcd}
$$

A function $d : X \times X \to \mathbf{R}_+$ *satisfying 1-3 above is called a* metric *on* X*.*

A metric gives a notion of distance between elements of X.

Normed Spaces and Norms

Definition 2. *Let V be a vector space over $\mathbf{R}$. A* norm *on V is a function $\|\cdot\|: V \to \mathbf{R}_+$ satisfying*

1. $\|x\| \geq 0 \ \forall x \in V$

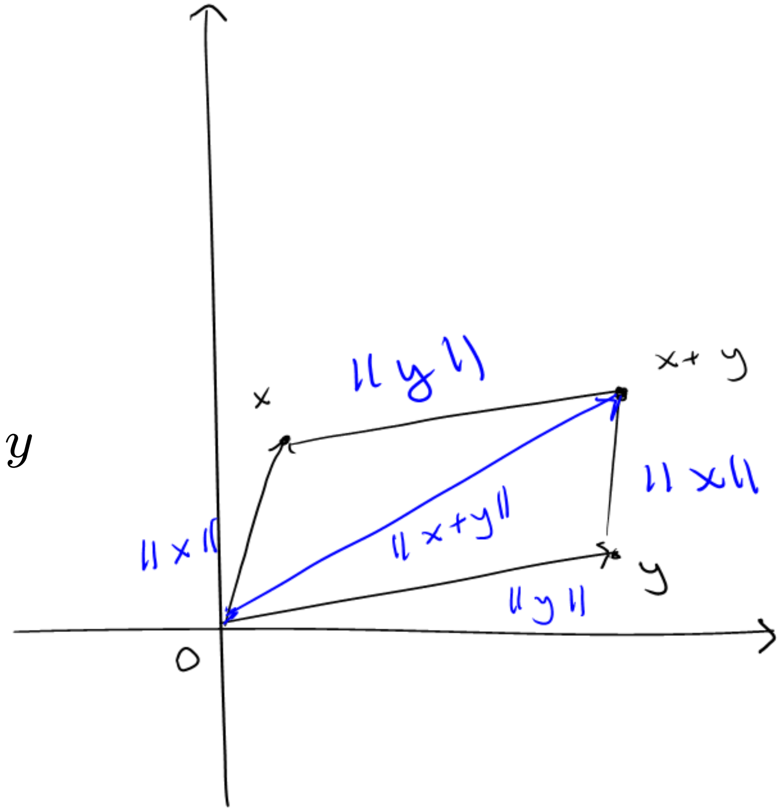
vector additive identity

2. $\|x\| = 0 \Leftrightarrow x = 0 \ \forall x \in V$

3. triangle inequality*:*

$$\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in V$$

$$\begin{tikzcd}
& x \arrow[dr, "y"] & \\
0 \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr] & & x+y \\
& y \arrow[ur, "x"'] &
\end{tikzcd}$$



4. $\|\alpha x\| = |\alpha|\|x\| \ \forall \alpha \in \mathbf{R}, x \in V$

A normed vector space *is a vector space over* $\mathbf{R}$ *equipped with a norm.*

A norm gives a notion of length of a vector in V.

$x \in \mathbb{R}^n \quad x = (x_1, \ldots, x_n) \quad \|x\| = \left[\sum_{i=1}^{n} x_i^2 \right]^{1/2}$

Normed Spaces and Norms

Example: In $\mathbf{R}^n$, standard notion of distance between two vectors x and y measures length of difference $x - y$, i.e., $d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

Theorem 1. *Let $(V, \|\cdot\|)$ be a normed vector space. Let $d : V \times V \to \mathbf{R}_+$ be defined by*

$$d(v, w) = \|v - w\|$$

Then (V, d) is a metric space.

$x = y \Rightarrow x + w = y + w$?

Proof. We must verify that d satisfies all the properties of a metric.

1. Let $v, w \in V$. Then by definition, $d(v,w) = \|v - w\| \geq 0$ (why?), and

$$\begin{aligned} d(v,w) = 0 \quad &\Leftrightarrow \quad \|v - w\| = 0 \\ &\Leftrightarrow \quad v - w = 0 \\ &\Leftrightarrow \quad (v + (-w)) + w = w \\ &\Leftrightarrow \quad v + ((-w) + w) = w \\ &\Leftrightarrow \quad v + 0 = w \\ &\Leftrightarrow \quad v = w \end{aligned}$$

vector additive identity

symmetry: 2. First, note that for any $x \in V$, $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$, so $0 \cdot x = 0$. Then $0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x =$

vector additive identity

$0 = x + (-1) \cdot x$, so we have $(-1) \cdot x = (-x)$. Then let $v, w \in V$.

$$
\begin{aligned}
d(v, w) &= \|v - w\| \\
&= |-1| \|v - w\| \\
&= \|(-1)(v + (-w))\| \\
&= \|(-1)v + (-1)(-w)\| \\
&= \|-v + w\| \\
&= \|w + (-v)\| \\
&= \|w - v\| \\
&= d(w, v)
\end{aligned}
$$

$d(x, y) = \| x - y \|$

triangle inequality

3. Let $u, w, v \in V$.

$$
\begin{aligned}
d(u, w) &= \|u - w\| \\
&= \|u + (-v + v) - w\| \\
&= \|(u - v) + (v - w)\| \\
&\leq \|u - v\| + \|v - w\| \\
&= d(u, v) + d(v, w)
\end{aligned}
$$

Thus d is a metric on V. $\square$

$\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

Normed Spaces and Norms

Examples

$\mathbb{R}^n$ with standard norm

- $\mathbf{E}^n$: n-dimensional Euclidean space.

$$V = \mathbf{R}^n, \ \|x\|_2 = |x| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

- $V = \mathbf{R}^n$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (the "taxi cab" norm or L^1 norm)

- $V = \mathbf{R}^n$, $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ (the maximum norm, or sup norm, or L^∞ norm)

Recall: $C([0,1])$ - continuous functions $f : [0,1] \to \mathbb{R}$

- $V =$ $C([0,1])$, $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$
- $V =$ $C([0,1])$, $\|f\|_2 = \sqrt{\int_0^1 (f(t))^2\, dt}$
- $V =$ $C([0,1])$, $\|f\|_1 = \int_0^1 |f(t)|\, dt$

$v, w \in \mathbb{R}^n \qquad \langle v, w \rangle = v \cdot w = \sum_{i=1}^{n} v_i w_i$

Normed Spaces and Norms

Theorem 2 (Cauchy-Schwarz Inequality).
If $v, w \in \mathbf{R}^n$*, then*

$$\left(\sum_{i=1}^{n} v_i w_i\right)^2 \leq \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right)$$

$$|\langle v, w \rangle|^2 = |v \cdot w|^2 \leq |v|^2 |w|^2 = \|v\|^2 \|w\|^2$$

$$|\langle v, w \rangle| = |v \cdot w| \leq |v||w| = \|v\| \, \|w\|$$

- learn some proof
- triangle inequality of $\|\cdot\|_2$ in $\mathbb{R}^n$ follows from C-S inequality (nice exercise)

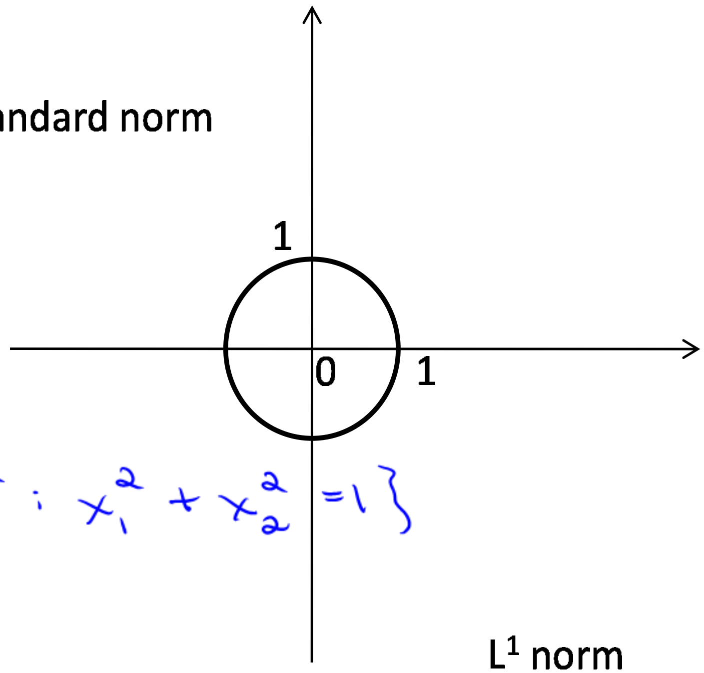
Equivalent Norms

A given vector space may have many different norms: if $\|\cdot\|$ is a norm on a vector space V, so are $2\|\cdot\|$ and $3\|\cdot\|$ and $k\|\cdot\|$ for any $k > 0$.

Less trivially, $\mathbf{R}^n$ supports many different norms as in the examples above. Different norms on a given vector space yield different geometric properties.

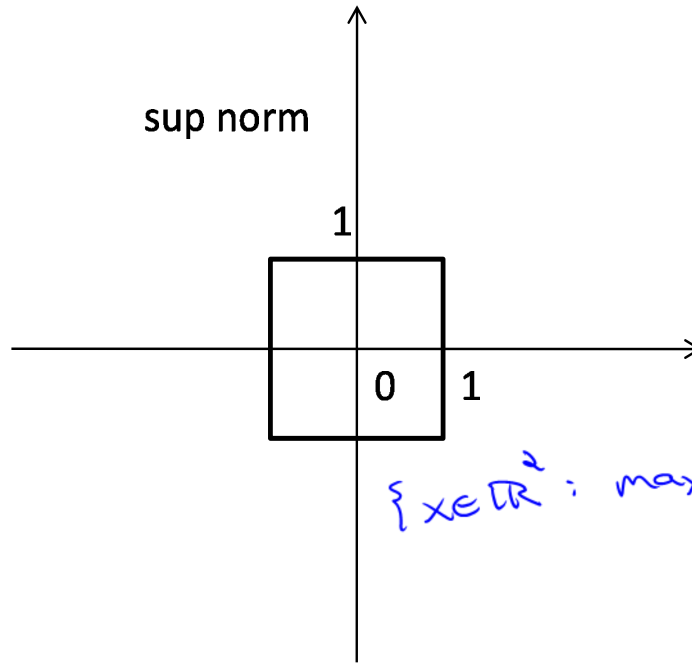
$\{ x \in \mathbb{R}^2 : \|x\| = 1 \}$ for different norms :

