Generalized Convex Games

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1 Introduction

Economists have long argued that the existence of increasing returns to scale or unique complementary inputs may lead to indeterminacy in how the gains to team production are shared among team members. Cooperative games provide a framework in which to formalize and explore this intuition. In games with side payments, the notion of a convex game introduced by Shapley (1971) provides a natural way to formalize these ideas. A coalition form game (N, v) with side payments is a *convex game* if for all coalitions S and T, $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$. This condition arises when each player provides some number of units of a homogeneous input and production displays increasing returns to scale. It also arises when each player possesses a unique input and the inputs are complementary (Topkis (1981)). Shapley showed that the core of a convex game is nonempty, and that its extreme points can be computed by the greedy algorithm, that is, by listing the players in some order and giving each player in turn his or her marginal contribution $v(S \cup \{i\}) - v(S)$ to the coalition S of preceding players.

The fact that any ordering of the players in the greedy algorithm yields a payoff vector in the core suggests that the core places weak restrictions on the way the fruits of cooperation are shared in convex games. Sharkey (1982) further explores this idea. He introduces the notion of a "large" core, which is characterized by the property that for every unblocked (but not necessarily feasible) payoff vector y there exists a payoff vector x in the core such that $x \leq y$. He shows that the core in a convex game is large in this sense.

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Ichiishi (1990) has studied comparative statics of the core in convex games, showing that when the marginal contribution of each coalition is an increasing function of some exogenous parameters, then the core increases in those parameters as well. More precisely, if $(N, v(S;t))$ is a family of convex games in which the difference $v(T';t) - v(T;t)$ is increasing in t for all coalitions T' and T such that $T \subset T'$, then the core is increasing in t in the sense that every core allocation in the game with a smaller value of the parameter is dominated by some core allocation in the game with any larger parameter value, and every core allocation in the game with a larger parameter value dominates some core allocation in the game with any smaller parameter.

When the definition of convex games is viewed from a different angle, the connections between convexity, production complementarities, and comparative statics of the core become quite natural. Suppose the set of coalitions is ordered by set inclusion, so that given two coalitions S and T, $S \leq T$ if and only if $S \subset T$. Then under this order, the union of two coalitions is their least upper bound, and their intersection is their greatest lower bound, that is, $S \cup T = S \vee T$ and $S \cap T = S \wedge T$. With this observation in mind, it is easy to see that "convexity" is simply equivalent to the supermodularity of the characteristic function v with respect to this ordering of coalitions. Similarly, in a parameterized family of convex games, to say that $v(T';t) - v(T;t)$ is nondecreasing in t for all $T \subset T'$ is equivalent to saying that the characteristic function has increasing differences in $(S;t)$. An optimization problem in which the objective function is supermodular in the choice variables and has increasing differences in the choice variables and parameters is characterized by complementarities in that increases in one of the choice variables or parameters increase the marginal benefit of all other choice variables. Moreover, Topkis (1978) shows that the solutions to such an optimization problem are nondecreasing in the parameters. Ichiishi's result is then similar in spirit to Topkis's result: Ichiishi shows that if the value function is supermodular in coalitions and has increasing differences in coalitions and some exogenous parameters, then the core is monotone nondecreasing with respect to these parameters.

In this paper, we show that this analysis of increasing returns and complementarities in cooperative games can be extended to nontransferable utility games using the same general ideas used to extend Topkis's results on supermodular optimization problems to an ordinal setting (Milgrom and Shannon (1994)). In the process, we define classes of generalized convex games for which analogues of the results for convex games carry over. These games are characterized by the condition that a demand by a player from her coalition partners that cannot be blocked by these partners also cannot be blocked by any larger coalition. These games encompass the existing extensions of convexity to nontransferable utility games. Finally, we illustrate these results with an application to the theory of public goods.

2 Generalized Convex Games

A cooperative game without side payments $\Gamma = (N, v)$ is described by a set N of players and a function v such that $v(S) \subset \mathbb{R}^S$ and $v(\emptyset) = \emptyset$. The set $v(S)$ describes the utility allocation vectors available to S if its members transact on their own. We assume that $v(S)$ is closed, downward comprehensive, bounded above and normalized so that for all $n \in N$, $v(\{n\}) = (-\infty, 0]$. Given $x \in \mathbb{R}^N$, x_S denotes the projection of x onto $\mathbb{R}^S$.

With these definitions, the **core** is defined as follows:

$$\text{Core}(N, v) = \{x \in v(N) : \nexists S \subset N \text{ and } z_S \in v(S) \text{ s.t. } z_S \gg x_S\}.$$

In words, Core(N, v) is the set of points in $v(N)$ that are *unblocked* by any coalition S. Using this notation, we can also talk about Core(S, v) for any subcoalition $S \subset N$.

In this paper, we will limit our analysis to games in which the concepts of Pareto optimality and weak Pareto optimality coincide for every coalition. That is, we consider only games with the property that for all z, if there exists $x_S \in v(S)$ such that $x_S > z$, then there exists $y_S \in v(S)$ such that $y_S \gg z$. In particular, any game with transferable utility has this property. Essentially, this condition means that the boundary of any coalition's set of feasible utility allocations is downward sloping. For such games, the core can be equivalently characterized as

$$\text{Core}(N, v) = \{x \in v(N) : \nexists S \subset N \text{ and } z_S \in v(S) \text{ s.t. } z_S > x_S\}.$$

Games with side payments can be embedded in the class of games without side payments, since games with side payments are just those games for which the function v takes the form $v(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq \bar{v}(S)\}$ for some function $\bar{v} : 2^S \setminus \emptyset \to \mathbb{R}$. Convex games are then those for which this function $\bar{v}$ has the property that for all coalitions S and T, $\bar{v}(S) + \bar{v}(T) \leq \bar{v}(S \cap T) + \bar{v}(S \cup T)$. As noted in the introduction, if we regard the set of coalitions as a lattice ordered by set inclusion, this is equivalent to saying that $\bar{v}$ is a supermodular function (Topkis (1987)). The supermodularity of $\bar{v}$ when side-payments are possible expresses a kind of increasing returns to coalition inclusiveness: the marginal value of a coalition S to a disjoint coalition T is larger the more inclusive is T. To see this, suppose S and T are disjoint, and let $T' \supset T$ be a larger coalition also disjoint from S. If the value function $\bar{v}$ is supermodular, then

$$\bar{v}(S \cup T) + \bar{v}(T') \leq \bar{v}(T) + \bar{v}(S \cup T')$$

or

$$\bar{v}(S \cup T) - \bar{v}(T) \leq \bar{v}(S \cup T') - \bar{v}(T').$$

This suggests that the demands that a coalition S can successfully make of a coalition T for its participation might be a monotone nondecreasing function of the inclusiveness of T. With this idea in mind, given disjoint coalitions S and T, we will say that x_S is **acceptable** to a coalition T if there is some vector x_T such that $(x_S, x_T) \in v(S \cup T)$ and $\nexists z_T \in v(T)$ such that $z_T > x_T$. A game (N, v) is then a **weak generalized convex game** if for all disjoint coalitions T and S, for all $T \subset T'$, and all x_S, if x_S is acceptable to T then x_S is also acceptable to T'.

Theorem 1. *A game with side payments is a weak generalized convex game if and only if it is a convex game.*

Proof: Observe that for transferable utility games, x_S is acceptable to T if and only if $\bar{v}(S \cup T) - \bar{v}(T) \geq \sum_{i \in S} x_i$. Now suppose that $(N, \bar{v})$ is a convex game, $S \cap T' = \emptyset$, and x_S is acceptable to $T \subset T'$. Then $\sum_{i \in S} x_i \leq \bar{v}(S \cup T) - \bar{v}(T) \leq \bar{v}(S \cup T') - \bar{v}(T')$, where the last inequality follows because this is a convex game. Hence x_S is acceptable to T' and so (N, v) is a generalized convex game.

Conversely, suppose (N, v) is a generalized convex game. Let $x_i = [\bar{v}(S \cup T) - \bar{v}(T)]/|S|$. Then x_S is acceptable to T and hence, by generalized convexity, x_S is acceptable to T'. Thus $\bar{v}(S \cup T) - \bar{v}(T) = \sum_{i \in S} x_i \leq \bar{v}(S \cup T') - \bar{v}(T')$. So $(N, \bar{v})$ is a convex game. ■

In convex games, the core is nonempty. Given an allocation in the core of the game (S, v), if a player is added to this coalition, there will be many feasible ways to divide up the increased value of this larger coalition among the members. One way is simply to give this new player his entire marginal contribution and give the original players their original allocations. In a convex game this distribution scheme is a core allocation in the new game. This result may not hold for weak generalized convex games without side payments as defined above, however. In section 3, we show that weak generalized convex games encompass both ordinally convex games and cardinally convex games. Sharkey (1981) has shown that the greedy algorithm may fail for cardinally convex games which are not ordinally convex, and he gives a stronger notion of cardinal convexity, which is in fact also stronger than ordinal convexity, under which the greedy algorithm is valid. In light of the results of section 3, Sharkey's example means that we will need a stronger condition than weak generalized convexity to ensure that the greedy algorithm holds.

We will say that a game (N, v) is a **generalized convex game** if it is a weak generalized convex game which satisfies the additional property that for all coalitions S, T and T' such that $S \cap T = \emptyset, S \cap T' = \emptyset$, and $T \cap T' = \emptyset$, if x_S is unblocked by S, $(x_S, x_T) \in v(S \cup T)$ and $(x_S, x_{T'}) \in v(S \cup T')$, then $(x_S, x_T, x_{T'}) \in v(S \cup T \cup T')$. For games with side payments, both notions of generalized convexity are equivalent, and are simply equivalent to convexity.

Theorem 2. *A game with side payments is a generalized convex game if and only if it is a convex game.*

Proof: Since a generalized convex game is a weak generalized game by definition, it is also a convex game by Theorem 1. Now let (N, v) be a convex game, and let S,T, and T' be coalitions such $S \cap T = S \cap T' = T \cap T' = \emptyset$. Let x_S be unblocked by S, $(x_S, x_T) \in v(S \cup T)$ and $(x_S, x_{T'}) \in v(S \cup T')$.

Thus $\sum_{i\in S} x_i + \sum_{i\in T} x_i \le v(S\cup T)$ and $\sum_{i\in S} x_i + \sum_{i\in T'} x_i \le v(S\cup T')$. Adding these inequalities and using convexity implies that

$$2\sum_{i\in S} x_i + \sum_{i\in T} x_i + \sum_{i\in T'} \le v(S\cup T) + v(S\cup T') \le v(S\cup T\cup T') + v(S).$$

So

$$v(S\cup T\cup T') \ge \sum_{i\in S\cup T\cup T'} x_i + (\sum_{i\in S} x_i - v(S)) \ge \sum_{i\in S\cup T\cup T'} x_i$$

since x_S is unblocked by S. Thus $(x_S, x_T, x_{T'}) \in v(S\cup T\cup T')$. ■

Furthermore, the greedy algorithm is valid for these games, as the following theorem shows.

Theorem 3. *Suppose (N,v) is a generalized convex game. Let $x_S \in \mathrm{Core}(S,v)$ and $i \notin S$ and define $x_i = \max\{y : (x_S, y) \in v(S\cup\{i\})\}$. Then $(x_S, x_i) \in \mathrm{Core}(S\cup\{i\}, v)$.*

Proof: By construction of x_i, (x_S, x_i) is not blocked by the coalition $S' \equiv S\cup\{i\}$. To see this, suppose there exists $(y_S, y_i) \in v(S')$ such that $(y_S, y_i) \gg (x_S, x_i)$. Since $v(S')$ is downward comprehensive, $(x_S, y_i) \in v(S')$ and $y_i > x_i$, contrary to the definition of x_i.

Now suppose that the allocation is blocked by some coalition $T \subset S'$. If $i \notin T$, then x_S is blocked by T, which contradicts the fact that $x \in \mathrm{Core}(S,v)$. Suppose $i \in T$, and let $T' = T\setminus\{i\}$. Then $\exists(z_{T'}, z_i) \in v(T)$ such that $z_{T'} > x_{T'}$ and $z_i > x_i$. So $(x_{T'}, z_i) \in v(T'\cup\{i\})$, since v is downward comprehensive, which implies by generalized convexity that $(x_{S\setminus T}, x_{T'}, z_i) \in v(S\cup\{i\})$, contradicting the definition of x_i. Thus $x \in \mathrm{Core}(S\cup\{i\}, v)$. ■

This result means that simply iterating this procedure of giving each successive player her entire marginal contribution to the existing coalition of previous players will yield an allocation in the core, that is, the greedy algorithm provides a means for computing core allocations in generalized convex games.

Theorem 4. *Let (N,v) be a generalized convex game, and let x be the allocation constructed iteratively where*

$$x_i = \max\{y : (x_1, \ldots, x_{i-1}, y) \in v(\{1,\ldots,i\})\}.$$

Then $x \in \mathrm{Core}(N,v)$. In particular, $\mathrm{Core}(N,v)$ is nonempty.

Proof: By construction, $x_1 \in \mathrm{Core}(\{1\}, v)$. Applying Theorem 2, if $x_{\{1,\ldots,i\}} \in \mathrm{Core}(\{1,\ldots,i\}, v)$, then $x_{\{1,\ldots,i+1\}} \in \mathrm{Core}(\{1,\ldots,i+1\}, v)$. Hence, by induction, $x \in \mathrm{Core}(N,v)$. ■

Since the greedy algorithm can be carried out for any ordering of the players, the core in a generalized convex game provides a great deal of scope regarding how the returns to scale are shared among the players. In convex games, Sharkey (1981) showed that the core is **large** in the sense that for every unblocked vector y, there is some core allocation x such that $x \le y$. The core is also large in generalized convex games, as the following result verifies.

Theorem 5. *Let* (N, v) *be a generalized convex game. If* y *is unblocked in* (N, v)*, then there exists* $x \in \text{Core}(N, v)$ *such that* $x \leq y$*.*

Proof: Let the vector $y \in \mathbb{R}^N$ be unblocked in the game (N, v). Fixing y, define a new game (N, w), where

$$w(S) = \cup_{T \supset S}\{x_S : (x_S, y_{T \setminus S}) \in v(T)\}.$$

Note that $v(S) \subset w(S)$ for all $S \subset N$, and that $w(N) = v(N)$. Thus $\text{Core}(N, w) \subset \text{Core}(N, v)$. Moreover, if $x \in w(N)$ and $x_i > y_i$ for some player i, then x is blocked by the coalition $N \setminus \{i\}$, so if $x \in \text{Core}(N, w)$ then $x \leq y$. Thus it suffices to show that $\text{Core}(N, w)$ is nonempty. To show that $\text{Core}(N, w)$ is nonempty, we will show that (N, w) is a generalized convex game.

First, (N, w) is a weak generalized convex game. To see this, let S and R be disjoint coalitions and j a member of neither, and let $R' = R \cup \{j\}$. Suppose that x_S is acceptable to R in the game (N, w). We must show that x_S is acceptable to R'. Since x_S is acceptable to R, there exists x_R such that $(x_S, x_R) \in w(S \cup R)$ and such that there is no $z_R > x_R$ with $z_R \in w(R)$. So there exists $T \supset S \cup R$ such that $(x_S, x_R, y_{T \setminus (S \cup R)}) \in v(S \cup R \cup (T \setminus (S \cup R)))$. Define $x_j = \max\{\hat{x}_j : (x_S, x_R, \hat{x}_j) \in w(S \cup R')\}$ and $z_j = \max\{\hat{x}_j : (x_R, \hat{x}_j) \in w(R')\}$. Note that x_S is unacceptable to R' only if $z_j > x_j$. By the definitions of z_j and w, there exists $T' \supset R$ such that $(x_R, z_j, y_{T' \setminus R}) \in v(R \cup (T' \setminus R))$. Since (N, v) is a generalized convex game, $(x_S, x_R, z_j, y_{(T' \cup T) \setminus (R \cup S)}) \in v(T' \cup T \cup S)$. Then $x_j \geq z_j$, since by definition $(x_S, x_R, z_j) \in w(S \cup R')$. Hence x_S is acceptable to R, which implies that (N, w) is a weak generalized convex game.

