

**Economics 204 Summer/Fall 2022**  
**Lecture 9—Thursday August 4, 2022**

**Section 3.3. Quotient Vector Spaces<sup>1</sup>**

Given a vector space  $X$  over a field  $F$  and a vector subspace  $W$  of  $X$ , define an equivalence relation by

$$x \sim y \iff x - y \in W$$

Form a new vector space  $X/W$ : the set of elements of  $X/W$  is

$$\{[x] : x \in X\}$$

where  $[x]$  denotes the equivalence class of  $x$  with respect to  $\sim$ .  $X/W$  is read “ $X \bmod W$ ”. Note that the vectors in  $X/W$  are *sets* of vectors in  $X$ : for  $x \in X$ ,

$$[x] = \{x + w : w \in W\}$$

We claim that  $X/W$  can be viewed as a vector space over  $F$ . Define the vector space operations  $+$ ,  $\cdot$  in  $X/W$  as follows:

$$\begin{aligned} [x] + [y] &= [x + y] \\ \alpha[x] &= [\alpha x] \end{aligned}$$

The exercise below asks you to verify that these operations are well-defined. Then  $X/W$  is a vector space over  $F$  with these definitions for  $+$  and  $\cdot$ .

**Exercise:** Verify that  $\sim$  above is an equivalence relation and that vector addition and scalar multiplication are well-defined, i.e.

$$\begin{aligned} [x] = [x'], [y] = [y'] &\Rightarrow [x + y] = [x' + y'] \\ [x] = [x'], \alpha \in F &\Rightarrow [\alpha x] = [\alpha x'] \end{aligned}$$

**Example:** Let  $X = \mathbf{R}^3$  and let  $W = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$ . Then for  $x, y \in \mathbf{R}^3$ ,

$$\begin{aligned} x \sim y &\iff x - y \in W \\ &\iff x_1 - y_1 = 0, x_2 - y_2 = 0 \\ &\iff x_1 = y_1, x_2 = y_2 \end{aligned}$$

and

$$[x] = \{x + w : w \in W\} = \{(x_1, x_2, z) : z \in \mathbf{R}\}$$

So the equivalence class corresponding to  $x$  is the line in  $\mathbf{R}^3$  through  $x$  parallel to the axis of the third coordinate. See Figure 1. What is  $X/W$ ? Intuitively this equivalence relation ignores the third coordinate, and we can identify the equivalence class  $[x]$  with the vector  $(x_1, x_2) \in \mathbf{R}^2$ . The next two results show how to formalize this connection.

---

<sup>1</sup>The first part of this material is not in de la Fuente.

**Theorem 1** *If  $X$  is a vector space with  $\dim X = n$  for some  $n \in \mathbf{N}$  and  $W$  is a vector subspace of  $X$ , then*

$$\dim(X/W) = \dim X - \dim W$$

**Proof:** (Sketch) Begin with a basis  $\{w_1, \dots, w_c\}$  for  $W$ , and a basis  $\{[x_1], \dots, [x_k]\}$  for  $X/W$ . Show that

$$\{w_1, \dots, w_c\} \cup \{x_1, \dots, x_k\}$$

is a basis for  $X$ . ■

**Theorem 2** *Let  $X$  and  $Y$  be vector spaces over the same field  $F$  and  $T \in L(X, Y)$ . Then  $\text{Im } T$  is isomorphic to  $X/\ker T$ .*

**Proof:** Notice that if  $X$  is finite-dimensional, then

$$\begin{aligned} \dim(X/\ker T) &= \dim X - \dim \ker T \quad (\text{by the previous theorem}) \\ &= \text{Rank } T \quad (\text{by the Rank-Nullity Theorem}) \\ &= \dim \text{Im } T \end{aligned}$$

so  $X/\ker T$  is isomorphic to  $\text{Im } T$ .

We prove that this is true in general, and that the isomorphism is natural.

Define  $\tilde{T} : X/\ker T \rightarrow \text{Im } T$  by

$$\tilde{T}([x]) = T(x)$$

We first need to check that this is well-defined, that is, that if  $[x] = [x']$  then  $\tilde{T}([x]) = \tilde{T}([x'])$ .

$$\begin{aligned} [x] = [x'] &\Rightarrow x \sim x' \\ &\Rightarrow x - x' \in \ker T \\ &\Rightarrow T(x - x') = 0 \\ &\Rightarrow T(x) = T(x') \end{aligned}$$

so  $\tilde{T}$  is well-defined.

