

Econ 204 2022

Lecture 9

Outline

1. Quotient Vector Spaces
2. Matrix Representations of Linear Transformations
3. Change of Basis and Similarity
4. Eigenvalues and Eigenvectors
5. Diagonalization

Quotient Vector Spaces

Given a vector space X and a vector subspace W of X , define an equivalence relation by

$$x \sim y \iff x - y \in W$$

Form a new vector space X/W : the set of vectors is

$$\{[x] : x \in X\}$$

where $[x]$ denotes the equivalence class of x with respect to \sim .

X/W is read “ $X \bmod W$ ”.

Note that the vectors in X/W are **sets** of vectors in X : for $x \in X$,

$$[x] = \{x + w : w \in W\}$$

Quotient Vector Spaces

We claim that X/W can be viewed as a vector space over F . Define the vector space operations $+$, \cdot in X/W as follows:

Define

$$\begin{aligned}[x] + [y] &= [x + y] \\ \alpha[x] &= [\alpha x]\end{aligned}$$

Exercise: Verify that \sim is an equivalence relation and that vector addition and scalar multiplication are well-defined.

Then X/W is a vector space over F with these definitions for $+$ and \cdot .

Quotient Vector Spaces

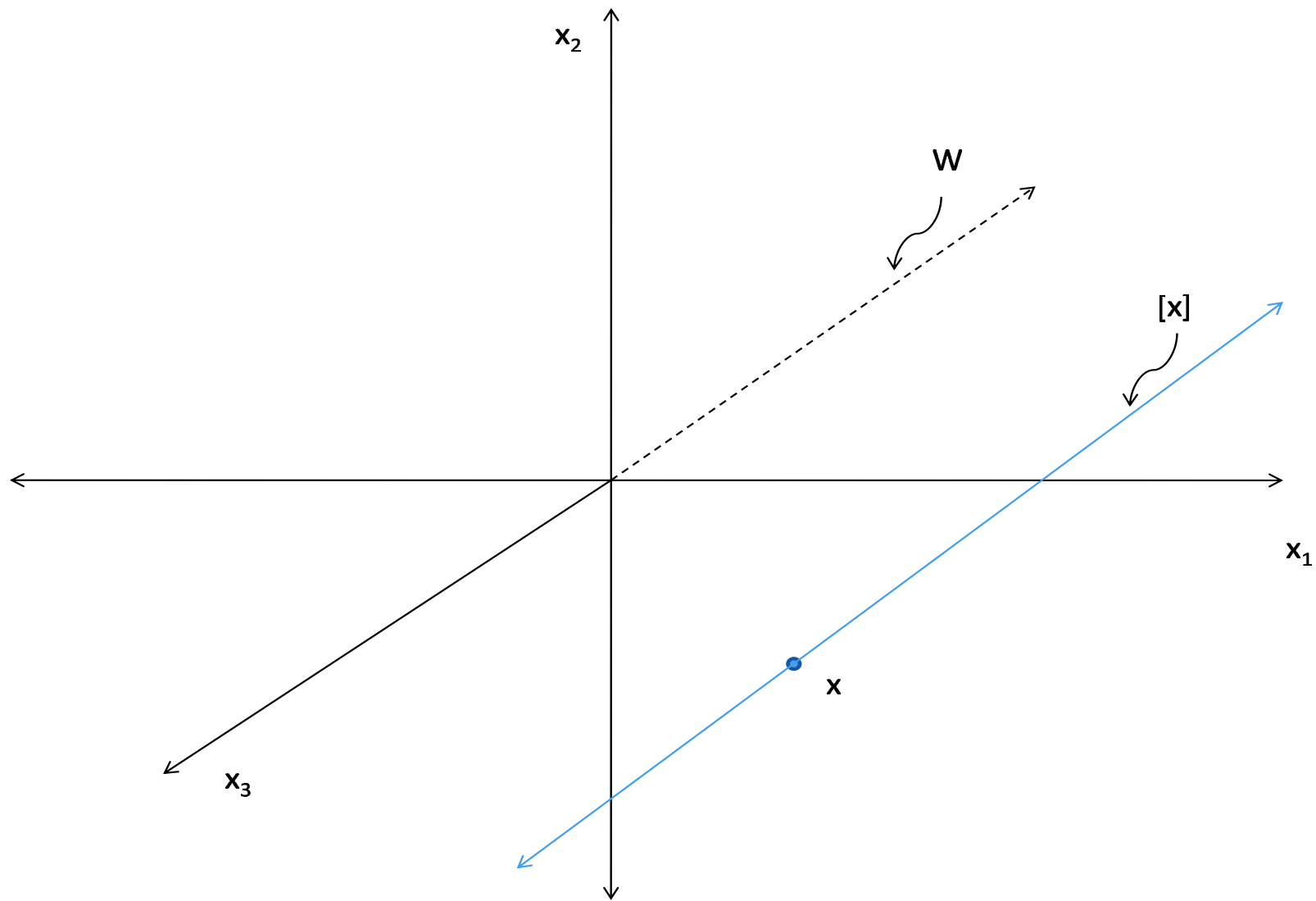
Example: Let $X = \mathbf{R}^3$ and let $W = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$.
Then for $x, y \in \mathbf{R}^3$,