To see that (N, w) is a generalized convex game, suppose S, R and R' are coalitions such that $S \cap R = S \cap R' = R \cap R' = \emptyset$. Let $(x_S, x_R) \in w(S \cup R)$ and $(x_S, x_{R'}) \in w(S \cup R')$. So there exist T, T' such that $(x_S, x_R, y_{T \setminus (S \cup R)}) \in v(T)$ and $(x_S, x_{R'}, y_{T' \setminus (S \cup R')}) \in v(T')$. Since (N, v) is a generalized convex game, $(x_S, x_R, x_{R'}, y_{(T \cup T') \setminus Q}) \in v(T \cup T')$, where $Q = S \cup R \cup R'$. So $(x_S, x_R, x_{R'}) \in w(S \cup R \cup R')$, and (N, w) is a generalized convex game. By Theorem 4, $\text{Core}(N, w)$ is nonempty. ■

Next, suppose that instead of a single cooperative game, we have a parameterized family of such games $(N, v(\cdot; t))$. In this family, we say that $\text{Core}(N, v(\cdot; t))$ is **ascending** in t if for all $s \leq t$ and every point in $x \in \text{Core}(N, v(\cdot; s))$, there exists a point $y \in \text{Core}(N, v(\cdot; t))$ with $y \geq x$. If in addition for every point $y \in \text{Core}(N, v(\cdot; t))$, there exists a point $x \in \text{Core}(N, v(\cdot; s))$ such that $y \geq x$, then we say that the $\text{Core}(N, v(\cdot; t))$ is **strongly ascending**. Ichiishi (1990) has shown that if the games in this family are convex games and if for all coalitions S and T with $S \subset T$, $v(T; t) - v(S; t)$ is nondecreasing in t, then the core is strongly ascending in t. As noted in the introduction, this condition is simply the requirement that the value function be supermodular in S and have increasing differences in $(S; t)$.

This analogy with comparative statics in optimization problems suggests that the appropriate generalization of this condition of increasing differences to an ordinal setting will be some version of the single crossing property. Building on this analogy, we will say that the parameterized family of generalized convex games $\{(N, v(\cdot; t)\}$ satisfies the **single crossing property** if for all $t' > t$, whenever x_S is acceptable to T in the game $(N, v(\cdot; t))$, then x_S is also acceptable to T in the game $(N, v(\cdot; t'))$. This notion is a natural extension of the single crossing property for optimization problems (Milgrom and Shannon (1994)). Moreover, in a parameterized family of generalized convex games satisfying the single crossing property, the core exhibits this monotone comparative statics property described above.

Theorem 6. *Suppose that* $\{(N, v(\cdot; t)\}$ *is a parameterized family of generalized convex games. If* $(N, v(\cdot; t))$ *satisfies the single crossing property, then* $\mathrm{Core}(N, v(\cdot; t))$ *is strongly ascending in* t.

Proof: Fix $x \in \mathrm{Core}(N, v(\cdot; t))$ and let $t' \geq t$. We must show that there exists $y \in \mathrm{Core}(N, v(\cdot; t'))$ such that $y \geq x$. Define $w(S) \equiv v(S, t') \cup \{z_S \in \mathbb{R}^S | z_S \leq x_S\}$. It is routine to verify that (N, w) is a generalized convex game. Also, using the single crossing property, $w(N) = v(N, t')$. Therefore, $\mathrm{Core}(N, w) \subset \mathrm{Core}(N, v(\cdot; t'))$ is nonempty. Also, for every $y \in \mathrm{Core}(N, w)$, $y \geq x$, for otherwise there is some coalition S such that $y_S \ll x_S \in w(S)$.

Now choose $y \in \mathrm{Core}(N, v(\cdot; t'))$. By the single crossing property, y is unblocked in $(N, v(\cdot; t))$, so because $\mathrm{Core}(N, v(\cdot; t))$ is large, there exists $x \in \mathrm{Core}(N, v(\cdot; t))$ such that $x \leq y$. So $\mathrm{Core}(N, v(\cdot; t))$ is strongly ascending. ∎

In the following sections, we show that the notions introduced in this section encompass both ordinal and cardinal convex games, and give an example involving public goods provision in which such ordinal complementarities among the players arise naturally.

3 Cardinally and Ordinally Convex Games

The notions of generalized convex games we present here are not the first extensions of the theory of convex games to games without side payments. At least two other extensions can be found in the literature: cardinally convex games introduced by Sharkey (1981), and ordinally convex games, introduced by Vilkov (1977). Both classes of games specialize to convex games in the transferable utility case, yet these classes are not nested, that is, one can find examples of games that are cardinally but not ordinally convex, and conversely, there are games that are ordinally but not cardinally convex (see Ichiishi (1991)).

In order to formally define cardinally convex games, let $v'(S) = v(S) \times \{0_{N \setminus S}\}$, so that the elements of $v'(S)$ are $|N|$-vectors formed from the elements of $v(S)$ by adding zeroes for the players who are not members of the coalition S. Then the game (N, v) is called **cardinally convex** if for all coalitions S and T, $v'(S) + v'(T) \subset v'(S \cap T) + v'(S \cup T)$. These games are always weak generalized convex games, as the following result shows.

Theorem 7. *Suppose that (N, v) is a cardinally convex game and that for all coalitions S, $v(S)$ is closed and convex. Then (N, v) is a weak generalized convex game.*

Proof: Let $A, B \subset N$, where $A \cap B = \emptyset$, and suppose x_A is acceptable to B. It suffices to show that for $n \notin A \cup B$, x_A is acceptable to $B \cup \{n\}$. Since x_A is acceptable to B, there exists x_B such that $(x_A, x_B) \in v(A \cup B)$ and there exists no $z_B \in v(B)$ such that $z_B > x_B$. Let $S = A \cup B$ and $T = B \cup \{n\}$. Let $x_n = \sup\{y_n | (x_A, x_B, y_n) \in v(S \cup T) \equiv v(A \cup B \cup \{n\})\}$. If x_A is unacceptable to T, then $\exists z_T \in v(T)$ such that $z_T > x_T \equiv (x_B, x_n)$. Since $v(B)$ is closed and convex and disjoint from $\{y_B | y_B > x_B\}$, there exists $\lambda_B > 0$ such that $\lambda_B \cdot x_B \geq \max\{\lambda_B \cdot y_B | y_B \in v(B)\}$. Let $\lambda_{N \setminus B} = 0$. Then

$$\max_{\substack{z \in v'(S)+v'(T) \\ z \geq (x_A, 0)}} \lambda \cdot z = \max_{\substack{x \in v'(S), y \in v'(T) \\ x+y \geq (x_A, 0)}} \lambda \cdot (x + y) = \max_{\substack{x \in v'(S) \\ x \geq (x_A, 0)}} \lambda \cdot x + \max_{y \in v'(T)} \lambda \cdot y$$

because $y \in v'(T)$ implies $y_A = 0$. Then by choice of λ,

$$\max_{\substack{x \in v'(S) \\ x \geq (x_A, 0)}} \lambda \cdot x \geq \max_{x \in v'(S \cap T)} \lambda \cdot x.$$

Moreover, because x_A is unacceptable to T,

$$\max_{y \in v'(T)} \lambda \cdot y > \max_{\substack{y \in v'(S \cup T) \\ y \geq x_A}} \lambda \cdot y.$$

This implies

$$\begin{aligned} \max_{\substack{z \in v'(S)+v'(T) \\ z \geq (x_A, 0)}} \lambda \cdot z &= \max_{\substack{x \in v'(S) \\ x \geq (x_A, 0)}} \lambda \cdot x + \max_{y \in v'(T)} \lambda \cdot y \\ &> \max_{x \in v'(S \cap T)} \lambda \cdot x + \max_{\substack{y \in v'(S \cup T) \\ y \geq x_A}} \lambda \cdot y = \max_{\substack{z \in v'(S \cup T)+v'(S \cap T) \\ z \geq x_A}} \lambda \cdot z. \end{aligned}$$

But this is a contradiction, since $v'(S) + v'(T) \subset v'(S \cap T) + v'(S \cup T)$. ■

Of course since the greedy algorithm may fail for cardinally convex games which are not ordinally convex, such games are not always generalized convex games.

