Econ 204 – Problem Set 2[1]

Due Tuesday August 2, 2022

1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Given $A, B \subseteq \mathbb{R}^n$, we define the sum of these two sets by:

$$A + B = \{a + b \mid a \in A, b \in B\}$$

   Prove or find a counterexample to the following statements:

   (a) If either A or B is an open set, then $A + B$ is an open set.

   (b) If both A and B are closed sets, $A + B$ is a closed set.

3. Let (X, d) be a metric space. Assume $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous on (X, d) and $(\mathbb{R}, |\cdot|)$, with $|\cdot|$ the absolute-value norm.

   (a) Show that $f + g : X \to \mathbb{R}$ is uniformly continuous, where $(f + g)(x) = f(x) + g(x)$.

   (b) Show that $\max\{f, g\} : X \to \mathbb{R}$ is uniformly continuous, where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.

   (c) Give a counterexample to the following statement: $f \cdot g : X \to \mathbb{R}$ *is uniformly continuous on* (X, d) *and* $(\mathbb{R}, |\cdot|)$, where $f \cdot g = f(x) \cdot g(x)$.

4. A function $f : X \to Y$ is open if $\forall A \subset X$ such that A is open, $f(A)$ is open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotonic.

5. Let (X, d) be a metric space and $A \subseteq X$. Show that

$$\overline{A} = \{x \in X \mid d(x, A) = 0\}$$

   where the distance between a point y and a set B is given by $d(y, B) = \inf_{b \in B}\{d(y, b)\}$.

   Conclude that a set A is closed iff there exists a continuous function $f : X \to \mathbb{R}$ such that $A = f^{-1}(0)$.

6. For some metric space (X, d), take any two sets $A, B \subset X$ such that $\text{int}A = \text{int}B = \emptyset$, and A is closed. Prove that $\text{int}(A \cup B) = \emptyset$.

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu