Econ 204 2022

Lecture 8

Outline

0. Characterizing continuity notions for 1. Bases correspondences

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- 2. Linear Transformations
- 3. Isomorphisms

Continuity for Correspondences

Definition 5. A correspondence $\Psi : X \to 2^Y$ is called closedvalued if $\Psi(x)$ is a closed subset of Y for all x; Ψ is called compact-valued if $\Psi(x)$ is compact for all x.

For closed-valued correspondences these concepts can be more tightly connected.

- A closed-valued and upper hemicontinuous correspondence must have closed graph.
- For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

Upper Hemicontinuity and Closed Graph **Theorem 4** (Not in de la Fuente). Suppose $X \subseteq E^n$ and $Y \subseteq E^m$, and $\Psi : X \to 2^Y$.

(i) If Ψ is closed-valued and uhc, then Ψ has closed graph.

(ii) If Ψ has closed graph and there is an open set W with $x_0 \in W$ and a compact set Z such that $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$, then Ψ is uhc at x_0 . $\mathcal{T} \subseteq \mathcal{V}$

(iii) If Y is compact, then Ψ has closed graph $\iff \Psi$ is closedvalued and uhc. *Proof.* (i) Suppose Ψ is closed-valued and uhc. If Ψ does not have closed graph, we can find a sequence $(x_n, y_n) \rightarrow (x_0, y_0)$, where (x_n, y_n) lies in the graph of Ψ (so $y_n \in \Psi(x_n)$) but (x_0, y_0) does not lie in the graph of Ψ (so $y_0 \notin \Psi(x_0)$). Since Ψ is closed-valued, $\Psi(x_0)$ is closed. Since $y_0 \notin \Psi(x_0)$, there is some $\varepsilon > 0$ such that

 $\Psi(x_0) \cap B_{2\varepsilon}(y_0) = \emptyset$



SO

$$\Psi(x_0) \subseteq \mathbf{E}^m \setminus B_{\varepsilon}[y_0] \quad \blacktriangleleft \quad \mathsf{e}_{\mathsf{e}}$$

Let $V = \mathbf{E}^m \setminus B_{\varepsilon}[y_0]$. Then V is open, and $\Psi(x_0) \subseteq V$. Since Ψ is uhc, there is an open set U with $x_0 \in U$ such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

Since $(x_n, y_n) \rightarrow (x_0, y_0)$, $x_n \in U$ for n sufficiently large, so

$$y_n \in \Psi(x_n) \subseteq V \;\; = \;\; \mathcal{R} \;\; \lor \; \mathcal{B}_{arepsilon}$$

Thus for *n* sufficiently large, $||y_n - y_0|| \ge \varepsilon$, which implies that $y_n \nleftrightarrow y_0$, and $(x_n, y_n) \nleftrightarrow (x_0, y_0)$, a contradiction. Thus Ψ is closed-graph.

(ii) Now, suppose Ψ has closed graph and there is an open set W with $x_0 \in W$ and a compact set Z such that $\forall Z \subseteq \forall$

$$x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$$

Since Ψ has closed graph, it is closed-valued. Let V be any open set such that $V \supseteq \Psi(x_0)$. We need to show there exists an open set U with $x_0 \in U$ such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

If not, we can find a sequence $x_n \to x_0$ and $y_n \in \Psi(x_n)$ such that $y_n \notin V \ \forall n$. Since $x_n \to x_0$, $x_n \in W \cap X$ for all n sufficiently large, and thus $\Psi(x_n) \subseteq Z$ for n sufficiently large. Since Z is compact, we can find a convergent subsequence $y_{n_k} \to y'$. Then

$$(x_{n_k}, y_{n_k}) \to (x_0, y')$$
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Since Ψ has closed graph, $y' \in \Psi(x_0)$, so $y' \in V$. Since V is open, $y_{n_k} \in V$ for all k sufficiently large, a contradiction. Thus, Ψ is uhc at x_0 .