$$\begin{aligned}x \sim y &\iff x - y \in W \\&\iff x_1 - y_1 = 0, x_2 - y_2 = 0 \\&\iff x_1 = y_1, x_2 = y_2\end{aligned}$$

and

$$[x] = \{x + w : w \in W\} = \{(x_1, x_2, z) : z \in \mathbf{R}\}$$

So the equivalence class corresponding to x is the line in \mathbf{R}^3 through x parallel to the axis of the third coordinate.



Example, cont.

What is X/W ? Intuitively this equivalence relation ignores the third coordinate, and we can identify the equivalence class $[x]$ with the vector $(x_1, x_2) \in \mathbf{R}^2$.

The next two results show how to formalize this connection.

Quotient Vector Spaces

Theorem 1. *If X is a vector space with $\dim X = n$ for some $n \in \mathbb{N}$ and W is a vector subspace of X , then*

$$\dim(X/W) = \dim X - \dim W$$

Proof. (Sketch) Begin with a basis $\{w_1, \dots, w_c\}$ for W , and a basis $\{[x_1], \dots, [x_k]\}$ for X/W . Show that

$$\{w_1, \dots, w_c\} \cup \{x_1, \dots, x_k\}$$

is a basis for X .



Quotient Vector Spaces

Theorem 2. *Let X and Y be vector spaces over the same field F and $T \in L(X, Y)$. Then $\text{Im } T$ is isomorphic to $X/\ker T$.*

Proof. Notice that if X is finite-dimensional, then

$$\begin{aligned}\dim(X/\ker T) &= \dim X - \dim \ker T && \text{(by the previous theorem)} \\ &= \text{Rank } T && \text{(by the Rank-Nullity Theorem)} \\ &= \dim \text{Im } T\end{aligned}$$

so $X/\ker T$ is isomorphic to $\text{Im } T$. (why??)

We prove that this is true in general, and that the isomorphism is natural.

Define $\tilde{T} : X / \ker T \rightarrow \operatorname{Im} T$ by

$$\tilde{T}([x]) = T(x)$$

We first need to check that this is well-defined, that is, that if $[x] = [x']$ then $\tilde{T}([x]) = \tilde{T}([x'])$.

$$\begin{aligned} [x] = [x'] &\Rightarrow x \sim x' \\ &\Rightarrow x - x' \in \ker T \\ &\Rightarrow T(x - x') = 0 \\ &\Rightarrow T(x) = T(x') \end{aligned}$$

so \tilde{T} is well-defined.

Clearly, $\tilde{T} : X / \ker T \rightarrow \operatorname{Im} T$. It is easy to check that \tilde{T} is linear,

so $\tilde{T} \in L(X/\ker T, \operatorname{Im} T)$. Next we show that \tilde{T} is an isomorphism.

$$\begin{aligned}\tilde{T}([x]) = \tilde{T}([y]) &\Rightarrow T(x) = T(y) \\ &\Rightarrow T(x - y) = 0 \\ &\Rightarrow x - y \in \ker T \\ &\Rightarrow x \sim y \\ &\Rightarrow [x] = [y]\end{aligned}$$

so \tilde{T} is one-to-one.

$$\begin{aligned}y \in \operatorname{Im} T &\Rightarrow \exists x \in X \text{ s.t. } T(x) = y \\ &\Rightarrow \tilde{T}([x]) = y\end{aligned}$$

so \tilde{T} is onto, hence \tilde{T} is an isomorphism. □

Example: Consider $T \in L(\mathbf{R}^3, \mathbf{R}^2)$ defined by

$$T(x_1, x_2, x_3) = (x_1, x_2)$$

Then

$$\ker T = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$$

is the x_3 -axis.

