

**Economics 204 Summer/Fall 2022**  
**Lecture 11—Monday August 8, 2022**

**Sections 4.1-4.3 (Unified)**

**Definition 1** Let  $f : I \rightarrow \mathbf{R}$ , where  $I \subseteq \mathbf{R}$  is an open interval.  $f$  is *differentiable* at  $x \in I$  if

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = a$$

for some  $a \in \mathbf{R}$ .

This is equivalent to  $\exists a \in \mathbf{R}$  such that:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(x+h) - (f(x) + ah)}{h} = 0 \\ \Leftrightarrow & \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(x+h) - (f(x) + ah)}{h} \right| < \varepsilon \\ \Leftrightarrow & \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \frac{|f(x+h) - (f(x) + ah)|}{|h|} < \varepsilon \\ \Leftrightarrow & \lim_{h \rightarrow 0} \frac{|f(x+h) - (f(x) + ah)|}{|h|} = 0 \end{aligned}$$

Recall that the limit considers  $h$  near zero, but not  $h = 0$ .

**Definition 2** If  $X \subseteq \mathbf{R}^n$  is open,  $f : X \rightarrow \mathbf{R}^m$  is *differentiable* at  $x \in X$  if<sup>1</sup>

$$\exists T_x \in L(\mathbf{R}^n, \mathbf{R}^m) \text{ s.t. } \lim_{h \rightarrow 0, h \in \mathbf{R}^n} \frac{|f(x+h) - (f(x) + T_x(h))|}{|h|} = 0 \quad (1)$$

$f$  is *differentiable* if it is differentiable at all  $x \in X$ .

Note that  $T_x$  is uniquely determined by Equation (1).  $h$  is a small, nonzero element of  $\mathbf{R}^n$ ;  $h \rightarrow 0$  from any direction, from above, below, along a spiral, etc. The definition requires that one linear operator  $T_x$  works no matter how  $h$  approaches zero. In this case,  $f(x) + T_x(h)$  is the best linear approximation to  $f(x+h)$  for small  $h$ .

**Notation:**

- $y = O(|h|^n)$  as  $h \rightarrow 0$  – read “ $y$  is big-Oh of  $|h|^n$ ” – means

$$\exists K, \delta > 0 \text{ s.t. } |h| < \delta \Rightarrow |y| \leq K|h|^n$$

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<sup>1</sup>Recall  $|\cdot|$  denotes the Euclidean distance.

- $y = o(|h|^n)$  as  $h \rightarrow 0$  – read “ $y$  is little-oh of  $|h|^n$ ” – means

$$\lim_{h \rightarrow 0} \frac{|y|}{|h|^n} = 0$$

Note that the statement  $y = O(|h|^{n+1})$  as  $h \rightarrow 0$  implies  $y = o(|h|^n)$  as  $h \rightarrow 0$ .

Also note that if  $y$  is either  $O(|h|^n)$  or  $o(|h|^n)$ , then  $y \rightarrow 0$  as  $h \rightarrow 0$ ; the difference in whether  $y$  is “big-Oh” or “little-oh” tells us something about the *rate* at which  $y \rightarrow 0$ .

Using this notation, note that  $f$  is differentiable at  $x \Leftrightarrow \exists T_x \in L(\mathbf{R}^n, \mathbf{R}^m)$  such that

$$f(x+h) = f(x) + T_x(h) + o(h) \text{ as } h \rightarrow 0$$

### Notation:

- $df_x$  is the linear transformation  $T_x$
- $Df(x)$  is the matrix of  $df_x$  with respect to the standard basis.  
This is called the *Jacobian* or *Jacobian matrix* of  $f$  at  $x$
- $E_f(h) = f(x+h) - (f(x) + df_x(h))$  is the *error term*

Using this notation,

$$f \text{ is differentiable at } x \Leftrightarrow E_f(h) = o(h) \text{ as } h \rightarrow 0$$