(iii) Follows from (i) and (ii).

Sequential Characterizations

Upper and lower hemicontinuity can be given sequential characterizations that are useful in applications.

Theorem 5 (Thm. 11.2). Suppose $X \subseteq E^n$ and $Y \subseteq E^m$. A compact-valued correspondence $\Psi : X \to 2^Y$ is unc at $x_0 \in X$ if and only if, for every sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$, and every sequence $\{y_n\}$ such that $y_n \in \Psi(x_n)$ for every n, there is a convergent subsequence $\{y_{n_k}\}$ such that $\lim y_{n_k} \in \Psi(x_0)$.

Proof. See de la Fuente.

Note that this characterization of upper hemicontinuity requires the correspondence to have compact values.

Sequential Characterizations

Theorem 6 (Thm. 11.3). A correspondence $\Psi : X \to 2^Y$ is lhc at $x_0 \in X$ if and only if, for every sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$, and every $y_0 \in \Psi(x_0)$, there exists a companion sequence $\{y_n\}$ with $y_n \in \Psi(x_n)$ for every n such that $y_n \to y_0$.

Proof. See de la Fuente.

Linear Combinations and Spans

Definition 1. Let X be a vector space over a field F. A linear combination of $x_1, \ldots, x_n \in X$ is a vector of the form

$$y = \sum_{i=1}^{n} \alpha_i x_i$$
 where $\alpha_1, \dots, \alpha_n \in F$

 α_i is the coefficient of x_i in the linear combination.

If $V \subseteq X$, the span of V, denoted span V, is the set of all linear combinations of elements of V.

A set $V \subseteq X$ spans X if span V = X.

Linear Dependence and Independence Definition 2. A set $V \subseteq X$ is linearly dependent if there exist

 $v_1, \ldots, v_n \in V$ and $\alpha_1, \ldots, \alpha_n \in F$ not all zero such that

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

A set $V \subseteq X$ is linearly independent if it is not linearly dependent.

Thus $V \subseteq X$ is linearly independent if and only if

$$\sum_{i=1}^{n} \alpha_{i} v_{i} = 0, \quad v_{i} \in V \ \forall i \Rightarrow \alpha_{i} = 0 \ \forall i$$

Definition 3. A Hamel basis (often just called a basis) of a vector space X is a linearly independent set of vectors in X that spans X.

Example: $\{(1,0), (0,1)\}$ is a basis for \mathbb{R}^2 (this is the standard basis).

 $x, \beta \in \mathbb{R}$: $\chi(1,0) + \beta(0,1) = (x,\beta)$

Example, cont: $\{(1,1), (-1,1)\}$ is another basis for \mathbb{R}^2 : Suppose $(x, y) = \alpha(1, 1) + \beta(-1, 1)$ for some $\alpha, \beta \in \mathbf{R}$ $x = \alpha - \beta \qquad \simeq (\alpha - \beta, \alpha + \beta)$ $y = \alpha + \beta$ $x + y = 2\alpha$ $\Rightarrow \alpha = \frac{x+y}{2}$ $y-x = 2\beta$ $\Rightarrow \beta = \frac{y-x}{2}$ $(x,y) = \frac{x+y}{2}(1,1) + \frac{y-x}{2}(-1,1)$

Since (x, y) is an arbitrary element of \mathbb{R}^2 , $\{(1, 1), (-1, 1)\}$ spans \mathbb{R}^2 . If (x, y) = (0, 0),

$$\alpha = \frac{0+0}{2} = 0, \quad \beta = \frac{0-0}{2} = 0$$

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so the coefficients are all zero, so $\{(1,1), (-1,1)\}$ is linearly independent. Since it is linearly independent and spans \mathbb{R}^2 , it is a basis.

Example: {(1,0,0), (0,1,0)} is not a basis of \mathbb{R}^3 , because it does not span \mathbb{R}^3 . If (x, y, z) and (x, y, z) and (x, y, z) and (x, y, z) and (x, y, z). **Example:** {(1,0), (0,1), (1,1)} is not a basis for \mathbb{R}^2 .

