Announcements

- PS 3 due tomorrow

Econ 204 2022

Lecture 9

Outline

0. Kernel, Rank, Isomorphisms (cont)
1. Quotient Vector Spaces
2. Matrix Representations of Linear Transformations
3. Change of Basis and Similarity
4. Eigenvalues and Eigenvectors
5. Diagonalization

$\ker T = \{x \in X : T(x) = 0\}$ $\quad$ $\operatorname{Rank} T = \dim \operatorname{Im} T$

$\ker T \subseteq X$ $\quad$ $\operatorname{Im} T = T(X) \subseteq Y$

Rank-Nullity Theorem

Theorem 8 (Thms. 2.9, 2.7, 2.6: The Rank-Nullity Theorem)**.** *Let X be a finite-dimensional vector space, $T \in L(X,Y)$. Then* Im T *and* ker T *are vector subspaces of Y and X respectively, and*

$$\dim X = \underbrace{\dim \ker T}_{\text{nullity of } T} + \operatorname{Rank} T$$

Sketch:
- Show $\operatorname{Im} T$, $\ker T$ are vector subspaces
- take $\{v_1, \dots, v_k\}$ a basis for $\ker T$
- extend to $\{v_1, \dots, v_k, w_1, \dots, w_r\}$ a basis for X
- Show $\{T(w_1), \dots, T(w_r)\}$ is a basis for $\operatorname{Im} T$

Kernel and Rank

Theorem 9 (Thm. 2.13)**.** *$T \in L(X, Y)$ is one-to-one if and only if $\ker T = \{0\}$.*

$\Rightarrow$: *Proof.* Suppose T is one-to-one. Suppose $x \in \ker T$. Then $T(x) = 0$. But since T is linear, $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$. Since T is one-to-one, $x = 0$, so $\ker T = \{0\}$.

$\Leftarrow$: Conversely, suppose that $\ker T = \{0\}$. Suppose $T(x_1) = T(x_2)$. Then

$$\begin{aligned} T(x_1 - x_2) &= T(x_1) - T(x_2) \\ &= 0 \end{aligned}$$

which says $x_1 - x_2 \in \ker T$, so $x_1 - x_2 = 0$, so $x_1 = x_2$. Thus, T is one-to-one. □

Invertible Linear Transformations

Definition 7. $T \in L(X, Y)$ *is* invertible *if there exists a function* $S : Y \to X$ *such that*

$$\begin{aligned} S(T(x)) &= x \quad \forall x \in X \\ T(S(y)) &= y \quad \forall y \in Y \end{aligned}$$

$S \circ T = id_X$
$T \circ S = id_Y$

Denote S *by* T^{-1}.

Note that T is invertible if and only if it is one-to-one and onto. This is just the condition for the existence of an inverse *function.* The linearity of the inverse follows from the linearity of T.

(we will show this)

Invertible Linear Transformations

Theorem 10 (Thm. 2.11). *If $T \in L(X,Y)$ is invertible, then $T^{-1} \in L(Y,X)$, i.e. T^{-1} is linear.*

Proof. Suppose $\alpha, \beta \in F$ and $v, w \in Y$. Since T is invertible, there exist unique $v', w' \in X$ such that

$$\begin{array}{rclrcl} T(v') & = & v & T^{-1}(v) & = & v' \\ T(w') & = & w & T^{-1}(w) & = & w' \end{array} .$$

Then

$$\begin{aligned} T^{-1}(\alpha v + \beta w) &= T^{-1}\left(\alpha \overset{\overset{v}{\shortparallel}}{T(v')} + \beta \overset{\overset{w}{\shortparallel}}{T(w')}\right) && \text{(definition)} \\ &= T^{-1}\left(T(\alpha v' + \beta w')\right) && \text{(T linear)} \\ &= \alpha v' + \beta w' && \text{(defn of } T^{-1}\text{)} \\ &= \alpha T^{-1}(v) + \beta T^{-1}(w) && \text{(defn of } v', w'\text{)} \end{aligned}$$

so $T^{-1} \in L(Y, X)$. $\square$

Linear Transformations and Bases

Theorem 11 (Thm. 3.2)**.** *Let X and Y be two vector spaces over the same field F, and let $V = \{v_\lambda : \lambda \in \Lambda\}$ be a basis for X. Then a linear transformation $T \in L(X,Y)$ is completely determined by its values on V, that is:*