Now compute  $Df(x) = (a_{ij})$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbf{R}^n$ . Look in direction  $e_j$  (note that  $|\gamma e_j| = |\gamma|$ ).

$$\begin{aligned} o(\gamma) &= f(x + \gamma e_j) - (f(x) + T_x(\gamma e_j)) \\ &= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \\ &= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} \gamma a_{1j} \\ \vdots \\ \gamma a_{mj} \end{pmatrix} \right) \end{aligned}$$

For  $i = 1, \dots, m$ , let  $f^i$  denote the  $i^{th}$  component of the function  $f$ :

$$\begin{aligned} f^i(x + \gamma e_j) - (f^i(x) + \gamma a_{ij}) &= o(\gamma) \\ \text{so } a_{ij} &= \frac{\partial f^i}{\partial x_j}(x) \end{aligned}$$

**Theorem 3 (Thm. 3.3)** Suppose  $X \subseteq \mathbf{R}^n$  is open and  $f : X \rightarrow \mathbf{R}^m$  is differentiable at  $x \in X$ . Then  $\frac{\partial f^i}{\partial x_j}$  exists at  $x$  for  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and

$$Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_n}(x) \end{pmatrix}$$

i.e. the Jacobian at  $x$  is the matrix of partial derivatives at  $x$ .

**Remark:** If  $f$  is differentiable at  $x$ , then all first-order partial derivatives  $\frac{\partial f^i}{\partial x_j}$  exist at  $x$ . However, the converse is false: existence of all the first-order partial derivatives does not imply that  $f$  is differentiable. The missing piece is continuity of the partial derivatives:

**Theorem 4 (Thm. 3.4)** If all the first-order partial derivatives  $\frac{\partial f^i}{\partial x_j}$  ( $1 \leq i \leq m$ ,  $1 \leq j \leq n$ ) exist and are continuous at  $x$ , then  $f$  is differentiable at  $x$ .

### Directional Derivatives:

Suppose  $X \subseteq \mathbf{R}^n$  open,  $f : X \rightarrow \mathbf{R}^m$  is differentiable at  $x$ , and  $|u| = 1$ .

$$\begin{aligned} f(x + \gamma u) - (f(x) + T_x(\gamma u)) &= o(\gamma) \text{ as } \gamma \rightarrow 0 \\ \Rightarrow f(x + \gamma u) - (f(x) + \gamma T_x(u)) &= o(\gamma) \text{ as } \gamma \rightarrow 0 \\ \Rightarrow \lim_{\gamma \rightarrow 0} \frac{f(x + \gamma u) - f(x)}{\gamma} &= T_x(u) = Df(x)u \end{aligned}$$

i.e. the directional derivative in the direction  $u$  (with  $|u| = 1$ ) is

$$Df(x)u \in \mathbf{R}^m$$

**Theorem 5 (Thm. 3.5, Chain Rule)** Let  $X \subseteq \mathbf{R}^n$ ,  $Y \subseteq \mathbf{R}^m$  be open,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow \mathbf{R}^p$ . Let  $x_0 \in X$  and  $F = g \circ f$ . If  $f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $f(x_0)$ , then  $F = g \circ f$  is differentiable at  $x_0$  and

$$\begin{aligned} dF_{x_0} &= dg_{f(x_0)} \circ df_{x_0} \\ &\quad (\text{composition of linear transformations}) \\ DF(x_0) &= Dg(f(x_0))Df(x_0) \\ &\quad (\text{matrix multiplication}) \end{aligned}$$

**Remark:** The statement is exactly the same as in the univariate case, except we replace the univariate derivative by a linear transformation. The proof is more or less the same, with a bit of linear algebra added.

**Theorem 6 (Thm. 1.7, Mean Value Theorem, Univariate Case)** *Let  $a, b \in \mathbf{R}$ . Suppose  $f : [a, b] \rightarrow \mathbf{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

*that is, such that*

$$f(b) - f(a) = f'(c)(b - a)$$

**Proof:** Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then  $g(a) = 0 = g(b)$ . See Figure 1. Note that for  $x \in (a, b)$ ,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so it suffices to find  $c \in (a, b)$  such that  $g'(c) = 0$ .