Clearly,  $\tilde{T} : X/\ker T \rightarrow \text{Im } T$ . It is easy to check that  $\tilde{T}$  is linear, so  $\tilde{T} \in L(X/\ker T, \text{Im } T)$ . Next we show that  $\tilde{T}$  is an isomorphism.

$$\begin{aligned} \tilde{T}([x]) = \tilde{T}([y]) &\Rightarrow T(x) = T(y) \\ &\Rightarrow T(x - y) = 0 \\ &\Rightarrow x - y \in \ker T \\ &\Rightarrow x \sim y \\ &\Rightarrow [x] = [y] \end{aligned}$$

so  $\tilde{T}$  is one-to-one.

$$\begin{aligned} y \in \text{Im } T &\Rightarrow \exists x \in X \text{ s.t. } T(x) = y \\ &\Rightarrow \tilde{T}([x]) = y \end{aligned}$$

so  $\tilde{T}$  is onto, hence  $\tilde{T}$  is an isomorphism. ■

**Example:** Consider  $T \in L(\mathbf{R}^3, \mathbf{R}^2)$  defined by

$$T(x_1, x_2, x_3) = (x_1, x_2)$$

Then  $\ker T = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$  is the  $x_3$ -axis. (Also notice  $\ker T = W$  from the previous example.)

Given  $x$ , the equivalence class  $[(x_1, x_2, x_3)]$  is just the line through  $x$  parallel to the  $x_3$ -axis.  $\tilde{T}([x]) = T(x_1, x_2, x_3) = (x_1, x_2)$ .

$$\text{Im } T = \mathbf{R}^2, \quad X/\ker T \cong \mathbf{R}^2 = \text{Im } T$$

as we suggested intuitively above (here the symbol  $\cong$  denotes isomorphism, that is, we write  $Y \cong Z$  if  $Y$  and  $Z$  are isomorphic.)

Every real vector space  $X$  with dimension  $n$  is isomorphic to  $\mathbf{R}^n$ . What's the isomorphism?

Let  $X$  be a finite-dimensional vector space over  $\mathbf{R}$  with  $\dim X = n$ . Fix any Hamel basis  $V = \{v_1, \dots, v_n\}$  of  $X$ . Any  $x \in X$  has a unique representation

$$x = \sum_{j=1}^n \beta_j v_j$$

(here, we allow  $\beta_j = 0$ ). (Generally, vectors are represented as column vectors, not row vectors.) Then given the representation of  $x$  above, we write

$$\text{crd}_V(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbf{R}^n$$

That is,  $\text{crd}_V(x)$  is the vector of coordinates of  $x$  with respect to the basis  $V$ .

$$\text{crd}_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \text{crd}_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \text{crd}_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\text{crd}_V$  is an isomorphism from  $X$  to  $\mathbf{R}^n$ .

## Matrix Representation of a Linear Transformation

Suppose  $T \in L(X, Y)$ ,  $\dim X = n$  and  $\dim Y = m$ . Fix bases

$$\begin{aligned} V &= \{v_1, \dots, v_n\} \text{ of } X \\ W &= \{w_1, \dots, w_m\} \text{ of } Y \end{aligned}$$

$T(v_j) \in Y$ , so

$$T(v_j) = \sum_{i=1}^m \alpha_{ij} w_i$$

Define

$$Mtx_{W,V}(T) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

Notice that the columns are the coordinates (expressed with respect to  $W$ ) of  $T(v_1), \dots, T(v_n)$ .

Observe

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{pmatrix}$$

so

$$\begin{aligned} Mtx_{W,V}(T) \cdot crd_V(v_j) &= crd_W(T(v_j)) \\ Mtx_{W,V}(T) \cdot crd_V(x) &= crd_W(T(x)) \quad \forall x \in X \end{aligned}$$

Multiplying a vector by a matrix does two things:

- Computes the action of  $T$
- Accounts for the change in basis

**Example:**  $X = Y = \mathbf{R}^2$ ,  $V = \{(1, 0), (0, 1)\}$ ,  $W = \{(1, 1), (-1, 1)\}$ ,  $T = id$ , that is,  $T(x) = x$  for all  $x$ .

$$Mtx_{W,V}(T) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$Mtx_{W,V}(T)$  is the matrix that *changes basis* from  $V$  to  $W$ . How do we compute it?

$$\begin{aligned} v_1 = (1, 0) &= \alpha_{11}(1, 1) + \alpha_{21}(-1, 1) \\ \alpha_{11} - \alpha_{21} &= 1 \\ \alpha_{11} + \alpha_{21} &= 0 \\ 2\alpha_{11} &= 1, \alpha_{11} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
\alpha_{21} &= -\frac{1}{2} \\
v_2 = (0, 1) &= \alpha_{12}(1, 1) + \alpha_{22}(-1, 1) \\
\alpha_{12} - \alpha_{22} &= 0 \\
\alpha_{12} + \alpha_{22} &= 1 \\
2\alpha_{12} &= 1, \alpha_{12} = \frac{1}{2} \\
\alpha_{22} &= \frac{1}{2}
\end{aligned}$$

$$Mtx_{W,V}(id) = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

**Theorem 3 (Thm. 3.5')** Let  $X$  and  $Y$  be vector spaces over the same field  $F$ , with  $\dim X = n$ ,  $\dim Y = m$ . Then  $L(X, Y)$ , the space of linear transformations from  $X$  to  $Y$ , is isomorphic to  $F_{m \times n}$ , the vector space of  $m \times n$  matrices over  $F$ . If  $V = \{v_1, \dots, v_n\}$  is a basis for  $X$  and  $W = \{w_1, \dots, w_m\}$  is a basis for  $Y$ , then

$$Mtx_{W,V} \in L(L(X, Y), F_{m \times n})$$

and  $Mtx_{W,V}$  is an isomorphism from  $L(X, Y)$  to  $F_{m \times n}$ .

**Theorem 4 (From Handout)** Let  $X, Y, Z$  be finite-dimensional vector spaces over the same field  $F$  with bases  $U, V, W$  respectively. Let  $S \in L(X, Y)$  and  $T \in L(Y, Z)$ . Then

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations.

**Proof:** See handout. ■

Note that  $Mtx_{W,V}$  is a function from  $L(X, Y)$  to the space  $F_{m \times n}$  of  $m \times n$  matrices, while  $Mtx_{W,V}(T)$  is an  $m \times n$  matrix.

The theorem can be summarized by the following “Commutative Diagram:”

$$\begin{array}{ccccc}
& & S & & T \\
& X & \rightarrow & Y & \rightarrow & Z \\
crd_U \downarrow & & & \downarrow crd_V & & \downarrow crd_W \\
& \mathbf{R}^n & \rightarrow & \mathbf{R}^m & \rightarrow & \mathbf{R}^r \\
& & Mtx_{V,U}(S) & & Mtx_{W,V}(T) & 
\end{array}$$

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The  $crd$  arrows go in both directions because  $crd$  is an isomorphism.

### Section 3.5. Change of Basis and Similarity

Let  $X$  be a finite-dimensional vector space with basis  $V$ . If  $T \in L(X, X)$  it is customary to use the same basis in the domain and range. In this case,

$$Mtx_V(T) \text{ denotes } Mtx_{V,V}(T)$$

**Question:** If  $W$  is another basis for  $X$ , how are  $Mtx_V(T)$  and  $Mtx_W(T)$  related?

$$\begin{aligned} Mtx_{V,W}(id) \cdot Mtx_W(T) \cdot Mtx_{W,V}(id) &= Mtx_{V,W}(id) \cdot Mtx_{W,V}(T \circ id) \\ &= Mtx_{V,V}(id \circ T \circ id) \\ &= Mtx_V(T) \end{aligned}$$

and

$$\begin{aligned} Mtx_{V,W}(id) \cdot Mtx_{W,V}(id) &= Mtx_{V,V}(id) \\ &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{aligned}$$

So this says that

$$Mtx_V(T) = P^{-1} Mtx_W(T) P$$

for the invertible matrix

$$P = Mtx_{W,V}(id)$$

that is the change of basis matrix. On the other hand, if  $P$  is any invertible matrix, then  $P$  is also a change of basis matrix for appropriate corresponding bases (see handout).

**Definition 5** Square matrices  $A$  and  $B$  are *similar* if

$$A = P^{-1} B P$$

for some invertible matrix  $P$ .

**Theorem 6** Suppose that  $X$  is a finite-dimensional vector space.

1. If  $T \in L(X, X)$  then any two matrix representations of  $T$  are similar. That is, if  $U, W$  are any two bases of  $X$ , then  $Mtx_W(T)$  and  $Mtx_U(T)$  are similar.
2. Conversely, two similar matrices represent the same linear transformation  $T$ , relative to suitable bases. That is, given similar matrices  $A, B$  with  $A = P^{-1} B P$  and any basis  $U$ , there is a basis  $W$  and  $T \in L(X, X)$  such that

$$\begin{aligned} B &= Mtx_U(T) \\ A &= Mtx_W(T) \\ P &= Mtx_{U,W}(id) \\ P^{-1} &= Mtx_{W,U}(id) \end{aligned}$$

**Proof:** See Handout on Diagonalization and Quadratic Forms. ■

### Section 3.6. Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that  $\lambda$  is an eigenvalue of  $T$  if and only if  $\lambda$  is an eigenvalue for some matrix representation of  $T$  if and only if  $\lambda$  is an eigenvalue for every matrix representation of  $T$ .

**Definition 7** Let  $X$  be a vector space and  $T \in L(X, X)$ . We say that  $\lambda$  is an *eigenvalue* of  $T$  and  $v \neq 0$  is an *eigenvector corresponding to  $\lambda$*  if  $T(v) = \lambda v$ .

**Theorem 8 (Theorem 4 in Handout)** Let  $X$  be a finite-dimensional vector space, and  $U$  a basis. Then  $\lambda$  is an eigenvalue of  $T$  if and only if  $\lambda$  is an eigenvalue of  $Mtx_U(T)$ .  $v$  is an eigenvector of  $T$  corresponding to  $\lambda$  if and only if  $crd_U(v)$  is an eigenvector of  $Mtx_U(T)$  corresponding to  $\lambda$ .

**Proof:** By the Commutative Diagram Theorem,

$$\begin{aligned} T(v) = \lambda v &\Leftrightarrow crd_U(T(v)) = crd_U(\lambda v) \\ &\Leftrightarrow Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v)) \end{aligned}$$

■

#### Computing eigenvalues and eigenvectors:

Suppose  $\dim X = n$ ; let  $I$  be the  $n \times n$  identity matrix. Given  $T \in L(X, X)$ , fix a basis  $U$  and let

$$A = Mtx_U(T)$$

Find the eigenvalues of  $T$  by computing the eigenvalues of  $A$ :

$$\begin{aligned} Av = \lambda v &\iff (A - \lambda I)v = 0 \\ &\iff (A - \lambda I) \text{ is not invertible} \\ &\iff \det(A - \lambda I) = 0 \end{aligned}$$

We have the following facts:

- If  $A \in \mathbf{R}_{n \times n}$ ,

$$f(\lambda) = \det(A - \lambda I)$$

is an  $n^{th}$  degree polynomial in  $\lambda$  with real coefficients; it is called the *characteristic polynomial* of  $A$ .

- $f$  has  $n$  roots in  $\mathbf{C}$ , counting multiplicity:

$$f(\lambda) = (c_1 - \lambda)(c_2 - \lambda) \cdots (c_n - \lambda)$$

where  $c_1, \dots, c_n \in \mathbf{C}$  are the eigenvalues; the  $c_j$ 's are not necessarily distinct. Notice that  $f(\lambda) = 0$  if and only if  $\lambda \in \{c_1, \dots, c_n\}$ , so the roots are the solutions of the equation  $f(\lambda) = 0$ .

- the roots that are not real come in conjugate pairs:

$$f(a + bi) = 0 \Leftrightarrow f(a - bi) = 0$$

- if  $\lambda = c_j \in \mathbf{R}$ , there is a corresponding eigenvector in  $\mathbf{R}^n$ .
- if  $\lambda = c_j \notin \mathbf{R}$ , the corresponding eigenvectors are in  $\mathbf{C}^n \setminus \mathbf{R}^n$ .

## Diagonalization

**Definition 9** Suppose  $X$  is a finite-dimensional vector space with basis  $U$ . Given a linear transformation  $T \in L(X, X)$ , let

$$A = Mtx_U(T)$$

We say that  $A$  can be diagonalized (or is diagonalizable) if there is a basis  $W$  for  $X$  such that  $Mtx_W(T)$  is diagonal, i.e.

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Notice that the eigenvectors of  $Mtx_W(T)$  are exactly the standard basis vectors of  $\mathbf{R}^n$ . But  $w_j$  is an eigenvector of  $T$  corresponding to  $\lambda_j$  if and only if  $crd_W(w_j)$  is an eigenvector of  $Mtx_W(T)$ , and  $crd_W(w_j)$  is the  $j^{th}$  standard basis vector of  $\mathbf{R}^n$ , so  $W = \{w_1, \dots, w_n\}$  where  $w_j$  is an eigenvector corresponding to  $\lambda_j$ .

Then the action of  $T$  is clear: it stretches each basis element  $w_i$  by the factor  $\lambda_i$ .



