Econ 204 – Problem Set 4

Due Tuesday, August 9

1. Let $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ be defined by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

2. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(av) = af(v)$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but f is not linear.

3. Prove that if $(v_1, ..., v_n)$ spans V and $T \in L(V, W)$ is onto, then $(Tv_1, ..., Tv_n)$ spans W.

4. Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a linear transformation. Assume that $T^n = 0$ for some integer $n > 0$, and $T^l \neq 0$ for $l < n$.

   (a) Show that T is not invertible.

   (b) Show that $T + I$ is invertible. *Hint*: the inverse of $T + I$ is a polynomial of T.

5. Let U and V be vector spaces. Suppose $T : U \to V$ is a linear transformation and $v \in V$. Prove that, if the preimage $T^{-1}(v)$ is non-empty, and $u \in T^{-1}(v)$, then $T^{-1}(v) = \{u + z | z \in \ker T\} = u + \ker T$.

6. Let V be a finite dimensional vector space with dimension $n > 1$. Let $L(V, V)$ be the set of all linear transformation from V to V, which is a vector space (you don't have to prove this). Consider $C \subset L(V, V)$ the set of all non-invertible linear transformations from V to V. Is C a subspace of $L(V, V)$? Prove or provide a counterexample.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (4x - 2y, x + y)$. Let V be the standard basis and $W = \{(5, 3), (1, 1)\}$ be another basis of $\mathbb{R}^2$.

   (a) Find $Mtx_V(T)$.

   (b) Find $Mtx_W(T)$.

   (c) Compute $T(4, 3)$ using the matrix representation of W.