1(1,0) + 1(0,1) + (-1)(1,1) = (0,0)

so the set is not linearly independent.

Theorem 1 (Thm. 1.2'). Let V be a Hamel basis for X. Then every vector $x \in X$ has a unique representation as a linear combination of a finite number of elements of V (with all coefficients nonzero).*

Proof. Let $x \in X$. Since V spans X, we can write

$$x = \sum_{s \in S_1} \alpha_s v_s$$

where S_1 is finite, $\alpha_s \in F$, $\alpha_s \neq 0$, and $v_s \in V$ for each $s \in S_1$. Now, suppose

$$x = \sum_{s \in S_1} \alpha_s v_s = \sum_{s \in S_2} \beta_s v_s$$

*The unique representation of 0 is $0 = \sum_{i \in \emptyset} \alpha_i b_i$.

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where S_2 is finite, $\beta_s \in F$, $\beta_s \neq 0$, and $v_s \in V$ for each $s \in S_2$. Let $S = S_1 \cup S_2$, and define

$$\alpha_s = 0 \quad \text{for} \quad s \in S_2 \setminus S_1$$

$$\beta_s = 0 \quad \text{for} \quad s \in S_1 \setminus S_2$$

Then

$$0 = x - x$$

= $\sum_{s \in S_1} \alpha_s v_s - \sum_{s \in S_2} \beta_s v_s$
= $\sum_{s \in S} \alpha_s v_s - \sum_{s \in S} \beta_s v_s$
= $\sum_{s \in S} (\alpha_s - \beta_s) v_s$

Since V is linearly independent, we must have $\alpha_s - \beta_s = 0$, so $\alpha_s = \beta_s$, for all $s \in S$.

$$s \in S_1 \Leftrightarrow \alpha_s \neq \mathbf{0} \Leftrightarrow \beta_s \neq \mathbf{0} \Leftrightarrow s \in S_2$$

so $S_1 = S_2$ and $\alpha_s = \beta_s$ for $s \in S_1 = S_2$, so the representation is unique.

Theorem 2. Every vector space has a Hamel basis.

Proof. The proof uses the Axiom of Choice. Indeed, the theorem is equivalent to the Axiom of Choice. $\hfill \Box$

A closely related result, from which you can derive the previous result, shows that any linearly independent set V in a vector space X can be extended to a basis of X.

Theorem 3. If X is a vector space and $V \subseteq X$ is linearly independent, then there exists a linearly independent set $W \subseteq X$ such that

 $V \subseteq W \subseteq \operatorname{span} W = X$

Theorem 4. Any two Hamel bases of a vector space X have the same cardinality (are numerically equivalent).

Proof. The proof depends on the so-called Exchange Lemma, whose idea we sketch. Suppose that $V = \{v_{\lambda} : \lambda \in \Lambda\}$ and $W = \{w_{\gamma} : \gamma \in \Gamma\}$ are Hamel bases of X. Remove one vector v_{λ_0} from V, so that it no longer spans (if it did still span, then v_{λ_0} would be a linear combination of other elements of V, and V would not be linearly independent). If $w_{\gamma} \in \text{span}(V \setminus \{v_{\lambda_0}\})$ for every $\gamma \in \Gamma$, then since W spans, $V \setminus \{v_{\lambda_0}\}$ would also span, contradiction. Thus, we can choose $\gamma_0 \in \Gamma$ such that

$$w_{\gamma_0} \not\in \operatorname{span}\left(V \setminus \{v_{\lambda_0}\}\right)$$