standard norm



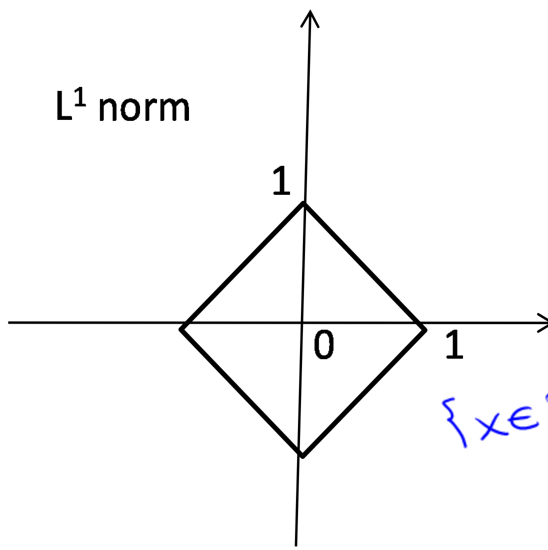
$\{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \}$

sup norm



$\{ x \in \mathbb{R}^2 : \max(|x_1|, |x_2|) = 1 \}$

L^1 norm



$\{ x \in \mathbb{R}^2 : |x_1| + |x_2| = 1 \}$

unit balls around 0 in different norms

Equivalent Norms

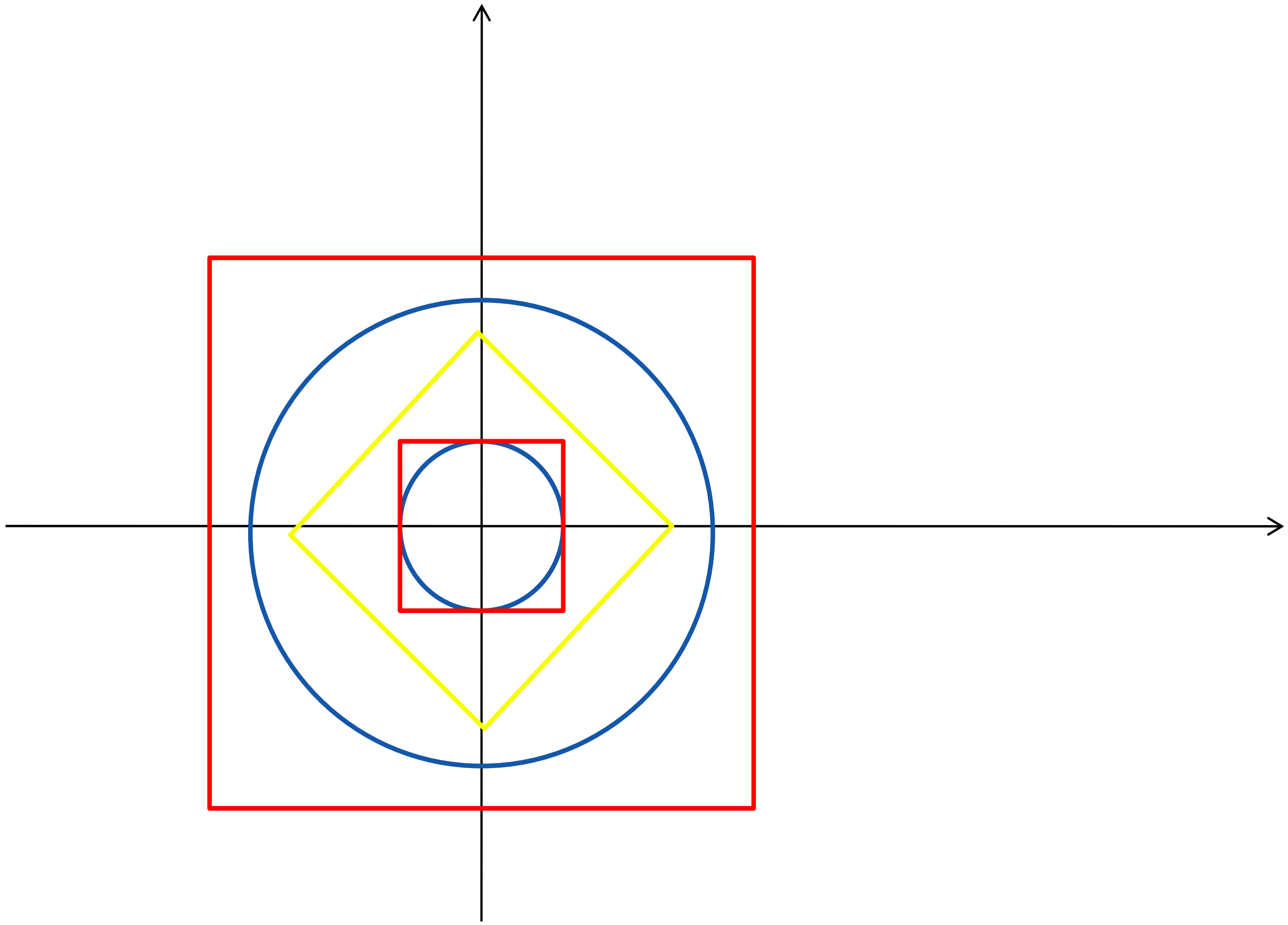
Definition 3. *Two norms* $\|\cdot\|$ *and* $\|\cdot\|^*$ *on the same vector space* V *are said to be* Lipschitz-equivalent *(or* equivalent *) if* $\exists m, M > 0$ *s.t.* $\forall x \in V$,