1. *Given any set $\{y_\lambda : \lambda \in \Lambda\} \subseteq Y$, $\exists T \in L(X,Y)$ s.t.*

$$T(v_\lambda) = y_\lambda \quad \forall \lambda \in \Lambda$$

2. *If $S, T \in L(X,Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$.*

$$T\left(\sum_{i=1}^{n} \alpha_i v_{\lambda_i}\right) = \sum_{i=1}^{n} \alpha_i T(v_{\lambda_i})$$

want $T(v_{\lambda_i}) = y_{\lambda_i} \ \forall i$

Proof. 1. If $x \in X$, x has a unique representation of the form

$$x = \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \quad \alpha_i \neq 0 \ i = 1, \ldots, n$$

for some n & $\lambda_1, \ldots, \lambda_n$

(Recall that if $x = 0$, then $n = 0$.) Define

$$T(x) = \sum_{i=1}^{n} \alpha_i \underbrace{y_{\lambda_i}}_{= T(v_{\lambda_i})}$$

(so $T(v_\lambda) = y_\lambda \ \forall \lambda$ by defn)

Then $T(x) \in Y$. The verification that T is linear is left as an exercise.

$T(\gamma z + \delta w) = \gamma T(z) + \delta T(w)$.

2. Suppose $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$. Given $x \in X$,

$$\begin{aligned}
S(x) &= S\left(\sum_{i=1}^{n} \alpha_i v_{\lambda_i}\right) \\
&= \sum_{i=1}^{n} \alpha_i S\left(v_{\lambda_i}\right) && \text{(S linear)} \\
&= \sum_{i=1}^{n} \alpha_i T\left(v_{\lambda_i}\right) && \left(\begin{array}{l}\text{S and T agree} \\ \text{on } \{v_\lambda : \lambda \in \Lambda\}\end{array}\right) \\
&= T\left(\sum_{i=1}^{n} \alpha_i v_{\lambda_i}\right) && \text{(T linear)} \\
&= T(x)
\end{aligned}$$

so $S = T$.

$\square$

Isomorphisms

Definition 8. *Two vector spaces* X *and* Y *over a field* F *are* isomorphic *if there is an invertible* $T \in L(X, Y)$*.*

$T \in L(X, Y)$ *is an* isomorphism *if it is invertible (one-to-one and onto).*

Isomorphic vector spaces are essentially indistinguishable as vector spaces.

Isomorphisms

Theorem 12 (Thm. 3.3). *Two vector spaces X and Y over the same field are isomorphic if and only if* $\dim X = \dim Y$.

$\Rightarrow$: *Proof.* Suppose X and Y are isomorphic, and let $T \in L(X, Y)$ be an isomorphism. Let

$$U = \{u_\lambda : \lambda \in \Lambda\}$$

be a basis of X, and let $v_\lambda = T(u_\lambda)$ for each $\lambda \in \Lambda$. Set

$$V = \{v_\lambda : \lambda \in \Lambda\}$$

Since T is one-to-one, U and $\underset{T(u)}{\overset{}{V}}$ have the same cardinality. If

If $y \in Y$, then there exists $x \in X$ such that

$$\begin{aligned} y &= T(x) && \text{(T is onto)} \\ &= T\left(\sum_{i=1}^{n} \alpha_{\lambda_i} u_{\lambda_i}\right) \\ &= \sum_{i=1}^{n} \alpha_{\lambda_i} T\left(u_{\lambda_i}\right) && \text{(linearity of T)} \\ &= \sum_{i=1}^{n} \alpha_{\lambda_i} v_{\lambda_i} && \text{(defn of } v_{\lambda_i}\text{)} \end{aligned}$$

which shows that V spans Y. To see that V is linearly independent, suppose

$$
\begin{aligned}
0 &= \sum_{i=1}^{m} \beta_i v_{\lambda_i} \\
&= \sum_{i=1}^{m} \beta_i T\left(u_{\lambda_i}\right) && \text{(defn of } v_{\lambda_i}\text{)} \\
&= T\left(\sum_{i=1}^{m} \beta_i u_{\lambda_i}\right) && \text{(T linear)}
\end{aligned}
$$

Since T is one-to-one, $\ker T = \{0\}$, so

$$\sum_{i=1}^{m} \beta_i u_{\lambda_i} = 0$$

Since U is a basis, we have $\beta_1 = \cdots = \beta_m = 0$, so V is linearly independent. Thus, V is a basis of Y; since U and V are numerically equivalent, $\dim X = \dim Y$.