\succ_{ω} Because $w_{\gamma_0} \in \operatorname{span} V$, we can write

$$w_{\gamma_0} = \sum_{i=0}^n \alpha_i v_{\lambda_i}$$

where α_0 , the coefficient of v_{λ_0} , is not zero (if it were, then we would have $w_{\gamma_0} \in \text{span}(V \setminus \{v_{\lambda_0}\})$). Since $\alpha_0 \neq 0$, we can solve for v_{λ_0} as a linear combination of w_{γ_0} and $v_{\lambda_1}, \ldots, v_{\lambda_n}$, so

$$\begin{array}{rcl} \operatorname{span}\left(\left(V \setminus \{v_{\lambda_0}\}\right) \cup \{w_{\gamma_0}\}\right) & \ni & \checkmark_{\lambda_0} \\ & \supseteq & \operatorname{span} V & \lnot & \operatorname{span}\left(\left(\lor \lor \lor \lor \lor \lor \lor\right) \lor & \lor \lor \lor \lor\right) \\ & = & X \end{array}$$

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$$\left(\left(V \setminus \{v_{\lambda_0}\}\right) \cup \{w_{\gamma_0}\}\right)$$

spans X. From the fact that $w_{\gamma_0} \not\in \text{span}\left(V \setminus \{v_{\lambda_0}\}\right)$ one can

show that

$$\left(\left(V \setminus \{v_{\lambda_0}\}\right) \cup \{w_{\gamma_0}\}\right)$$

is linearly independent, so it is a basis of X. Repeat this process to exchange every element of V with an element of W (when V is uncountable, this is done by a process called transfinite induction). At the end, we obtain a bijection from V to W, so that V and W are numerically equivalent.

Dimension

Definition 4. The dimension of a vector space X, denoted dim X, is the cardinality of any basis of X.

For $V \subseteq X$, |V| denotes the cardinality of the set V.

Dimension

Example: The set of all $m \times n$ real-valued matrices is a vector space over \mathbf{R} . A basis is given by

$$\{E_{ij}: 1 \leq i \leq m, 1 \leq j \leq n\}$$
 $E_{ij} \qquad matrix$

where

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$$\begin{array}{c} \text{Kl} \stackrel{\text{full}}{\underset{m}} \text{ for } k = i \text{ and } l = j \\ \text{for } k = i \text{ or } k = i \text{ or } k = j \\ 0 \text{ otherwise.} \end{array}$$

The dimension of the vector space of $m \times n$ matrices is mn.

Dimension and Dependence

Theorem 5 (Thm. 1.4). Suppose dim $X = n \in \mathbb{N}$. If $V \subseteq X$ and |V| > n, then V is linearly dependent.

If not, V is linearly independent, so V an
be extended to a basis
$$W d X$$
, and
 $V \subseteq W \Longrightarrow n < |V| \le |W|$

contradiction.

Dimension and Dependence **Theorem 6** (Thm. 1.5'). *Suppose* dim $X = n \in \mathbb{N}$, $V \subseteq X$, and |V| = n.

- If V is linearly independent, then V spans X, so V is a Hamel basis.
- If V spans X, then V is linearly independent, so V is a Hamel basis.
 1) Otherwise, extend V to a basis W of X, with V & W, so [W] > [W] = n
 contradiction.
 (2) Otherwise, choose V' & V a basis for X, and [W'] < [V] = n
 (2) Otherwise, choose V' & V a basis for X, and [V'] < [V] = n

Linear Transformations

Definition 5. Let X and Y be two vector spaces over the field F. We say $T: X \rightarrow Y$ is a linear transformation if



Equivalently: • T(ax) = aT(x) $\forall a \in F, \forall x \in X$ • $T(x, + x_2) = T(x_1) + T(x_2) \quad \forall x_1, x_2 \in X$

? $+: L(X,Y) \times L(X,Y) \longrightarrow L(X,Y)$ $\cdot: F \times L(X,Y) \longrightarrow L(X,Y)$ Linear Transformations

Theorem 7. L(X,Y) is a vector space over F.

Proof. First, define linear combinations in L(X, Y) as follows. For $T_1, T_2 \in L(X, Y)$ and $\alpha, \beta \in F$, define $\alpha T_1 + \beta T_2$ by

$$(\underline{\alpha T_1 + \beta T_2})(x) = \alpha T_1(x) + \beta T_2(x)$$

We need to show that $\alpha T_1 + \beta T_2 \in L(X, Y)$.

$$(\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2) \qquad (definition)$$

$$= \alpha T_1(\gamma x_1 + \delta x_2) + \beta T_2(\gamma x_1 + \delta x_2) \qquad (definition)$$

$$= \alpha (\gamma T_1(x_1) + \delta T_1(x_2)) + \beta (\gamma T_2(x_1) + \delta T_2(x_2)) (T_1, T_2 \text{ inear})$$

$$= \gamma (\alpha T_1(x_1) + \beta T_2(x_1)) + \delta (\alpha T_1(x_2) + \beta T_2(x_2)) (collect \text{ terms})$$

$$= \gamma (\alpha T_1 + \beta T_2) (x_1) + \delta (\alpha T_1 + \beta T_2) (x_2) \qquad (definition again)$$
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$$T(8x,+8xa) = T(x,) + ST(xa)$$

$$\forall x, g \quad \forall x, ya$$

so $\alpha T_1 + \beta T_2 \in L(X,Y)$.

The rest of the proof involves straightforward checking of the vector space axioms. $\hfill \Box$

Compositions of Linear Transformations

X, \forall , \mathcal{Z} vector spaces over some field \mathcal{F} Given $R \in L(X,Y)$ and $S \in L(Y,Z)$, $S \circ R : X \to Z$. We will show that $S \circ R \in L(X,Z)$, that is, the composition of two linear transformations is linear.

$$(S \circ R)(\alpha x_1 + \beta x_2) = S(R(\alpha x_1 + \beta x_2)) \quad (\text{ defn of } S \circ R)$$

$$= S(\alpha R(x_1) + \beta R(x_2)) \quad (R \text{ linear})$$

$$= \alpha S(R(x_1)) + \beta S(R(x_2)) \quad (S \text{ linear})$$

$$= \alpha (S \circ R)(x_1) + \beta (S \circ R)(x_2) \quad (\text{ defn of } S \circ R)$$

So $S \circ R \in L(X, Z).$

Kernel and Rank

Definition 6. Let $T \in L(X, Y)$.

- The image of T is $\operatorname{Im} T = T(X) \subseteq \mathbb{Y}$ • Can show ImT is a vector subspace of \mathbb{Y}
- The kernel of T is ker $T = \{x \in X : T(x) = 0\}$ (null space of $\overline{\checkmark}$)
- The rank of T is $\operatorname{Rank} T = \operatorname{dim}(\operatorname{Im} T)$

Recall: • W S X is a vector subspace if it is a vector space over F under +, • from X • W S X, W \$ X is a vector subspace17 (>) Y w, No E W, Y X, BEF, X W, + B Wo E W

Rank-Nullity Theorem

Theorem 8 (Thms. 2.9, 2.7, 2.6: The Rank-Nullity Theorem). Let X be a finite-dimensional vector space, $T \in L(X,Y)$. Then Im T and ker T are vector subspaces of Y and X respectively, and



Kernel and Rank

Theorem 9 (Thm. 2.13). $T \in L(X, Y)$ is one-to-one if and only if ker $T = \{0\}$.

- ⇒ Proof. Suppose T is one-to-one. Suppose $x \in \ker T$. Then T(x) = 0. But since T is linear, $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$. Since T is one-to-one, x = 0, so ker $T = \{0\}$.
- Conversely, suppose that ker $T = \{0\}$. Suppose $T(x_1) = T(x_2)$. Then

$$T(x_1 - x_2) = T(x_1) - T(x_2)$$

= 0

which says $x_1 - x_2 \in \ker T$, so $x_1 - x_2 = 0$, so $x_1 = x_2$. Thus, T is one-to-one.