$$m\|x\| \leq \|x\|^* \leq M\|x\|$$

Equivalently, $\exists m, M > 0$ *s.t.* $\forall x \in V, x \neq 0$,

$$0 < m \leq \frac{\|x\|^*}{\|x\|} \leq M < +\infty$$

this is an equivalence relation (nice exercise)



norms on $\mathbf{R}^n$ are equivalent

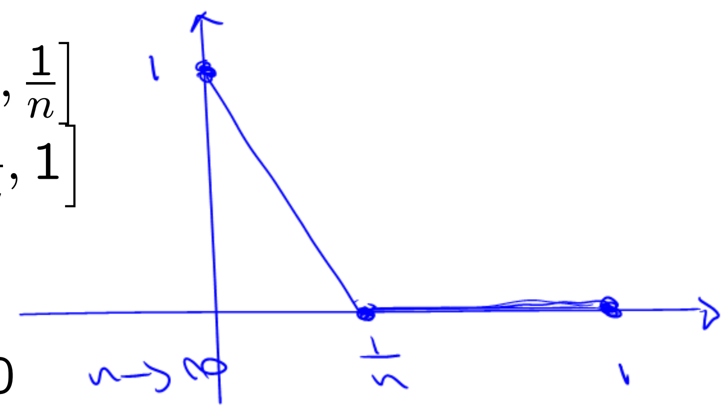
Equivalent Norms

In $\mathbf{R}^n$ (or any finite-dimensional normed vector space), all norms are equivalent. Roughly, up to a difference in scaling, for topological purposes there is a unique norm in $\mathbf{R}^n$.

Theorem 3. *All norms on $\mathbf{R}^n$ are equivalent.*

Infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0,1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt & \text{if } t \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } t \in \left(\frac{1}{n}, 1\right] \end{cases}$$



Then

$$\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{\frac{1}{2n}}{1} = \frac{1}{2n} \to 0$$

$$\|f_n\|_1 = \int_0^1 |f_n(t)|\,dt = \frac{1}{2n}$$

$$\|f_n\|_\infty = \sup\{|f_n(t)| : t \in [0,1]\} = 1 \quad \forall n$$

Metrics and Sets

Definition 4. *In a metric space* (X, d)*, a subset* $S \subseteq X$ *is* bounded *if* $\exists x \in X, \beta \in \mathbf{R}_+$ *such that* $\forall s \in S$*,* $d(s, x) \leq \beta$*.*

In a metric space (X, d), define for $\varepsilon > 0$

$$
\begin{aligned}
B_\varepsilon(x) &= \{y \in X : d(y, x) < \varepsilon\} \\
&= \text{“}\textit{open}\text{” } \textit{ball with center } x \textit{ and radius } \varepsilon \\
B_\varepsilon[x] &= \{y \in X : d(y, x) \leq \varepsilon\} \\
&= \text{“}\textit{closed}\text{” } \textit{ball with center } x \textit{ and radius } \varepsilon
\end{aligned}
$$

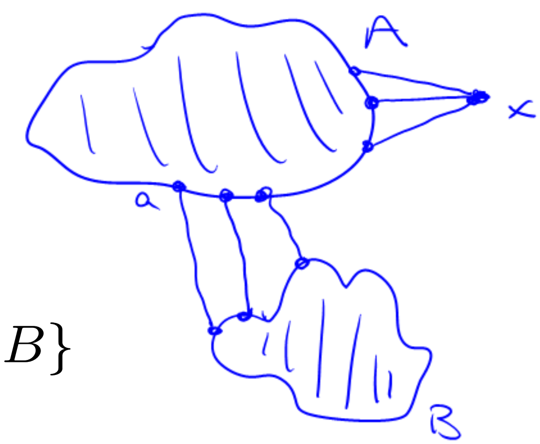
Metrics and Sets

We can use the metric d to define a generalization of "radius". In a metric space (X, d), define the *diameter* of a subset $S \subseteq X$ by

$$\text{diam }(S) = \sup\{d(s, s') : s, s' \in S\}$$

Similarly, we can define the distance from a point to a set, and distance between sets, as follows:

$$\begin{aligned} d(A, x) &= \inf_{a \in A} d(a, x) \\ d(A, B) &= \inf_{a \in A} d(B, a) \\ &= \inf\{d(a, b) : a \in A, b \in B\} \end{aligned}$$



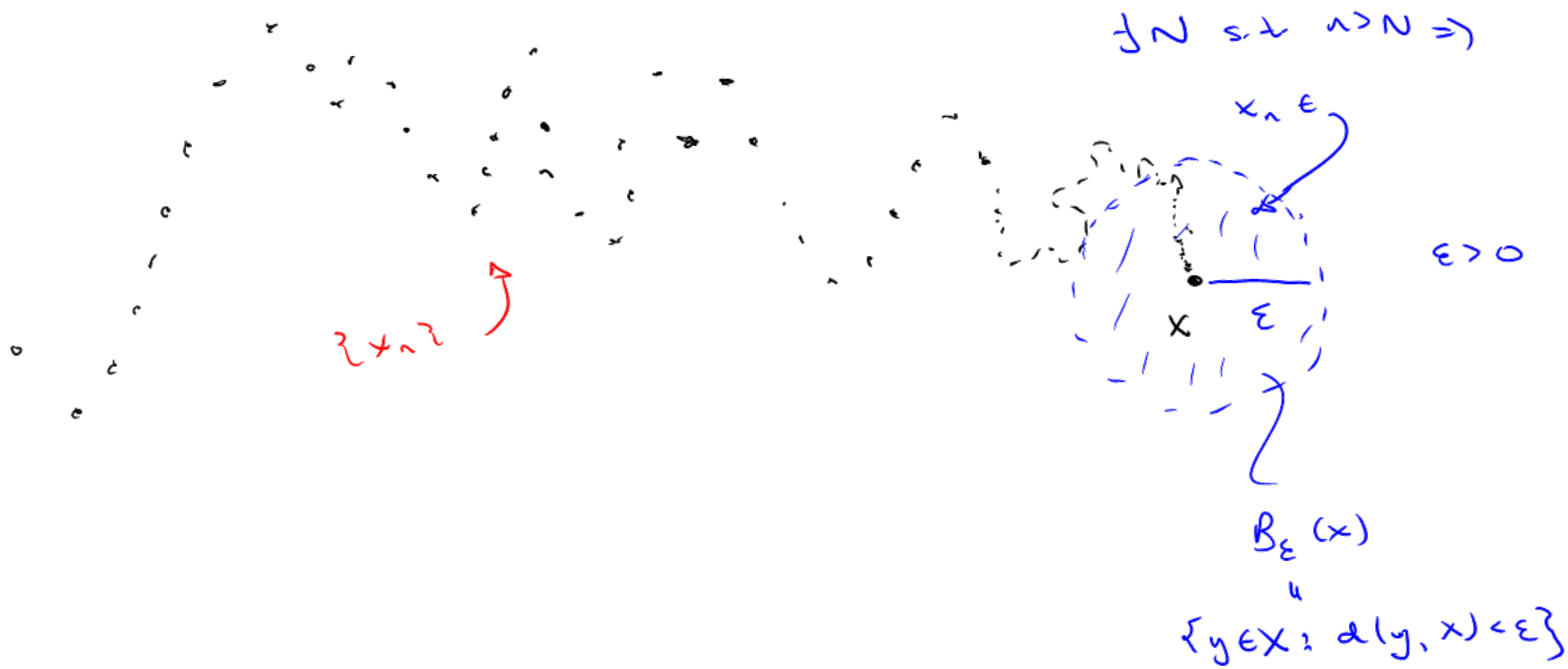
But $d(A, B)$ is **not** a metric.

Convergence of Sequences

Definition 5. *Let* (X, d) *be a metric space. A sequence* $\{x_n\} \subset X$ converges *to* $x \in X$ *(written* $x_n \to x$ *or* $\lim_{n\to\infty} x_n = x$*) if*

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbf{N} \ \text{s.t.} \ \forall \ n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$$

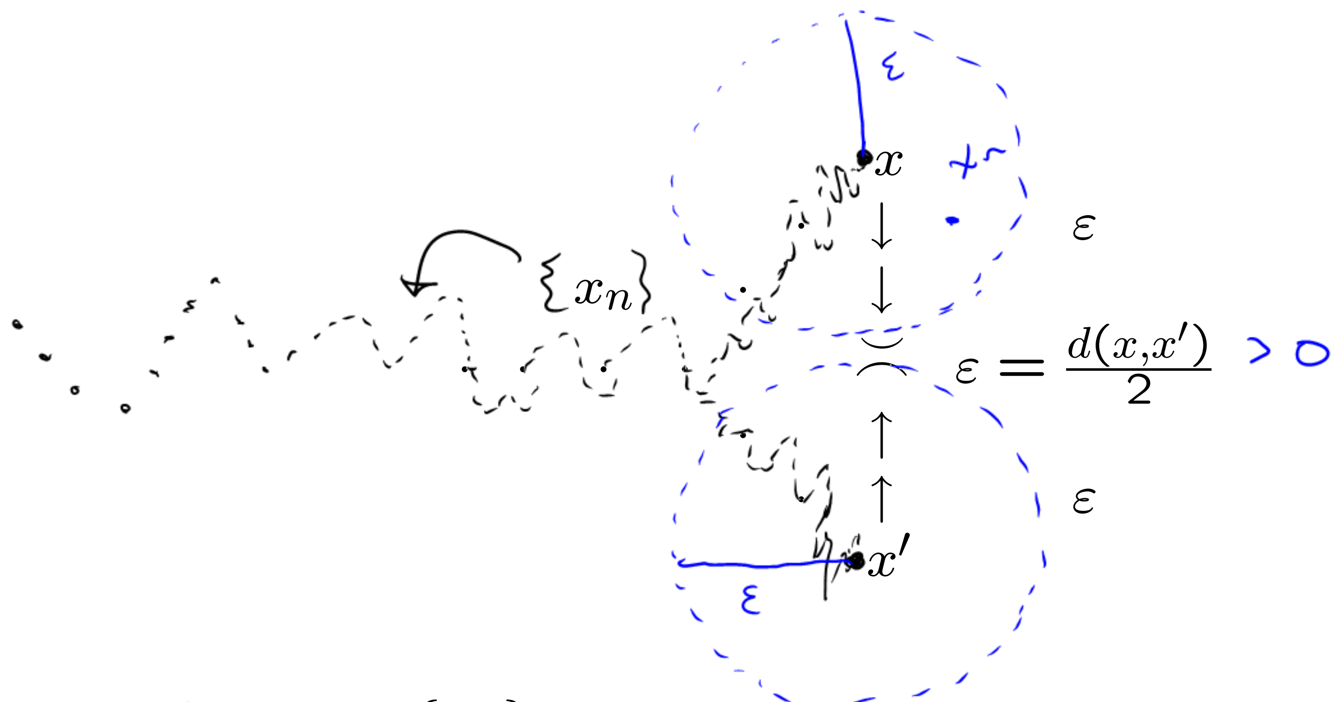
Notice that this is exactly the same as the definition of convergence of a sequence of real numbers, except we replace the standard measure of distance $|\cdot|$ in $\mathbf{R}$ by the general metric d.