$|\ddot{U}|$ $|\ddot{V}|$

$\Longleftarrow$: Now suppose $\dim X = \dim Y$. Let

$$U = \{u_\lambda : \lambda \in \Lambda\} \text{ and } V = \{v_\lambda : \lambda \in \Lambda\}$$

be bases of X and Y; note we can use the same index set Λ for both because $\dim X = \dim Y$. By Theorem 3.2, there is a unique

previous result

$T \in L(X, Y)$ such that $T(u_\lambda) = v_\lambda$ for all $\lambda \in \Lambda$. If $T(x) = 0$, then

T is 1-1:

$$
\begin{aligned}
0 &= T(x) \\
&= T\left(\sum_{i=1}^{n} \alpha_i u_{\lambda_i}\right) \\
&= \sum_{\mathrm{i}=1}^{n} \alpha_i T\left(u_{\lambda_i}\right) && \text{(T linear)} \\
&= \sum_{\mathrm{i}=1}^{n} \alpha_i v_{\lambda_i} && (T(u_{\lambda_i}) = v_{\lambda_i} \ \forall i) \\
&\Rightarrow \alpha_1 = \cdots = \alpha_n = 0 \text{ since } V \text{ is a basis} \\
&\Rightarrow x = 0 \quad = \sum_{i=1}^{n} \alpha_i u_{\lambda_i} \\
&\Rightarrow \ker T = \{0\} \\
&\Rightarrow T \text{ is one-to-one}
\end{aligned}
$$

T is onto:

If $y \in Y$, write $y = \sum_{i=1}^{m} \beta_i v_{\lambda_i}$. Let

$$x = \sum_{i=1}^{m} \beta_i u_{\lambda_i}$$

Then

$$\begin{aligned} T(x) &= T\left(\sum_{i=1}^{m} \beta_i u_{\lambda_i}\right) \\ &= \sum_{i=1}^{m} \beta_i T(u_{\lambda_i}) && \text{(T linear)} \\ &= \sum_{i=1}^{m} \beta_i v_{\lambda_i} \\ &= y \end{aligned}$$

so T is onto, so T is an isomorphism and X, Y are isomorphic. $\square$

$y \in [x] \iff x \sim y \iff x - y \in W$

$\iff \exists\, w' \in W$ s.t. $x - y = w'$

Quotient Vector Spaces

$\iff \exists\, w' \in W$ s.t. $\boxed{y = x - w'}$

Given a vector space X and a vector subspace W of X, define an equivalence relation by

$w' \in W \iff -w' \in W$

$$x \sim y \iff x - y \in W$$

Form a new vector space X/W: the set of vectors is

$\iff y = x + (-w')$

$$\{[x] : x \in X\}$$

where $[x]$ denotes the equivalence class of x with respect to $\sim$.

X/W is read "X mod W".

Note that the vectors in X/W are **sets** of vectors in X: for $x \in X$,

$$[x] = \{x + w : w \in W\}$$

$y = x + w$

$z = \alpha x + w'$

Quotient Vector Spaces

We claim that X/W can be viewed as a vector space over F. Define the vector space operations $+, \cdot$ in X/W as follows:

Define

$$
\begin{aligned}
[x] + [y] &= [x + y] \\
\alpha[x] &= [\alpha x]
\end{aligned}
$$

Exercise: Verify that $\sim$ is an equivalence relation and that vector addition and scalar multiplication are well-defined.*

Then X/W is a vector space over F with these definitions for $+$ and $\cdot$ (more exercise)

* Show: $\forall x, x', y, y' \in X, \forall \alpha \in F$:

$[x] = [x'], \ [y] = [y']$

$\Rightarrow [x+y] = [x'+y'], \ [\alpha x] = [\alpha x']$

Quotient Vector Spaces

Example: Let $X = \mathbf{R}^3$ and let $W = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$. Then for $x, y \in \mathbf{R}^3$,