Given x , the equivalence class $[x]$ is just the line through x parallel to the x_3 -axis.

$$\tilde{T}([x]) = T(x_1, x_2, x_3) = (x_1, x_2)$$

and

$$\operatorname{Im} T = \mathbf{R}^2, \quad X / \ker T \cong \mathbf{R}^2 = \operatorname{Im} T$$

as we suggested intuitively above (here the symbol \cong denotes isomorphism, that is, we write $Y \cong Z$ if Y and Z are isomorphic.)

Coordinate Representations

Every real vector space X with dimension n is isomorphic to \mathbf{R}^n . What's the isomorphism?

Let X be a finite-dimensional vector space over \mathbf{R} with $\dim X = n$. Fix any Hamel basis $V = \{v_1, \dots, v_n\}$ of X . Any $x \in X$ has a unique representation

$$x = \sum_{j=1}^n \beta_j v_j$$

(here, we allow $\beta_j = 0$).

$$\text{crd}_V(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbf{R}^n$$

$crd_V(x)$ is the vector of coordinates of x with respect to the basis V .

$$crd_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad crd_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad crd_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

crd_V is an isomorphism from X to \mathbf{R}^n

Matrix Representations of Linear Transformations

Suppose $T \in L(X, Y)$, $\dim X = n$, $\dim Y = m$. Fix bases

$$\begin{aligned} V &= \{v_1, \dots, v_n\} \text{ of } X \\ W &= \{w_1, \dots, w_m\} \text{ of } Y \end{aligned}$$

$T(v_j) \in Y$, so

$$T(v_j) = \sum_{i=1}^m \alpha_{ij} w_i$$

Define

$$Mtx_{W,V}(T) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

Matrix Representations of Linear Transformations

Notice that the columns are the coordinates (expressed with respect to W) of $T(v_1), \dots, T(v_n)$.

Observe

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{pmatrix}$$

so

$$\begin{aligned} Mtx_{W,V}(T) \cdot crd_V(v_j) &= crd_W(T(v_j)) \\ Mtx_{W,V}(T) \cdot crd_V(x) &= crd_W(T(x)) \quad \forall x \in X \end{aligned}$$

Matrix Representations

Multiplying a vector by a matrix does two things:

- Computes the action of T
- Accounts for the change in basis

Example: $X = Y = \mathbf{R}^2$, $V = \{(1, 0), (0, 1)\}$, $W = \{(1, 1), (-1, 1)\}$, $T = id$, that is, $T(x) = x$ for each x .

$$Mtx_{W,V}(T) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$Mtx_{W,V}(T)$ is the matrix that *changes basis* from V to W .

How do we compute it?

$$v_1 = (1, 0) = \alpha_{11}(1, 1) + \alpha_{21}(-1, 1)$$

$$\alpha_{11} - \alpha_{21} = 1$$

$$\alpha_{11} + \alpha_{21} = 0$$

$$2\alpha_{11} = 1, \alpha_{11} = \frac{1}{2}$$

$$\alpha_{21} = -\frac{1}{2}$$

$$v_2 = (0, 1) = \alpha_{12}(1, 1) + \alpha_{22}(-1, 1)$$

$$\alpha_{12} - \alpha_{22} = 0$$

$$\alpha_{12} + \alpha_{22} = 1$$

$$2\alpha_{12} = 1, \alpha_{12} = \frac{1}{2}$$

$$\alpha_{22} = \frac{1}{2}$$

So

$$Mtx_{W,V}(id) = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Matrix Representations

Theorem 3 (Thm. 3.5'). *Let X and Y be vector spaces over the same field F , with $\dim X = n$, $\dim Y = m$. Then $L(X, Y)$, the space of linear transformations from X to Y , is isomorphic to $F_{m \times n}$, the vector space of $m \times n$ matrices over F . If $V = \{v_1, \dots, v_n\}$ is a basis for X and $W = \{w_1, \dots, w_m\}$ is a basis for Y , then*

$$Mtx_{W,V} \in L(L(X, Y), F_{m \times n})$$

and $Mtx_{W,V}$ is an isomorphism from $L(X, Y)$ to $F_{m \times n}$.

Matrix Representations

Theorem 4 (From Handout). *Let X, Y, Z be finite-dimensional vector spaces over the same field F with bases U, V, W respectively. Let $S \in L(X, Y)$ and $T \in L(Y, Z)$. Then*

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations.

Proof. See handout. □

Note that $Mtx_{W,V}$ is a function from $L(X, Y)$ to the space $F_{m \times n}$ of $m \times n$ matrices, while $Mtx_{W,V}(T)$ is an $m \times n$ matrix.

Matrix Representations

The theorem can be summarized by the following “Commutative Diagram:”

$$\begin{array}{ccccc}
 & & S & & T \\
 & X & \rightarrow & Y & \rightarrow & Z \\
 \text{\textit{crd}}_U \updownarrow & & & \updownarrow \text{\textit{crd}}_V & & \updownarrow \text{\textit{crd}}_W \\
 & \mathbf{R}^n & \rightarrow & \mathbf{R}^m & \rightarrow & \mathbf{R}^r \\
 & & \text{\textit{Mtx}}_{V,U}(S) & & \text{\textit{Mtx}}_{W,V}(T) &
 \end{array}$$

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The *crd* arrows go in both directions because *crd* is an isomorphism.

Change of Basis

Let X be a finite-dimensional vector space with basis V . If $T \in L(X, X)$ it is customary to use the same basis in the domain and range. In this case, $Mtx_V(T)$ denotes $Mtx_{V,V}(T)$.

Question: If W is another basis for X , how are $Mtx_V(T)$ and $Mtx_W(T)$ related?

$$\begin{aligned}
Mtx_{V,W}(id) \cdot Mtx_W(T) \cdot Mtx_{W,V}(id) &= Mtx_{V,W}(id) \cdot Mtx_{W,V}(T \circ id) \\
&= Mtx_{V,V}(id \circ T \circ id) \\
&= Mtx_V(T)
\end{aligned}$$

and

$$\begin{aligned}
Mtx_{V,W}(id) \cdot Mtx_{W,V}(id) &= Mtx_{V,V}(id) \\
&= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
\end{aligned}$$

So this says that

$$Mtx_V(T) = P^{-1}Mtx_W(T)P$$

for the invertible matrix

$$P = Mtx_{W,V}(id)$$

that is the change of basis matrix.

On the other hand, if P is any invertible matrix, then P is also a change of basis matrix for appropriate corresponding bases (see handout).

Similarity

Definition 1. *Square matrices A and B are similar if*

$$A = P^{-1}BP$$

for some invertible matrix P .

Similarity

Theorem 5. *Suppose that X is a finite-dimensional vector space.*

- 1. If $T \in L(X, X)$ then any two matrix representations of T are similar. That is, if U, W are any two bases of X , then $Mtx_W(T)$ and $Mtx_U(T)$ are similar.*
- 2. Conversely, two similar matrices represent the same linear transformation T , relative to suitable bases. That is, given similar matrices A, B with $A = P^{-1}BP$ and any basis U , there is a basis W and $T \in L(X, X)$ such that*

$$\begin{aligned} B &= Mtx_U(T) \\ A &= Mtx_W(T) \\ P &= Mtx_{U,W}(id) \\ P^{-1} &= Mtx_{W,U}(id) \end{aligned}$$

Proof. See Handout on Diagonalization and Quadratic Forms.



Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that λ is an eigenvalue of T if and only if λ is an eigenvalue for some matrix representation of T if and only if λ is an eigenvalue for every matrix representation of T .