**Theorem 10 (Thm. 6.7')** *Let  $X$  be an  $n$ -dimensional vector space,  $T \in L(X, X)$ ,  $U$  any basis of  $X$ , and  $A = Mtx_U(T)$ . Then the following are equivalent:*

1.  *$A$  can be diagonalized*
2. *there is a basis  $W$  for  $X$  consisting of eigenvectors of  $T$*
3. *there is a basis  $V$  for  $\mathbf{R}^n$  consisting of eigenvectors of  $A$*

**Proof:** Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout. ■

**Theorem 11 (Thm. 6.8')** *Let  $X$  be a vector space and  $T \in L(X, X)$ .*

1. *If  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$  with corresponding eigenvectors  $v_1, \dots, v_m$ , then  $\{v_1, \dots, v_m\}$  is linearly independent.*
2. *If  $\dim X = n$  and  $T$  has  $n$  distinct eigenvalues, then  $X$  has a basis consisting of eigenvectors of  $T$ ; consequently, if  $U$  is any basis of  $X$ , then  $Mtx_U(T)$  is diagonalizable.*

**Proof:** This is an adaptation of the proof of Theorem 6.8 in de la Fuente. ■

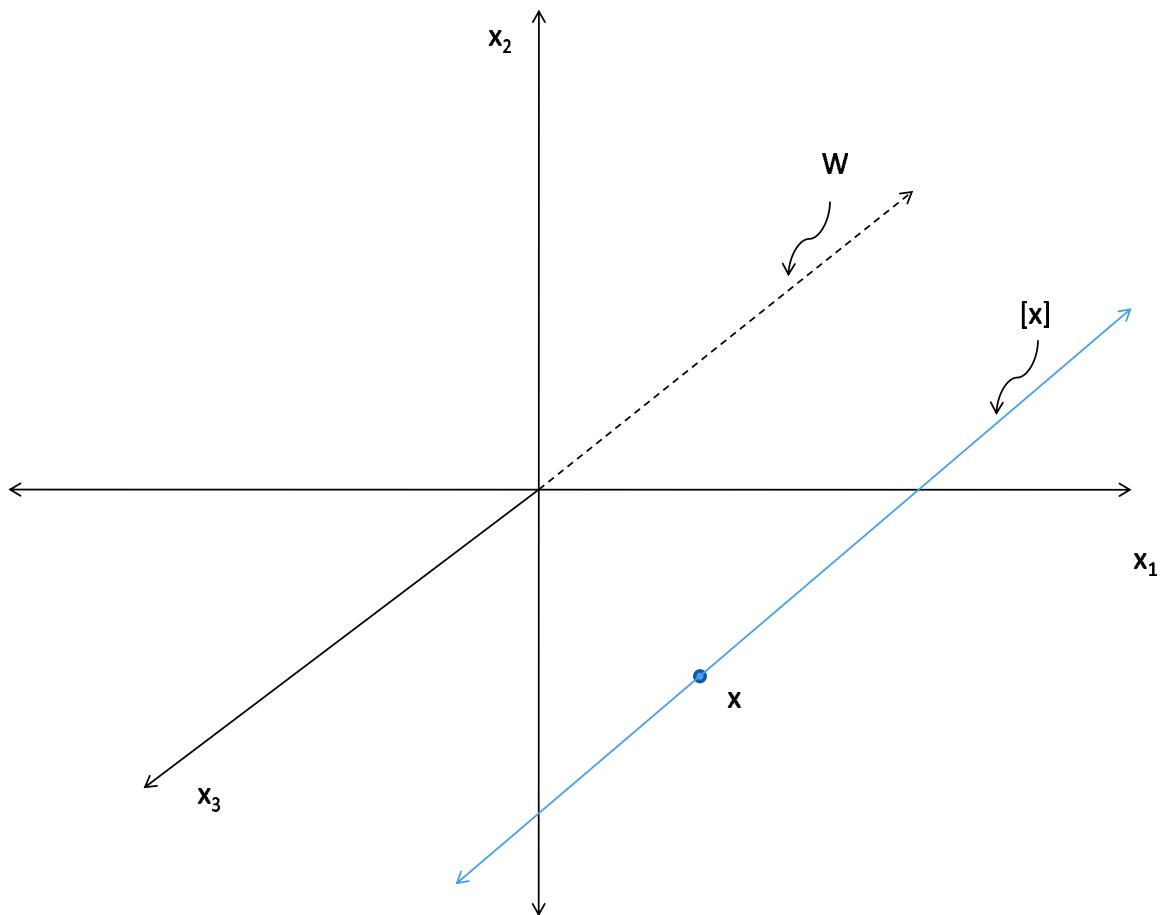


Figure 1: An illustration of  $X/W$  where  $X = \mathbf{R}^3$  and  $W = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$ . Here  $[x] = \{(x_1, x_2, z) : z \in \mathbf{R}\}$  is the line through  $x$  parallel to the axis of the third coordinate.