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Invertible Linear Transformations

Definition 7. $T \in L(X, Y)$ is invertible if there exists a function $S: Y \to X$ such that

S(T(x))	=	x	$\forall x \in X$	Sot = idx
T(S(y))	=	y	$\forall y \in Y$	TOS = idy

Denote S by T^{-1} .

Note that T is invertible if and only if it is one-to-one and onto. This is just the condition for the existence of an inverse *function*. The linearity of the inverse follows from the linearity of T.

Invertible Linear Transformations

Theorem 10 (Thm. 2.11). If $T \in L(X,Y)$ is invertible, then $T^{-1} \in L(Y,X)$, i.e. T^{-1} is linear.

Proof. Suppose $\alpha, \beta \in F$ and $v, w \in Y$. Since T is invertible, there exist unique $v', w' \in X$ such that

$$\begin{array}{rcl} T(v') &= v & T^{-1}(v) &= v' \\ T(w') &= w & T^{-1}(w) &= w' \end{array}$$

Then

$$T^{-1}(\alpha v + \beta w) = T^{-1} \left(\alpha T(v') + \beta T(w') \right) \quad (\text{definition})$$

$$= T^{-1} \left(T(\alpha v' + \beta w') \right) \quad (\tau \text{ cinear})$$

$$= \alpha v' + \beta w' \quad (\text{definition})$$

$$= \alpha T^{-1}(v) + \beta T^{-1}(w) \quad (\text{definition})$$

so
$$T^{-1} \in L(Y, X)$$
.

Linear Transformations and Bases

Theorem 11 (Thm. 3.2). Let X and Y be two vector spaces over the same field F, and let $V = \{v_{\lambda} : \lambda \in \Lambda\}$ be a basis for X. Then a linear transformation $T \in L(X,Y)$ is completely determined by its values on V, that is:

1. Given any set $\{y_{\lambda} : \lambda \in \Lambda\} \subseteq Y$, $\exists T \in L(X, Y)$ s.t.

$$T(v_{\lambda}) = y_{\lambda} \quad \forall \lambda \in \Lambda$$

2. If $S, T \in L(X, Y)$ and $S(v_{\lambda}) = T(v_{\lambda})$ for all $\lambda \in \Lambda$, then S = T.

Proof. 1. If $x \in X$, x has a unique representation of the form

$$x = \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \quad \alpha_i \neq 0 \ i = 1, \dots, n$$

(Recall that if x = 0, then n = 0.) Define

$$T(x) = \sum_{i=1}^{n} \alpha_i y_{\lambda_i} \xrightarrow{\tau(\vee \lambda_i)} (\sum_{j=1}^{n} \alpha_j y_{\lambda_i} \xrightarrow{\tau(\vee \lambda_i)} (\sum_$$

Then $T(x) \in Y$. The verification that T is linear is left as an exercise.

2. Suppose $S(v_{\lambda}) = T(v_{\lambda})$ for all $\lambda \in \Lambda$. Given $x \in X$,

$$S(x) = S\left(\sum_{i=1}^{n} \alpha_{i} v_{\lambda_{i}}\right)$$

$$= \sum_{i=1}^{n} \alpha_{i} S\left(v_{\lambda_{i}}\right) \qquad (S \cup e^{n})$$

$$= \sum_{i=1}^{n} \alpha_{i} T\left(v_{\lambda_{i}}\right) \qquad (S \cup e^{n})$$

$$= T\left(\sum_{i=1}^{n} \alpha_{i} v_{\lambda_{i}}\right) \qquad (T \cup e^{n})$$

$$= T(x)$$

so S = T.

Isomorphisms

Definition 8. Two vector spaces X and Y over a field F are isomorphic if there is an invertible $T \in L(X, Y)$.

 $T \in L(X, Y)$ is an isomorphism if it is invertible (one-to-one and onto).

Isomorphic vector spaces are essentially indistinguishable as vector spaces.