Definition 2. *Let X be a vector space and $T \in L(X, X)$. We say that λ is an eigenvalue of T and $v \neq 0$ is an eigenvector corresponding to λ if $T(v) = \lambda v$.*

Eigenvalues and Eigenvectors

Theorem 6 (Theorem 4 in Handout). *Let X be a finite-dimensional vector space, and U a basis. Then λ is an eigenvalue of T if and only if λ is an eigenvalue of $Mtx_U(T)$. v is an eigenvector of T corresponding to λ if and only if $crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corresponding to λ .*

Proof. By the Commutative Diagram Theorem,

$$\begin{aligned} T(v) = \lambda v &\Leftrightarrow crd_U(T(v)) = crd_U(\lambda v) \\ &\Leftrightarrow Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v)) \end{aligned}$$



Computing Eigenvalues and Eigenvectors

Suppose $\dim X = n$; let I be the $n \times n$ identity matrix. Given $T \in L(X, X)$, fix a basis U and let

$$A = Mtx_U(T)$$

Find the eigenvalues of T by computing the eigenvalues of A :

$$\begin{aligned} Av = \lambda v &\iff (A - \lambda I)v = 0 \\ &\iff (A - \lambda I) \text{ is not invertible} \\ &\iff \det(A - \lambda I) = 0 \end{aligned}$$

We have the following facts:

- If $A \in \mathbf{R}_{n \times n}$,

$$f(\lambda) = \det(A - \lambda I)$$

is an n^{th} degree polynomial in λ with real coefficients; it is called the *characteristic polynomial* of A .

- f has n roots in \mathbf{C} , counting multiplicity:

$$f(\lambda) = (c_1 - \lambda)(c_2 - \lambda) \cdots (c_n - \lambda)$$

where $c_1, \dots, c_n \in \mathbf{C}$ are the eigenvalues; the c_j 's are not necessarily distinct. Notice that $f(\lambda) = 0$ if and only if $\lambda \in \{c_1, \dots, c_n\}$, so the roots are the solutions of the equation $f(\lambda) = 0$.

- the roots that are not real come in conjugate pairs:

$$f(a + bi) = 0 \Leftrightarrow f(a - bi) = 0$$

- if $\lambda = c_j \in \mathbf{R}$, there is a corresponding eigenvector in \mathbf{R}^n .
- if $\lambda = c_j \notin \mathbf{R}$, the corresponding eigenvectors are in $\mathbf{C}^n \setminus \mathbf{R}^n$.

Diagonalization

Definition 3. Suppose X is a finite-dimensional vector space with basis U . Given a linear transformation $T \in L(X, X)$, let

$$A = Mtx_U(T)$$

We say that A can be diagonalized if there is a basis W for X such that $Mtx_W(T)$ is a diagonal matrix, that is,

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

So

A can be diagonalized $\iff A$ is similar to a diagonal matrix
 $\iff A = P^{-1}BP$ where B is diagonal

Suppose there is a basis W such that

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Then the standard basis vectors of \mathbf{R}^n are eigenvectors of $Mtx_W(T)$.

z_j is an eigenvector of T corresponding to $\lambda_j \iff crd_W(z_j)$ is an eigenvector of $Mtx_W(T)$ corresponding to λ_j .

So an eigenvector corresponding to λ_j is w_j , since $crd_W(w_j) = e_j$, the j^{th} standard basis vector in \mathbf{R}^n .

Thus $Mtx_W(T)$ is diagonal if and only if $W = \{w_1, \dots, w_n\}$ where w_j is an eigenvector of T corresponding to λ_j for each j .

Then the action of T is clear: it stretches each basis element w_i by the factor λ_i .

Diagonalization

Theorem 7 (Thm. 6.7'). *Let X be an n -dimensional vector space, $T \in L(X, X)$, U any basis of X , and $A = Mtx_U(T)$. Then the following are equivalent:*

- 1. A can be diagonalized*
- 2. there is a basis W for X consisting of eigenvectors of T*
- 3. there is a basis V for \mathbf{R}^n consisting of eigenvectors of A*

Proof. Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout. □

Diagonalization

Theorem 8 (Thm. 6.8'). *Let X be a vector space and $T \in L(X, X)$.*

- 1. If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors v_1, \dots, v_m , then $\{v_1, \dots, v_m\}$ is linearly independent.*
- 2. If $\dim X = n$ and T has n distinct eigenvalues, then X has a basis consisting of eigenvectors of T ; consequently, if U is any basis of X , then $Mtx_U(T)$ is diagonalizable.*

Proof. This is an adaptation of the proof of Theorem 6.8 in de la Fuente. □