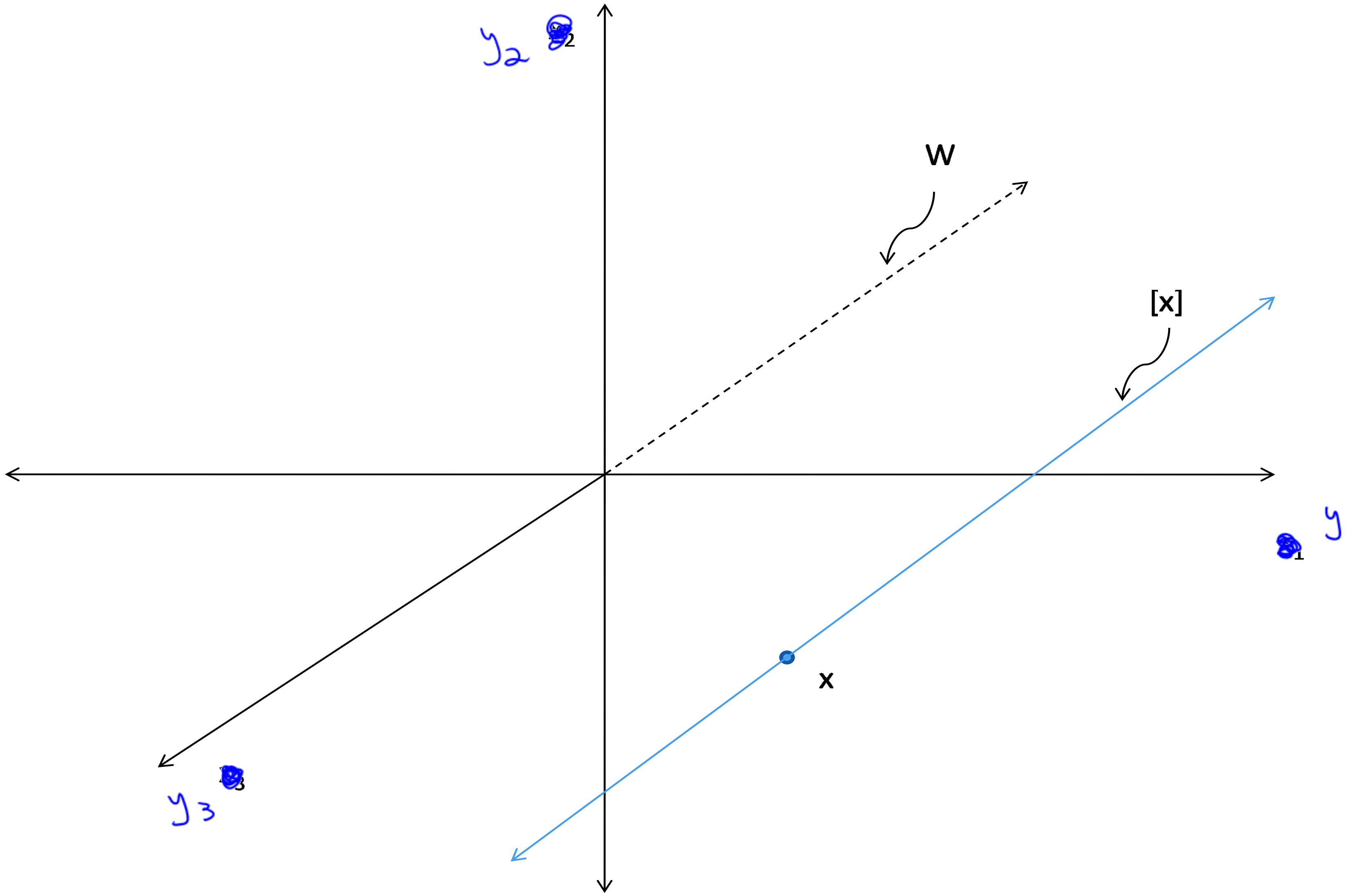
$W = \{ w \in \mathbb{R}^3 : w = (0, 0, w_3) \text{ for some } w_3 \in \mathbb{R} \}$

$$\begin{aligned} x \sim y \quad &\Longleftrightarrow \quad x - y \in W \\ &\Longleftrightarrow \quad x_1 - y_1 = 0, x_2 - y_2 = 0 \\ &\Longleftrightarrow \quad x_1 = y_1, x_2 = y_2 \end{aligned}$$

and

$$[x] = \{x + w : w \in W\} = \{(x_1, x_2, z) : z \in \mathbf{R}\}$$

So the equivalence class corresponding to x is the line in $\mathbf{R}^3$ through x parallel to the axis of the third coordinate.



y_2
W
[x]
y_1
x
y_3

Example, cont.

What is X/W? Intuitively this equivalence relation ignores the third coordinate, and we can identify the equivalence class $[x]$ with the vector $(x_1, x_2) \in \mathbf{R}^2$.

The next two results show how to formalize this connection.

Quotient Vector Spaces

Theorem 1. *If X is a vector space with* $\dim X = n$ *for some* $n \in \mathbf{N}$ *and W is a vector subspace of X, then*

$$\dim(X/W) = \dim X - \dim W$$

Proof. (Sketch) Begin with a basis $\{w_1, \ldots, w_c\}$ for W, and a basis $\{[x_1], \ldots, [x_k]\}$ for X/W. Show that

$$\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$$

is a basis for X.

$\square$

$[x] = \{ x + z : z \in \ker T \}$

Quotient Vector Spaces

Theorem 2. *Let X and Y be vector spaces over the same field F and $T \in L(X,Y)$. Then* $\operatorname{Im} T$ *is isomorphic to* $X/\ker T$.

Proof. Notice that if X is finite-dimensional, then

$$\begin{aligned} \dim(X/\ker T) &= \dim X - \dim \ker T \quad \text{(by the previous theorem)} \\ &= \operatorname{Rank} T \quad \text{(by the Rank-Nullity Theorem)} \\ &= \dim \operatorname{Im} T \end{aligned}$$

so $X/\ker T$ is isomorphic to $\operatorname{Im} T$. (why??)

We prove that this is true in general, and that the isomorphism is natural.

Define $\tilde{T} : X/\ker T \to \operatorname{Im} T$ by

$$\tilde{T}([x]) = T(x) \qquad \text{(natural)}$$

We first need to check that this is well-defined, that is, that if $[x] = [x']$ then $\tilde{T}([x]) = \tilde{T}([x'])$.

$x \not\equiv x'$

$$\begin{aligned} [x] = [x'] &\Rightarrow x \sim x' \\ &\Rightarrow x - x' \in \ker T \\ &\Rightarrow T(x - x') = 0 \quad = T(x) - T(x') \quad (T \text{ linear}) \\ &\Rightarrow T(x) = T(x') \end{aligned}$$

so $\tilde{T}$ is well-defined.

Clearly, $\tilde{T} : X/\ker T \to \operatorname{Im} T$. It is easy to check that $\tilde{T}$ is linear, so $\tilde{T} \in L(X/\ker T, \operatorname{Im} T)$. Next we show that $\tilde{T}$ is an isomorphism.

1-1:

$$\begin{aligned}
\tilde{T}([x]) = \tilde{T}([y]) &\Rightarrow T(x) = T(y) \\
&\Rightarrow T(x-y) = 0 \qquad \text{(T linear)} \\
&\Rightarrow x - y \in \ker T \\
&\Rightarrow x \sim y \\
&\Rightarrow [x] = [y]
\end{aligned}$$

so $\tilde{T}$ is one-to-one.

onto:

$$\begin{aligned}
y \in \operatorname{Im} T &\Rightarrow \exists x \in X \text{ s.t. } T(x) = y \\
&\Rightarrow \tilde{T}([x]) = y = T(x)
\end{aligned}$$

so $\tilde{T}$ is onto, hence $\tilde{T}$ is an isomorphism. $\square$

Example: Consider $T \in L(\mathbf{R}^3, \mathbf{R}^2)$ defined by

$$T(x_1, x_2, x_3) = (x_1, x_2)$$

Then

$$\ker T = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$$ (= W from previous example)

is the x_3-axis.

Given x, the equivalence class $[x]$ is just the line through x parallel to the x_3-axis.

$$\tilde{T}([x]) = T(x_1, x_2, x_3) = (x_1, x_2)$$

and

$$\operatorname{Im} T = \mathbf{R}^2, \quad X/\ker T \cong \mathbf{R}^2 = \operatorname{Im} T$$

as we suggested intuitively above (here the symbol $\cong$ denotes isomorphism, that is, we write $Y \cong Z$ if Y and Z are isomorphic.)

Coordinate Representations

(over $\mathbb{R}$)

Every real vector space X with dimension n is isomorphic to $\mathbf{R}^n$. What's the isomorphism?

Let X be a finite-dimensional vector space over $\mathbf{R}$ with $\dim X = n$. Fix any Hamel basis $V = \{v_1, \ldots, v_n\}$ of X. Any $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} \beta_j v_j$$

(here, we allow $\beta_j = 0$).

<u>Define</u>:

$$crd_V(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbf{R}^n$$

$crd_V : X \to \mathbb{R}^n$

"coordinate representation of x with respect to V"

$V = \{v_1, \ldots, v_n\}$ basis for X

$crd_V(x)$ is the vector of coordinates of x with respect to the basis V.

Notice:

$$crd_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad crd_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \ldots \quad crd_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

crd_V is an isomorphism from X to $\mathbf{R}^n$

Matrix Representations of Linear Transformations

Suppose $T \in L(X, Y)$, $\dim X = n$, $\dim Y = m$. Fix bases

$$\begin{aligned} V &= \{v_1, \ldots, v_n\} \text{ of } X \\ W &= \{w_1, \ldots, w_m\} \text{ of } Y \end{aligned}$$

$T(v_j) \in Y$, so

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$$

for some α_{ij}, $i = 1, \ldots, m$
(again allow $\alpha_{ij} = 0$)

Define

$$Mtx_{W,V}(T) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

"matrix representation of T with respect to V and W"

coordinates of $T(v_1)$ w.r.t. W .... coordinates of $T(v_n)$ w.r.t. W

Matrix Representations of Linear Transformations

Notice that the columns are the coordinates (expressed with respect to W) of $T(v_1), \ldots, T(v_n)$.

Observe

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \underset{crd_V(v_1)}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} = \underset{crd_W(T(v_1))}{\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{pmatrix}}$$

so

$$\begin{aligned} Mtx_{W,V}(T) \cdot crd_V(v_j) &= crd_W(T(v_j)) \quad \forall j \\ \Rightarrow \quad Mtx_{W,V}(T) \cdot crd_V(x) &= crd_W(T(x)) \ \ \forall x \in X \end{aligned}$$

Matrix Representations

Multiplying a vector by a matrix does two things:

- Computes the action of T
- Accounts for the change in basis

basis for X basis for Y

Example: $X = Y = \mathbf{R}^2$, $V = \{(1,0),(0,1)\}$, $W = \{(1,1),(-1,1)\}$, $T = id$, that is, $T(x) = x$ for each x.

$$Mtx_{W,V}(T) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$Mtx_{W,V}(T)$ is the matrix that *changes basis* from V to W.

How do we compute it? Look for coord. representations of $T(v_1) = v_1$ and $T(v_2) = v_2$

$$
\begin{aligned}
T(v_1) = v_1 = (1,0) &= \alpha_{11}(1,1) + \alpha_{21}(-1,1) \\
\alpha_{11} - \alpha_{21} &= 1 \\
\alpha_{11} + \alpha_{21} &= 0 \\
2\alpha_{11} &= 1, \alpha_{11} = \frac{1}{2} \\
\alpha_{21} &= -\frac{1}{2} \\
T(v_2) = v_2 = (0,1) &= \alpha_{12}(1,1) + \alpha_{22}(-1,1) \\
\alpha_{12} - \alpha_{22} &= 0 \\
\alpha_{12} + \alpha_{22} &= 1 \\
2\alpha_{12} &= 1, \alpha_{12} = \frac{1}{2} \\
\alpha_{22} &= \frac{1}{2}
\end{aligned}
$$

$$\Rightarrow \operatorname{crd}_W(v_1) = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = T(v_1)$$

$$\Rightarrow \operatorname{crd}_W(v_2) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = T(v_2)$$

$T : X \to Y, \quad T = id$

So

$$Mtx_{W,V}(\overset{T}{id}) = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$crd_W(T(v_1)) = crd_W(v_1) \qquad crd_W(T(v_2)) = crd_W(v_2)$

Matrix Representations

Theorem 3 (Thm. 3.5'). *Let X and Y be vector spaces over the same field F, with* $\dim X = n$, $\dim Y = m$. *Then $L(X,Y)$, the space of linear transformations from X to Y, is isomorphic to $F_{m\times n}$, the vector space of $m \times n$ matrices over F. If $V = \{v_1, \ldots, v_n\}$ is a basis for X and $W = \{w_1, \ldots, w_m\}$ is a basis for Y, then*

$$Mtx_{W,V} \in L(L(X,Y), F_{m\times n})$$

and $Mtx_{W,V}$ is an isomorphism from $L(X,Y)$ to $F_{m\times n}$.

Matrix Representations

Theorem 4 (From Handout)**.** *Let* X, Y, Z *be finite-dimensional vector spaces over the same field* F *with bases* U, V, W *respectively. Let* $S \in L(X, Y)$ *and* $T \in L(Y, Z)$*. Then*

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations.

Proof. See handout. $\square$

Note that $Mtx_{W,V}$ is a function from $L(X, Y)$ to the space $F_{m \times n}$ of $m \times n$ matrices, while $Mtx_{W,V}(T)$ is an $m \times n$ matrix.

Matrix Representations

The theorem can be summarized by the following "Commutative Diagram:"

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, leftrightarrow, "crd_U"'] & Y \arrow[r, "T"] \arrow[d, leftrightarrow, "crd_V"] & Z \arrow[d, leftrightarrow, "crd_W"] \\ \mathbf{R}^n \arrow[r, "Mtx_{V,U}(S)"'] & \mathbf{R}^m \arrow[r, "Mtx_{W,V}(T)"'] & \mathbf{R}^r \end{tikzcd}$$

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The crd arrows go in both directions because crd is an isomorphism.

Change of Basis

Let X be a finite-dimensional vector space with basis V. If $T \in L(X, X)$ it is customary to use the same basis in the domain and range. In this case, $Mtx_V(T)$ denotes $Mtx_{V,V}(T)$.

Question: If W is another basis for X, how are $Mtx_V(T)$ and $Mtx_W(T)$ related?

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[rrr, bend left, "T"] & X \arrow[r, "T"] & X \arrow[r, "id"] & X \\
\text{basis } V \arrow[u] & \text{basis } W \arrow[u] & \text{basis } W \arrow[u] & \text{basis } V \arrow[u]
\end{tikzcd}$$

$$
\begin{aligned}
Mtx_{V,W}(id) \cdot Mtx_W(T) \cdot Mtx_{W,V}(id) &= Mtx_{V,W}(id) \cdot Mtx_{W,V}(T \circ id) \\
&= Mtx_{V,V}(id \circ T \circ id) \\
&= Mtx_V(T)
\end{aligned}
$$

and

$$
\begin{aligned}
Mtx_{V,W}(id) \cdot Mtx_{W,V}(id) &= Mtx_{V,V}(id) \\
&= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
\end{aligned}
$$

$crd_V(v_1)$ - - - - - $crd_V(v_n)$

So this says that

$$Mtx_V(T) = P^{-1} Mtx_W(T) P$$

for the invertible matrix

$$P = Mtx_{W,V}(id)$$

that is the change of basis matrix.

On the other hand, if P is any invertible matrix, then P is also a change of basis matrix for appropriate corresponding bases (see handout).

Similarity

(nxn)

Definition 1. *Square matrices A and B are* similar *if*

$$A = P^{-1}BP$$

for some invertible matrix P.

Similarity

Theorem 5. *Suppose that X is a finite-dimensional vector space.*

1. *If $T \in L(X, X)$ then any two matrix representations of T are similar. That is, if U, W are any two bases of X, then $Mtx_W(T)$ and $Mtx_U(T)$ are similar.*

2. *Conversely, two similar matrices represent the same linear transformation T, relative to suitable bases. That is, given similar matrices A, B with $A = P^{-1}BP$ and any basis U, there is a basis W and $T \in L(X, X)$ such that*

$$\begin{aligned} B &= Mtx_U(T) \\ A &= Mtx_W(T) \\ P &= Mtx_{U,W}(id) \\ P^{-1} &= Mtx_{W,U}(id) \end{aligned}$$

Proof. See Handout on Diagonalization and Quadratic Forms. $\square$

Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that λ is an eigenvalue of T if and only if λ is an eigenvalue for some matrix representation of T if and only if λ is an eigenvalue for every matrix representation of T.

Definition 2. *Let X be a vector space and $T \in L(X,X)$. We say that λ is an* eigenvalue *of T and $v \neq 0$ is an* eigenvector corresponding to *λ if $T(v) = \lambda v$.*

Eigenvalues and Eigenvectors

Theorem 6 (Theorem 4 in Handout)**.** *Let X be a finite-dimensional vector space, and U a basis. Then λ is an eigenvalue of T if and only if λ is an eigenvalue of $Mtx_U(T)$. v is an eigenvector of T corresponding to λ if and only if $crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corresponding to λ.*

Proof. By the Commutative Diagram Theorem,

$$\begin{aligned} T(v) = \lambda v \;&\Leftrightarrow\; crd_U(T(v)) = crd_U(\lambda v) = \lambda\, crd_u(v) \\ &\Leftrightarrow\; Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v)) \end{aligned}$$

$\square$

$A = mtx_u(T)$, $\quad x = crd_u(v)$,

$\Leftrightarrow \quad Ax = \lambda x$

Computing Eigenvalues and Eigenvectors

Suppose $\dim X = n$; let I be the $n \times n$ identity matrix. Given $T \in L(X, X)$, fix a basis U and let

$$A = Mtx_U(T)$$

Find the eigenvalues of T by computing the eigenvalues of A:

$$\begin{aligned} \exists\, v \neq 0 \text{ s.t. } Av = \lambda v \quad &\iff \quad (A - \lambda I)v = 0 \quad \text{for some } v \neq 0 \\ &\iff \quad (A - \lambda I) \text{ is not invertible} \\ &\iff \quad \det(A - \lambda I) = 0 \end{aligned}$$

We have the following facts:

- If $A \in \mathbf{R}_{n\times n}$,

$$f(\lambda) = \det(A - \lambda I)$$

  is an n^{th} degree polynomial in λ with real coefficients; it is called the *characteristic polynomial* of A.

- f has n roots in $\mathbf{C}$, counting multiplicity:

$$f(\lambda) = (c_1 - \lambda)(c_2 - \lambda) \cdots (c_n - \lambda)$$

  (may have $c_i = c_j$, $i \neq j$)

  where $c_1, \ldots, c_n \in \mathbf{C}$ are the eigenvalues; the c_j's are not necessarily distinct. Notice that $f(\lambda) = 0$ if and only if $\lambda \in \{c_1, \ldots, c_n\}$, so the roots are the solutions of the equation $f(\lambda) = 0$.

- the roots that are not real come in conjugate pairs:

$$f(a+bi) = 0 \Leftrightarrow f(a-bi) = 0$$

- if $\lambda = c_j \in \mathbf{R}$, there is a corresponding eigenvector in $\mathbf{R}^n$.

- if $\lambda = c_j \notin \mathbf{R}$, the corresponding eigenvectors are in $\mathbf{C}^n \setminus \mathbf{R}^n$.

Diagonalization

Definition 3. *Suppose X is a finite-dimensional vector space with basis U. Given a linear transformation $T \in L(X, X)$, let*

$$A = Mtx_U(T)$$

We say that A can be diagonalized *if there is a basis W for X such that $Mtx_W(T)$ is a diagonal matrix, that is,*

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

So

$$\begin{aligned} A \text{ can be diagonalized} \quad & \Longleftrightarrow \quad A \text{ is similar to a diagonal matrix} \\ & \Longleftrightarrow \quad A = P^{-1}BP \text{ where } B \text{ is diagonal} \end{aligned}$$

$$\parallel$$

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \lambda_n \\ 0 & \cdots & \cdots & \end{pmatrix}$$

Suppose there is a basis W such that

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$\Rightarrow$ $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $mtx_W(T)$ and T

Then the standard basis vectors of $\mathbf{R}^n$ are eigenvectors of $Mtx_W(T)$.

In general:

z_j is an eigenvector of T corresponding to $\lambda_j \iff crd_W(z_j)$ is an eigenvector of $Mtx_W(T)$ corresponding to λ_j.

So an eigenvector of T corresponding to λ_j is w_j, since $crd_W(w_j) = e_j$, the j^{th} standard basis vector in $\mathbf{R}^n$.

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \cdots & & \lambda_n \end{pmatrix}$$

Thus $Mtx_W(T)$ is diagonal if and only if $W = \{w_1, \ldots, w_n\}$ where w_j is an eigenvector of T corresponding to λ_j for each j.

Then the action of T is clear: it stretches each basis element w_i by the factor λ_i.

Diagonalization

Theorem 7 (Thm. 6.7'). *Let X be an n-dimensional vector space, $T \in L(X, X)$, U any basis of X, and $A = Mtx_U(T)$. Then the following are equivalent:*

1. *A can be diagonalized*

2. *there is a basis W for X consisting of eigenvectors of T*

3. *there is a basis V for $\mathbf{R}^n$ consisting of eigenvectors of A*

Proof. Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout. $\square$

Diagonalization

Theorem 8 (Thm. 6.8'). *Let X be a vector space and $T \in L(X, X)$.*

1. *If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors $v_1, \ldots, v_m$, then $\{v_1, \ldots, v_m\}$ is linearly independent.*

2. *If $\dim X = n$ and T has n distinct eigenvalues, then X has a basis consisting of eigenvectors of T; consequently, if U is any basis of X, then $Mtx_U(T)$ is diagonalizable.*

Proof. This is an adaptation of the proof of Theorem 6.8 in de la Fuente. $\square$