Econ 204 – Problem Set 5

Due Friday, August 12

1. Let $C = C([a, b])$ be the set of continuous functions from $[a, b]$ to $\mathbb{R}$ equipped with the sup norm.

   (a) Define, for any $t \in [a, b]$, the function $\mathcal{A}_t : C \to \mathbb{R}$ as

   $$\mathcal{A}_t(f) = f(t)$$

   Prove that $\mathcal{A}_t$ is 1-Lipschitz, but not L-Lipschitz for any $L < 1$. [1].

   (b) Define $\mathcal{I} : C \to \mathbb{R}$ as

   $$\mathcal{I}(f) = \int_a^b f(t)dt$$

   Find the constant K such that $\mathcal{I}$ is K-Lipschitz, but not L-Lipschitz for any $L < K$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 (twice continuously differentiable) function. The function and its second derivative are bounded, namely there exist $M, N > 0$ such that $\sup_{x\in\mathbb{R}} |f(x)| \leq M$ and $\sup_{x\in\mathbb{R}} |f''(x)| \leq N$. Show that $\sup_{x\in\mathbb{R}} |f'(x)| \leq 2\sqrt{MN}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, the image of the derivative function, is an interval (possibly a singleton).

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n \in \mathbb{N}$ with $|f_n'(x)| \leq 1$ for all n and $x \in \mathbb{R}$. Assume that

   $$\lim_{n\to\infty} f_n(x) = g(x),$$

   for all $x \in \mathbb{R}$. Prove that $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

5. Suppose $\Psi : X \to 2^X$ is a non-empty and compact-valued upper-hemicontinuous correspondence. The metric space X is compact. Show that there exists a non-empty compact set $C \subset X$ such that $\Psi(C) = C$ (you can use the exercises that are proved in the sections).

[1] We say that a function $f : X \to Y$ is L-Lipschitz, or Lipschitz with constant L, if $||f(x) - f(y)||_Y \leq L||x - y||_X \forall x, y \in X$