In order to define ordinal convexity, construct the characteristic function $v''(S) \equiv v(S) \times \mathbb{R}^{N \setminus S}$. Thus $v''(S)$ is the cylinder set in $\mathbb{R}^N$ associated with $v(S)$. The game (N, v) is called **ordinally convex** if for all coalitions S and T, $v''(S) \cap v''(T) \subset v''(S \cap T) \cup v''(S \cup T)$. Ordinally convex games are also generalized convex games, as the following result shows.

Theorem 8. *If (N, v) is an ordinally convex game, then (N, v) is a generalized convex game.*

Proof: First, we show that (N, v) is a weak generalized convex game. To see this, let $A, B \subset N$, where $A \cap B = \emptyset$, and $n \notin A \cup B$. It suffices to show that there exists no y_A that is acceptable to B but not to $B \cup \{n\}$. Suppose there were. Then there exists $x_A < y_A$ that is unacceptable to

$B \cup \{n\}$ and such that for some $x_B \notin v(B)$, $(x_A, x_B) \in v(A \cup B)$. Let $S = A \cup B$ and $T = B \cup \{n\}$. Let $x_n \equiv \sup\{y_n : (x_A, x_B, y_n) \in v(S \cup T) \equiv v(A \cup B \cup \{n\})\}$. Since x_A is not acceptable to T, $\exists z_T \in v(T)$ such that $z_T > x_T \equiv (x_B, x_n)$. So $(x_A, x_B, z_n) \in v''(S) \cap v''(T)$ since $(x_A, x_B) \in v(S)$ and, by the downward comprehensiveness of $v(T)$, $(x_B, z_n) \in v(T)$. However, $x_B \notin v(B) = v(S \cap T)$ and by definition of x_n, $(x_A, x_B, z_n) \notin v(S \cup T)$, so $(x_A, x_B, z_n) \notin v''(S \cup T) \cup v''(S \cap T)$, contrary to the hypothesis of ordinal convexity. So (N, v) is a weak generalized convex game.

Now suppose x_S is unblocked by S, $(x_S, x_T) \in v(S \cup T)$, and $(x_S, x_{T'}) \in v(S \cup T')$, where $S \cap T = S \cap T' = T \cap T' = \emptyset$. Then

$$(x_S, x_T, x_{T'}) \in v''(S \cup T) \cap v''(S \cup T').$$

Then by ordinal convexity,

$$(x_S, x_T, x_{T'}) \in v''(S) \cup v''(S \cup T \cup T').$$

Since $x_S \notin v(S)$, this means that $(x_S, x_T, x_{T'}) \in v(S \cup T \cup T')$. Thus (N, v) is a generalized convex game. ■

4 Public and Private Goods Provision

Generalized convex games can arise from models involving both private and public goods provision. For example, Topkis (1987) studied side payment games in which each member n of a coalition S takes an individual action a_n to contribute to the creation of a divisible, transferable private good (e.g., money) which can then be shared among the players. More precisely, suppose there is some lattice X_n of feasible actions for player n with a minimum action denoted by 0, and suppose the value of coalition S is

$$\bar{v}(S) \equiv \max\{f(a_S, 0_{N \setminus S}) : (a_S, 0_{N \setminus S}), a_n \in X_n \text{ for } n \in S\}.$$

Topkis showed that if f is supermodular, so that for all action profiles a and a', $f(a) + f(a') \leq f(a \vee a') + f(a \wedge a')$, then the resulting game (N, v) is a convex game. Such a game is also a generalized convex game by Theorem 2.

Here, we study the polar case in which there is no possibility of side payments among agents, and the parties contribute personally costly effort to create a pure public good. Various special cases of this model have been studied in the literature on the core of economies with public goods (see Roberts (1974) and the references therein). Despite its quite different economic interpretation, this model, like the private goods model, leads to a generalized convex game.