Case I: If  $g(x) = 0$  for all  $x \in [a, b]$ , choose an arbitrary  $c \in (a, b)$ , and note that  $g'(c) = 0$ , so we are done.

Case II: Suppose  $g(x) > 0$  for some  $x \in [a, b]$ . Since  $g$  is continuous on  $[a, b]$ , it attains its maximum at some point  $c \in (a, b)$ . Since  $g$  is differentiable at  $c$  and  $c$  is an interior point of the domain of  $g$ , we have  $g'(c) = 0$ , and we are done.

Case III: If  $g(x) < 0$  for some  $x \in [a, b]$ , the argument is similar to that in Case II. ■

**Remark:** The Mean Value Theorem is useful for estimating bounds on functions and error terms in approximation of functions.

**Notation:**

$$\ell(x, y) = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$$

is the line segment from  $x$  to  $y$ .

**Theorem 7 (Mean Value Theorem)** *Suppose  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is differentiable on an open set  $X \subseteq \mathbf{R}^n$ ,  $x, y \in X$  and  $\ell(x, y) \subseteq X$ . Then there exists  $z \in \ell(x, y)$  such that*

$$f(y) - f(x) = Df(z)(y - x)$$

**Remark:** This statement is different from Theorem 3.7 in de la Fuente. Notice that the statement is exactly the same as in the univariate case. For  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ , we can apply the Mean Value Theorem to each component, to obtain  $z_1, \dots, z_m \in \ell(x, y)$  such that

$$f^i(y) - f^i(x) = Df^i(z_i)(y - x)$$

However, we cannot find a single  $z$  which works for every component. Note that each  $z_i \in \ell(x, y) \subset \mathbf{R}^n$ ; there are  $m$  of them, one for each component in the range.

The following result plays the same role in estimating function values and error terms for functions taking values in  $\mathbf{R}^m$  as the Mean Value Theorem plays for functions from  $\mathbf{R}$  to  $\mathbf{R}$ .

**Theorem 8** *Suppose  $X \subset \mathbf{R}^n$  is open and  $f : X \rightarrow \mathbf{R}^m$  is differentiable. If  $x, y \in X$  and  $\ell(x, y) \subseteq X$ , then there exists  $z \in \ell(x, y)$  such that*

$$\begin{aligned} |f(y) - f(x)| &\leq |df_z(y - x)| \\ &\leq \|df_z\| |y - x| \end{aligned}$$

**Remark:** To understand why we don't get equality, consider  $f : [0, 1] \rightarrow \mathbf{R}^2$  defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

$f$  maps  $[0, 1]$  to the unit circle in  $\mathbf{R}^2$ . Note that  $f(0) = f(1) = (1, 0)$ , so  $|f(1) - f(0)| = 0$ . However, for any  $z \in [0, 1]$ ,

$$\begin{aligned} |df_z(1 - 0)| &= |2\pi(-\sin 2\pi z, \cos 2\pi z)| \\ &= 2\pi \sqrt{\sin^2 2\pi z + \cos^2 2\pi z} \\ &= 2\pi \end{aligned}$$

## Section 4.4. Taylor's Theorem

**Theorem 9 (Thm. 1.9, Taylor's Theorem in  $\mathbf{R}^1$ )** *Let  $f : I \rightarrow \mathbf{R}$  be  $n$ -times differentiable, where  $I \subseteq \mathbf{R}$  is an open interval. If  $x, x + h \in I$ , then*

$$f(x + h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)h^k}{k!} + E_n$$

where  $f^{(k)}$  is the  $k^{th}$  derivative of  $f$  and

$$E_n = \frac{f^{(n)}(x + \lambda h)h^n}{n!} \text{ for some } \lambda \in (0, 1)$$