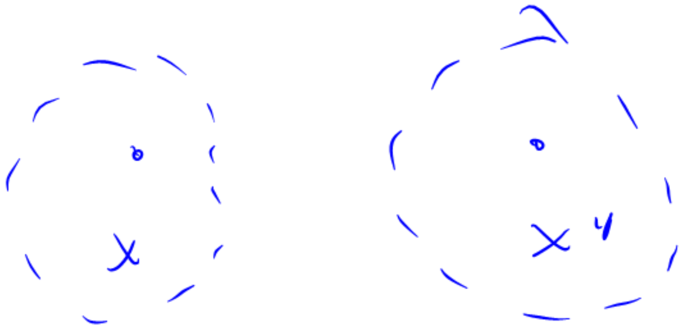
$\{x_n\}$
$\exists N$ s.t. $n > N \Rightarrow$
$x_n \in$
$\varepsilon > 0$
x
ε
$B_\varepsilon(x)$
$\{y \in X : d(y, x) < \varepsilon\}$

Uniqueness of Limits

Theorem 4 (Uniqueness of Limits)**.** *In a metric space* (X, d)*, if* $x_n \to x$ *and* $x_n \to x'$*, then* $x = x'$*.*



Proof. Suppose $\{x_n\}$ is a sequence in X, $x_n \to x$, $x_n \to x'$, $x \neq x'$.



Since $x \neq x'$, $d(x, x') > 0$. Let

$$\varepsilon = \frac{d(x, x')}{2} \quad > 0$$

Then there exist $N(\varepsilon)$ and $N'(\varepsilon)$ such that

$$n > N(\varepsilon) \;\Rightarrow\; d(x_n, x) < \varepsilon \qquad (x_n \to x)$$
$$n > N'(\varepsilon) \;\Rightarrow\; d(x_n, x') < \varepsilon \qquad (x_n \to x')$$

Choose

$$n > \max\{N(\varepsilon), N'(\varepsilon)\}$$

Fix $n > \max\{N, N'\}$

Then

$$\begin{aligned} d(x,x') &\leq d(x,x_n) + d(x_n,x') \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \\ &= d(x,x') \\ \Rightarrow d(x,x') &< d(x,x') \end{aligned}$$

a contradiction. $\square$

Cluster Points

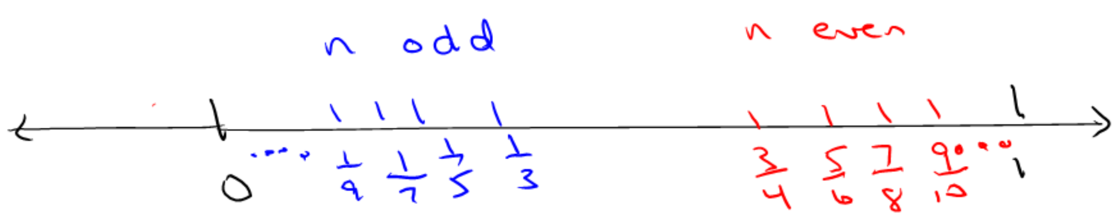
Definition 6. *An element c* $\in X$ *is a* cluster point *of a sequence* $\{x_n\} \subseteq X$ *in a metric space* (X, d) *if* $\forall \varepsilon > 0$, $\{n : x_n \in B_\varepsilon(c)\}$ *is an infinite set. Equivalently,*

$$\forall \varepsilon > 0, \overset{\forall}{N} \in \mathbf{N} \ \exists n > N \ \textit{s.t.} \ x_n \in B_\varepsilon(c)$$

Example:

$$x_n = \begin{cases} 1 - \frac{1}{n} & \text{if } n \text{ even} \\ \frac{1}{n} & \text{if } n \text{ odd} \end{cases}$$

For n large and odd, x_n is close to zero; for n large and even, x_n is close to one. The sequence does not converge; the set of cluster points is $\{0, 1\}$.