Let $a_n \in \mathbb{R}_+$ denote the effort or personal resources contributed by individual n to creating the public good, and let p be the level of the public good provided. Individual n's utility given effort

a_n and level of public good p is denoted $u_n(a_n, p)$. The level of public good provided to coalition S is given by $p = f(a_S, 0_{N\setminus S})$. The characteristic function for the game is then defined by

$$v(S) = \{x_S \in \mathbb{R}^S | \exists a_S \text{ s.t. } x_i \leq u_i(a_i, f(a_S, 0_{N\setminus S}) \ \forall i \in S\}.$$

Since the appropriate subscript for the zero is always implied by the context, we will omit it below.

Theorem 9. *Suppose that f is a monotone nondecreasing function, and that for all players n, $u_n(a_n, p)$ is increasing in p and decreasing in a_n. Then (N, v) is a generalized convex game.*

Proof: Let $S \cap T = \emptyset$ and $n \notin S \cup T$. Suppose x_S is acceptable to T. Then there exist $\bar{a}_T$ and $\bar{a}_S$ for which the resulting utility levels are acceptable to T. Let $\bar{p}$ be the associated level of the public good for the coalition $S \cup T$. Define

$$x_n \equiv \max\{u_n(a_n, p) : u_j(a_j, p) \geq x_j \text{ for } j \in T, a_S \leq \bar{a}_S, \text{ and } p \leq f(a_S, a_T, a_n, 0)\}.$$

Observe that because $u_n(0, \bar{p})$ is attainable for n in this maximization and using the assumed properties of u_n and f, it follows that at the maximum, $p^* \geq \bar{p}$. Then using this fact and the assumptions about utility, it follows that at the optimum, $u_j(a_j, p^*) \geq x_j$ for $j \in S$. Hence, $(x_S, x_T, x_n) \in v(S \cup T \cup \{n\})$. It remains to show that there is no $(z_n, z_T) \in v(T \cup \{n\})$ such that $(z_n, z_T) > (x_n, x_T)$. Indeed, suppose $z_T \geq x_T$. Then

$$\begin{aligned} z_n &\leq \max\{u_n(a_n, p) | u_j(a_j, p) \geq z_j \text{ for } j \in T, a_S = 0, p \leq f(a_S, a_T, a_n, 0)\} \\ &\leq x_n \end{aligned}$$

by definition of x_n. Thus (N, v) is a weak generalized convex game.

To see that (N, v) is a generalized convex game, let $S \cap T = S \cap T' = T \cap T' = \emptyset$, and suppose $(x_S, x_T) \in v(S \cup T)$ and $(x_S, x_{T'}) \in v(S \cup T')$. So $\exists (a_S, a_T)$ and $(\hat{a}_S, a_{T'})$ such that $x_i \leq u_i(a_i, f(a_S, a_T))$ for all $i \in S \cup T$, $x_i \leq u_i(\hat{a}_i, f(\hat{a}_S, a'_T))$ for all $i \in S$, and $x_i \leq u_i(a_i, f(\hat{a}_S, a_{T'}))$ for all $i \in T'$. Consider the effort profile $(a_S \vee \hat{a}_S, a_T, a_{T'})$, where $a_S \vee \hat{a}_S = \{\max\{a_i, \hat{a}_i\}\}_{i \in S}$. Then for $i \in T \cup T'$, $x_i \leq u_i(a_i, f(\hat{a}_S \vee a_S, a_T, a_{T'}))$ since f is nondecreasing and u_i is nondecreasing in p. For $i \in S$, $a_i \vee \hat{a}_i \in \{a_i, \hat{a}_i\}$. Suppose $a_i \vee \hat{a}_i = a_i$. Then since f is nondecreasing,

$$\begin{aligned} x_i &\leq u_i(a_i, f(\hat{a}_S \vee a_S, a_T, a_{T'})) \\ &= u_i(\hat{a}_i \vee a_i, f(\hat{a}_S \vee a_S, a_T, a_{T'})). \end{aligned}$$

If $a_i \vee \hat{a}_i = \hat{a}_i$, a similar argument shows that $x_i \leq u_i(\hat{a}_i, f(\hat{a}_S \vee a_S, a_T, a_{T'}))$. This implies that $(x_S, x_T, x_{T'}) \in v(S \cup T \cup T')$. ■

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