**Motivation:** Let

$$\begin{aligned}
T_n(h) &= f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} \\
&= f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \cdots + \frac{f^{(n)}(x)h^n}{n!} \\
T_n(0) &= f(x) \\
T'_n(h) &= f'(x) + f''(x)h + \cdots + \frac{f^{(n)}(x)h^{n-1}}{(n-1)!} \\
T'_n(0) &= f'(x) \\
T''_n(h) &= f''(x) + \cdots + \frac{f^{(n)}(x)h^{n-2}}{(n-2)!} \\
T''_n(0) &= f''(x) \\
&\vdots \\
T_n^{(n)}(0) &= f^{(n)}(x)
\end{aligned}$$

so  $T_n(h)$  is the unique  $n^{th}$  degree polynomial such that

$$\begin{aligned}
T_n(0) &= f(x) \\
T'_n(0) &= f'(x) \\
&\vdots \\
T_n^{(n)}(0) &= f^{(n)}(x)
\end{aligned}$$

The proof of the formula for the remainder  $E_n$  is essentially the Mean Value Theorem; the problem in applying it is that the point  $x + \lambda h$  is not known in advance.

**Theorem 10 (Alternate Taylor's Theorem in  $\mathbf{R}^1$ )** *Let  $f : I \rightarrow \mathbf{R}$  be  $n$  times differentiable, where  $I \subseteq \mathbf{R}$  is an open interval and  $x \in I$ . Then*

$$f(x+h) = f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} + o(h^n) \text{ as } h \rightarrow 0$$

*If  $f$  is  $(n+1)$  times continuously differentiable (i.e. all derivatives up to order  $n+1$  exist and are continuous), then*

$$f(x+h) = f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} + O(h^{n+1}) \text{ as } h \rightarrow 0$$

**Remark:** The first equation in the statement of the theorem is essentially a restatement of the definition of the  $n^{th}$  derivative. The second statement is proven from Theorem 1.9, and the continuity of the derivative, hence the boundedness of the derivative on a neighborhood of  $x$ .

**Definition 11** Let  $X \subseteq \mathbf{R}^n$  be open. A function  $f : X \rightarrow \mathbf{R}^m$  is *continuously differentiable* on  $X$  if

- $f$  is differentiable on  $X$  and
- $df_x$  is a continuous function of  $x$  from  $X$  to  $L(\mathbf{R}^n, \mathbf{R}^m)$ , with operator norm  $\|df_x\|$

$f$  is  $C^k$  if all partial derivatives of order less than or equal to  $k$  exist and are continuous in  $X$ .

**Theorem 12 (Thm. 4.3)** Suppose  $X \subseteq \mathbf{R}^n$  is open and  $f : X \rightarrow \mathbf{R}^m$ . Then  $f$  is continuously differentiable on  $X$  if and only if  $f$  is  $C^1$ .

**Remark:** The notation in Taylor's Theorem is difficult. If  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ , the quadratic terms are not hard for  $m = 1$ ; for  $m > 1$ , we handle each component separately. For cubic and higher order terms, the notation is a mess.

#### Linear Terms:

**Theorem 13** Suppose  $X \subseteq \mathbf{R}^n$  is open and  $x \in X$ . If  $f : X \rightarrow \mathbf{R}^m$  is differentiable, then

$$f(x+h) = f(x) + Df(x)h + o(h) \text{ as } h \rightarrow 0$$

The previous theorem is essentially a restatement of the definition of differentiability.