Subsequences

If $\{x_n\}$ is a sequence and $n_1 < n_2 < n_3 < \cdots$ then $\{x_{n_k}\}$ is called a *subsequence*. of $\{x_n\}$

Note that a subsequence is formed by taking some of the elements of the parent sequence, *in the same order*.

$\forall k$

Example: $x_n = \frac{1}{n}$, so $\{x_n\} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. If $n_k = 2k$, then $\{x_{n_k}\} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$.

Cluster Points and Subsequences

Theorem 5 (2.4 in De La Fuente, plus ...). *Let (X, d) be a metric space, $c \in X$, and $\{x_n\}$ a sequence in X. Then c is a cluster point of $\{x_n\}$ if and only if there is a subsequence $\{x_{n_k}\}$* of $\{x_n\}$ *such that* $\lim_{k\to\infty} x_{n_k} = c$.

$\Rightarrow$: *Proof.* Suppose c is a cluster point of $\{x_n\}$. We inductively construct a subsequence that converges to c. For $k = 1$, $\{n : x_n \in B_1(c)\}$ is infinite, so nonempty; let

$$n_1 = \min\{n : x_n \in B_1(c)\}$$

Now, suppose we have chosen $n_1 < n_2 < \cdots < n_k$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k$$

$\{n : x_n \in B_{\frac{1}{k+1}}(c)\}$ is infinite, so it contains at least one element bigger than n_k, so let

$$n_{k+1} = \min\left\{n : n > n_k,\ x_n \in B_{\frac{1}{k+1}}(c)\right\}$$

Thus, we have chosen $n_1 < n_2 < \cdots < n_k < n_{k+1}$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k, k+1$$

Thus, by induction, we obtain a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \in B_{\frac{1}{k}}(c) \qquad \forall\, k$$

Given any $\varepsilon > 0$, by the Archimedean property, there exists $N(\varepsilon) > 1/\varepsilon$.

$$\begin{aligned} \forall \quad k > N(\varepsilon) \;&\Rightarrow\; x_{n_k} \in B_{\frac{1}{k}}(c) \\ &\Rightarrow\; x_{n_k} \in B_{\varepsilon}(c) \end{aligned}$$

$k > N \Rightarrow \frac{1}{k} < \varepsilon$

so

$$x_{n_k} \to c \text{ as } k \to \infty$$

$\Leftarrow$: Conversely, suppose that there is a subsequence $\{x_{n_k}\}$ converging to c. Given any $\varepsilon > 0$, there exists $K \in \mathbf{N}$ such that

$$k > K \Rightarrow d(x_{n_k}, c) < \varepsilon \Rightarrow x_{n_k} \in B_\varepsilon(c)$$

Therefore,

$$\{n : x_n \in B_\varepsilon(c)\} \supseteq \{n_{K+1}, n_{K+2}, n_{K+3}, \ldots\}$$

Since $n_{K+1} < n_{K+2} < n_{K+3} < \cdots$, this set is infinite, so c is a cluster point of $\{x_n\}$. $\square$

Sequences in $\mathbf{R}$ and $\mathbf{R}^m$

Definition 7. *A sequence of real numbers* $\{x_n\}$ *is* increasing *(*decreasing*) if* $x_{n+1} \geq x_n$ *(* $x_{n+1} \leq x_n$ *) for all* n*.*

Definition 8. *If* $\{x_n\}$ *is a sequence of real numbers,* $\{x_n\}$ tends to infinity *(written* $x_n \to \infty$ *or* $\lim x_n = \infty$*) if*

$$\forall K \in \mathbf{R}\ \exists N(K) \text{ s.t. } n > N(K) \Rightarrow x_n > K$$

Similarly define $x_n \to -\infty$ *or* $\lim x_n = -\infty$*.*

Increasing and Decreasing Sequences

Theorem 6 (Theorem 3.1'). *Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. Then*

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbf{N}\}$$

$$\left(\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbf{N}\} \right)$$

In particular, the limit exists.

work through proof in DLF - think about unbounded case

Lim Sups and Lim Infs

Consider a sequence $\{x_n\}$ of real numbers. Let

$$
\begin{aligned}
\alpha_n &= \sup\{x_k : k \geq n\} \\
&= \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \\
\beta_n &= \inf\{x_k : k \geq n\} \\
&= \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}
\end{aligned}
$$

Either $\alpha_n = +\infty$ for all n, or $\alpha_n \in \mathbf{R}$ $\forall n$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$.

Either $\beta_n = -\infty$ for all n, or $\beta_n \in \mathbf{R}$ $\forall n$ and $\beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots$.