Isomorphisms

Theorem 12 (Thm. 3.3). Two vector spaces X and Y over the same field are isomorphic if and only if dim $X = \dim Y$.

 \implies *Proof.* Suppose X and Y are isomorphic, and let $T \in L(X, Y)$ be an isomorphism. Let

 $U = \{u_{\lambda} : \lambda \in \Lambda\}$

be a basis of X, and let $v_{\lambda} = T(u_{\lambda})$ for each $\lambda \in \Lambda$. Set

$$V = \{v_{\lambda} : \lambda \in \Lambda\}$$

Since T is one-to-one, U and V have the same cardinality. If $\tau(u)$

 $y \in Y$, then there exists $x \in X$ such that

$$y = T(x) \qquad (T \text{ is onto})$$
$$= T\left(\sum_{i=1}^{n} \alpha_{\lambda_{i}} u_{\lambda_{i}}\right)$$
$$= \sum_{i=1}^{n} \alpha_{\lambda_{i}} T\left(u_{\lambda_{i}}\right) \qquad (\text{ transitions of } T)$$
$$= \sum_{i=1}^{n} \alpha_{\lambda_{i}} v_{\lambda_{i}} \qquad \text{tdefn of } v_{\lambda_{i}})$$

which shows that V spans Y. To see that V is linearly indepen-

dent, suppose

$$D = \sum_{i=1}^{m} \beta_{i} v_{\lambda_{i}}$$

$$= \sum_{i=1}^{m} \beta_{i} T(u_{\lambda_{i}}) \qquad (\text{defn d } \vee_{\lambda_{u}})$$

$$= T\left(\sum_{i=1}^{m} \beta_{i} u_{\lambda_{i}}\right) \qquad (\neg \text{ where })$$

Since T is one-to-one, ker $T = \{0\}$, so

$$\sum_{i=1}^{m} \beta_i u_{\lambda_i} = 0$$

Since U is a basis, we have $\beta_1 = \cdots = \beta_m = 0$, so V is linearly independent. Thus, V is a basis of Y; since U and V are numerically equivalent, dim $X = \dim Y$.

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 $\langle = :$ Now suppose dim $X = \dim Y$. Let

$$U = \{u_{\lambda} : \lambda \in \Lambda\} \text{ and } V = \{v_{\lambda} : \lambda \in \Lambda\}$$

be bases of X and Y; note we can use the same index set Λ for both because dim $X = \dim Y$. By Theorem 3.2, there is a unique

$$T \in L(X,Y) \text{ such that } T(u_{\lambda}) = v_{\lambda} \text{ for all } \lambda \in \Lambda. \text{ If } T(x) = 0, \text{ then}$$

$$T (x) = T(x)$$

$$= T\left(\sum_{i=1}^{n} \alpha_{i} u_{\lambda_{i}}\right)$$

$$= \sum_{i=1}^{n} \alpha_{i} T\left(u_{\lambda_{i}}\right) \qquad (T \text{ times})$$

$$= \sum_{i=1}^{n} \alpha_{i} v_{\lambda_{i}} \qquad (T(u_{\lambda_{i}}) = \sqrt{\lambda_{i}}, \sqrt{\lambda_{i}})$$

$$\Rightarrow \alpha_{1} = \cdots = \alpha_{n} = 0 \text{ since } V \text{ is a basis}$$

$$\Rightarrow x = 0 \qquad = \sum_{i=1}^{n} \lambda_{i} u_{\lambda_{i}}$$

$$\Rightarrow \text{ ker } T = \{0\}$$

$$\Rightarrow T \text{ is one-to-one}$$

T is onto:
If
$$y \in Y$$
, write $y = \sum_{i=1}^{m} \beta_i v_{\lambda_i}$. Let

$$x = \sum_{i=1}^{m} \beta_i u_{\lambda_i}$$

Then

so T is onto, so T is an isomorphism and X, Y are isomorphic.