Econ 204 – Problem Set 3[1]

Due 11:59PM Friday August 5 , 2022

1. Let (X, d) be a metric space. Given a subset $Y \subseteq X$, we define

$$Y^\varepsilon = \{x \in X \mid d(x, Y) < \varepsilon\}$$

Let $K \subseteq X$ be compact and A an open set such that $K \subseteq A$. Show that there exists $\varepsilon > 0$ such that $K^\varepsilon \subseteq A$.

2. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be bounded. Given $M > 0$, define $f_M : X \to \mathbb{R}$ by :

$$f_M(x) = \inf_{y \in X} \{f(y) + Md(x, y)\}$$

Show that:

(a) $\forall x \in X \; f_M(x) \leq f(x)$

(b) Show that f_M is M-Lipschitz

(c) Show that if f is Lipschitz and the lipschitz constant of f , M_f, is less or equal than M, then $f_M = f$

(d) Show that, given $x \in X$ and $M < M'$, we have that $f_M(x) \leq f_{M'}(x)$.

(e) Show that when $M \to \infty$, then $f_M(x) \to f(x)$ in every point $x \in X$ such that f is continuous.

(f) Show that if f is continuous and X is compact,

$$\lim_{M \to \infty} \sup_{x \in X} \{d(f_M(x), f(x))\} = 0$$

3. Let $U \subseteq \mathbb{R}^d$ be an open set and $f : [0, 1] \to U$ be continuous. For each $n \in \mathbb{N}$, define the n-polygonal approximation of f to be the function $\gamma_n : [0, 1] \to \mathbb{R}^d$ given by:

$$\gamma_n(t) = f\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right)$$

where $i \in \{1, \ldots, n\}$ is such that $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$.

(a) Show that γ_n is continuous for all $n \in \mathbb{N}$.

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

(b) Show that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 \gamma_n(t) \in U$ for all $t \in [0,1]$.

4. Let (X,d) be a metric space. Given $x \in X$, we define the connected component of x in X as the set

$$C(x) = \bigcup_{\substack{U \subseteq X \text{ s.t } x \in U \\ U \text{ is connected}}} U$$

Prove that:

(a) For every $x \in X$, $C(x)$ is a non-empty connected set.

(b) For every two elements $x, y \in X$, they either share a connected component $C(x) = C(y)$ or their connected components are disjoint $C(x) \cap C(y) = \varnothing$.

(c) Conclude that there exists a subset $\mathcal{A} \subseteq X$ such that $X = \dot{\bigcup}_{x \in \mathcal{A}} C(x)$, where $\dot{\cup}$ represents the disjoint union.

5. Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ [0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}. \tag{1}$$

Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

6. Let (X,d) be a compact metric space.

(a) Show that there exists A a countable subset of X such that $\overline{A} = X$.

(b) We say that $x \in X$ is an isolated point if there exists $\delta > 0$ such that $B(x,\delta) = \{x\}$. Show that the set of isolated points of X is empty, finite or countable.