Lim Sups and Lim Infs

Definition 9.

$$\limsup_{n\to\infty} x_n = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases}$$

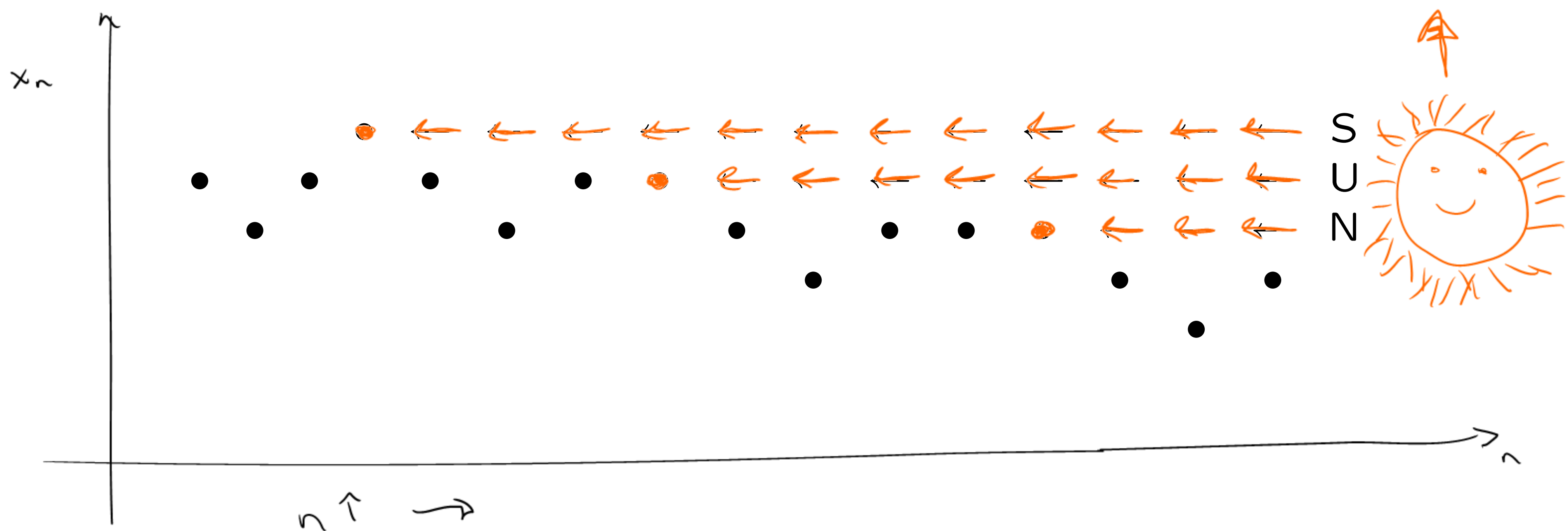
$$\liminf_{n\to\infty} x_n = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}$$

Theorem 7. *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\lim_{n\to\infty} x_n = \gamma \in \mathbf{R} \cup \{-\infty, \infty\}$$
$$\Leftrightarrow \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \gamma$$

Increasing and Decreasing Subsequences

Theorem 8 (Theorem 3.2, Rising Sun Lemma)**.** *Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.*



Proof. Let

$$S = \{s \in \mathbf{N} : x_s > x_n \quad \forall n > s\}$$

Either S is infinite, or S is finite. (or empty)

If S is infinite, let

$$\begin{aligned} n_1 &= \min S \\ n_2 &= \min (S \setminus \{n_1\}) \\ n_3 &= \min (S \setminus \{n_1, n_2\}) \\ &\vdots \\ n_{k+1} &= \min (S \setminus \{n_1, n_2, \ldots, n_k\}) \end{aligned}$$

Then $n_1 < n_2 < n_3 < \cdots$.

$$\begin{aligned}
x_{n_1} > x_{n_2} &\quad \text{since } n_1 \in S \text{ and } n_2 > n_1 \\
x_{n_2} > x_{n_3} &\quad \text{since } n_2 \in S \text{ and } n_3 > n_2 \\
&\quad \vdots \\
x_{n_k} > x_{n_{k+1}} &\quad \text{since } n_k \in S \text{ and } n_{k+1} > n_k \\
&\quad \vdots
\end{aligned}$$

so $\{x_{n_k}\}$ is a strictly decreasing subsequence of $\{x_n\}$.

If S is finite and nonempty, let $n_1 = (\max S) + 1$; if $S = \emptyset$, let $n_1 = 1$. Then

$$\begin{aligned}
n_1 \notin S &\quad \text{so} \quad \exists n_2 > n_1 \text{ s.t. } x_{n_2} \geq x_{n_1} \\
n_2 \notin S &\quad \text{so} \quad \exists n_3 > n_2 \text{ s.t. } x_{n_3} \geq x_{n_2} \\
&\quad \vdots \\
n_k \notin S &\quad \text{so} \quad \exists n_{k+1} > n_k \text{ s.t. } x_{n_{k+1}} \geq x_{n_k} \\
&\quad \vdots
\end{aligned}$$

so $\{x_{n_k}\}$ is a (weakly) increasing subsequence of $\{x_n\}$. $\square$

Bolzano-Weierstrass Theorem

Theorem 9 (Thm. 3.3, Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers contains a convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence of real numbers. By the Rising Sun Lemma, find an increasing or decreasing subsequence $\{x_{n_k}\}$. If $\{x_{n_k}\}$ is increasing, then by Theorem 3.1',

$$\lim x_{n_k} = \sup\{x_{n_k} : k \in \mathbf{N}\} \leq \sup\{x_n : n \in \mathbf{N}\} < \infty$$

since the sequence is bounded; since the limit is finite, the subsequence converges. Similarly, if the subsequence is decreasing, it converges. $\square$