**Theorem 14 (Corollary of 4.4)** Suppose  $X \subseteq \mathbf{R}^n$  is open and  $x \in X$ . If  $f : X \rightarrow \mathbf{R}^m$  is  $C^2$ , then

$$f(x+h) = f(x) + Df(x)h + O(|h|^2) \text{ as } h \rightarrow 0$$

#### Quadratic Terms:

We treat each component of the function separately, so consider  $f : X \rightarrow \mathbf{R}$ ,  $X \subseteq \mathbf{R}^n$  an open set. Let

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

$$f \in C^2 \Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

$$\Rightarrow D^2 f(x) \text{ is symmetric}$$

$$\Rightarrow D^2 f(x) \text{ has an orthonormal basis of eigenvectors}$$

and thus can be diagonalized

**Theorem 15 (Stronger Version of Thm. 4.4)** Let  $X \subseteq \mathbf{R}^n$  be open,  $f : X \rightarrow \mathbf{R}$ ,  $f \in C^2(X)$ , and  $x \in X$ . Then

$$f(x+h) = f(x) + Df(x)h + \frac{1}{2}h^\top (D^2f(x))h + o(|h|^2) \text{ as } h \rightarrow 0$$

If  $f \in C^3$ ,

$$f(x+h) = f(x) + Df(x)h + \frac{1}{2}h^\top (D^2f(x))h + O(|h|^3) \text{ as } h \rightarrow 0$$

**Remark:** de la Fuente assumes  $X$  is convex.  $X$  is said to be *convex* if, for every  $x, y \in X$  and  $\alpha \in [0, 1]$ ,  $\alpha x + (1 - \alpha)y \in X$ . Notice we don't need this. Since  $X$  is open,

$$x \in X \Rightarrow \exists \delta > 0 \text{ s.t. } B_\delta(x) \subseteq X$$

and  $B_\delta(x)$  is convex.

**Definition 16** We say  $f$  has a *saddle* at  $x$  if  $Df(x) = 0$  but  $f$  has neither a local maximum nor a local minimum at  $x$ .

**Corollary 17** Suppose  $X \subseteq \mathbf{R}^n$  is open and  $x \in X$ . If  $f : X \rightarrow \mathbf{R}$  is  $C^2$ , then there is an orthonormal basis  $\{v_1, \dots, v_n\}$  and corresponding eigenvalues  $\lambda_1, \dots, \lambda_n \in \mathbf{R}$  of  $D^2f(x)$  such that

$$\begin{aligned} f(x+h) &= f(x + \gamma_1 v_1 + \dots + \gamma_n v_n) \\ &= f(x) + \sum_{i=1}^n (Df(x)v_i) \gamma_i + \frac{1}{2} \sum_{i=1}^n \lambda_i \gamma_i^2 + o(|\gamma|^2) \end{aligned}$$

where  $\gamma_i = h \cdot v_i$ .

1. If  $f \in C^3$ , we may strengthen  $o(|\gamma|^2)$  to  $O(|\gamma|^3)$ .
2. If  $f$  has a local maximum or local minimum at  $x$ , then

$$Df(x) = 0$$

3. If  $Df(x) = 0$ , then

$$\begin{aligned} \lambda_1, \dots, \lambda_n > 0 &\Rightarrow f \text{ has a local minimum at } x \\ \lambda_1, \dots, \lambda_n < 0 &\Rightarrow f \text{ has a local maximum at } x \\ \lambda_i < 0 \text{ for some } i, \lambda_j > 0 \text{ for some } j &\Rightarrow f \text{ has a saddle at } x \\ \lambda_1, \dots, \lambda_n \geq 0, \lambda_i > 0 \text{ for some } i &\Rightarrow f \text{ has a local minimum} \\ &\quad \text{or a saddle at } x \\ \lambda_1, \dots, \lambda_n \leq 0, \lambda_i < 0 \text{ for some } i &\Rightarrow f \text{ has a local maximum} \\ &\quad \text{or a saddle at } x \\ \lambda_1 = \dots = \lambda_n = 0 &\text{ gives no information.} \end{aligned}$$



**Proof:** (Sketch) From our study of quadratic forms, we know the behavior of the quadratic terms is determined by the signs of the eigenvalues. If  $\lambda_i = 0$  for some  $i$ , then we know that the quadratic form arising from the second partial derivatives is identically zero in the direction  $v_i$ , and the higher derivatives will determine the behavior of the function  $f$  in the direction  $v_i$ . For example, if  $f(x) = x^3$ , then  $f'(0) = 0$ ,  $f''(0) = 0$ , but we know that  $f$  has a saddle at  $x = 0$ ; however, if  $f(x) = x^4$ , then again  $f'(0) = 0$  and  $f''(0) = 0$  but  $f$  has a local (and global) minimum at  $x = 0$ .■

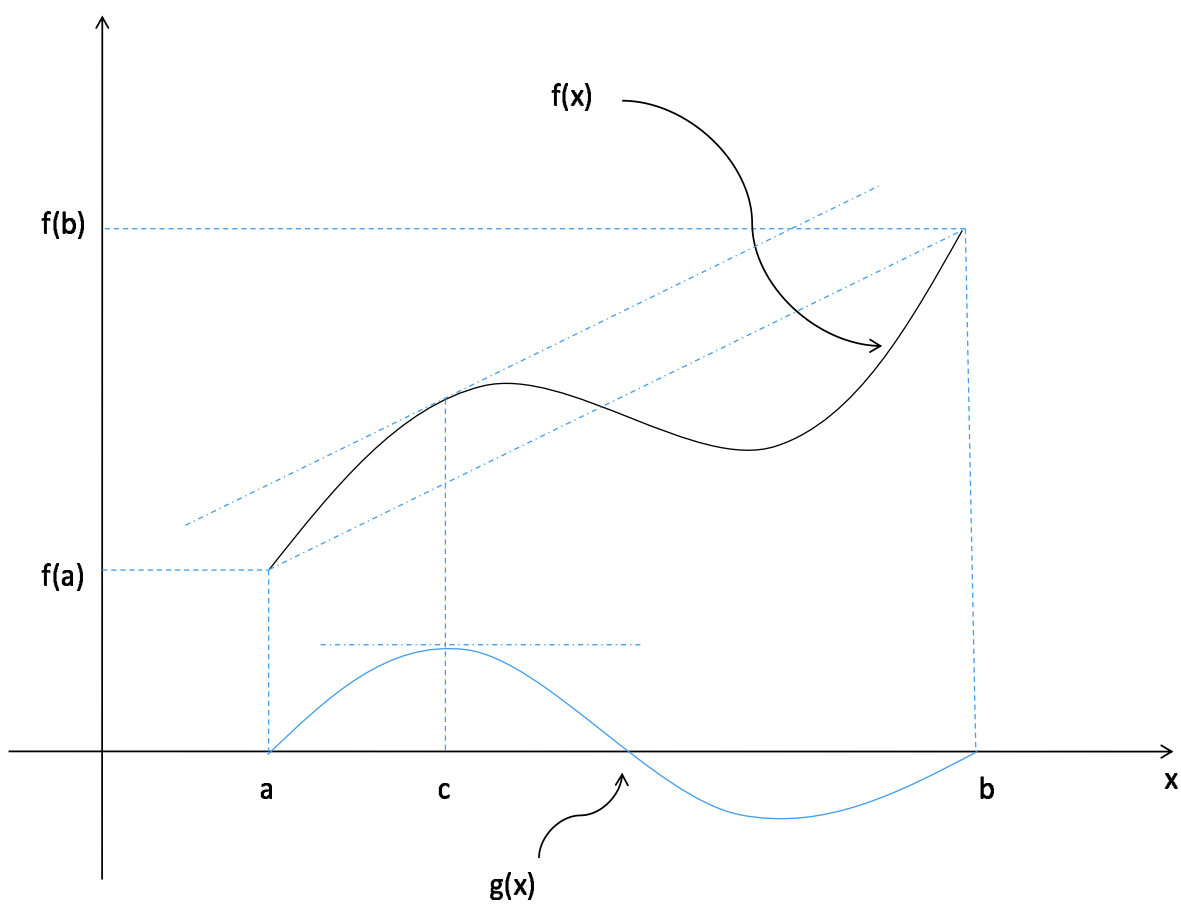


Figure 1: The Mean Value Theorem.