# AN ORDINAL THEORY OF GAMES WITH STRATEGIC COMPLEMENTARITIES

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## 1. Introduction

Many economic situations are characterized by the existence of some type of complementarity, whether it be between the actions of players in a game, the outputs of firms in an industry, or the products of several different industries. When goods in a market are substitutes, a price increase by several other firms may make it more profitable for a given firm to increase its price as well. In a model of bank runs in which a player must decide whether to withdraw funds, withdrawals by other customers may make it more profitable for him to withdraw as well. Technological innovation, in which many products are better thought of as systems, provides a wealth of further examples, as improvements in one part of the system often become more profitable with improvements in other parts of the system. For example, the introduction of steel rails made possible the use of longer trains which traveled faster and carried heavier loads, which was even more profitable with similar improvements in braking technology. Similarly, the development of more powerful engines and faster automobiles became even more profitable with the development of better brakes and safety equipment, as well as better roads.[1] Indeed, Rosenberg (1979) argues that the early industrial revolution can only be understood in terms of "the interactions of a few basic technologies that provided the essential foundation for other technological changes in a series of ever-widening concentric circles..." (Rosenberg, 1979, p.29). Moreover, complementarities characterize many macroeconomic models of multiple equilibria and coordination failure, as in Cooper and John (1988) and Heller (1986).

Several authors have explored the importance of notions of strategic complementarity in economic models, which is defined by Bulow et. al. as the situation wherein a more aggressive strategy by one player increases the marginal, rather than total, profit or payoff to other players (Bulow et. al., 1985). Some of the first work on types of functions which give rise to such strategic complementarity in games was done by Topkis. He develops many results on submodular functions and submodular optimization, as well as developing results on equilibria in certain games with submodular payoff functions (Topkis, 1976, 1978, 1979). Work on submodularity and submodular optimization, as well as related topics in lattice programming, has also been done by Veinott (1989). Furthermore, Vives (1989) and Sobel (1989) both discuss supermodular games, and give results on the existence of equilibria in supermodular games, and the existence of monotone equilibrium selections

[1] These examples are found in Rosenberg (1979); for a more in depth discussion, see his article.

in such games, as well as giving several examples of supermodular games in an economic setting. Lippman, Mamer, and McCardle (1987) use transfinitely iterated play to analyze equilibria and comparative statics in games with monotone increasing composite best reply functions, which is one of the implications of supermodularity.

Milgrom and Roberts (1989a) develop quite strong results in a model of modern manufacturing which exhibits strategic complementarity via supermodularity, and then give a relatively comprehensive theory of supermodular games in a following paper, detailing a number of strong results which derive from relatively weak order- and lattice-theoretic notions (Milgrom and Roberts, 1989b). They show that in any supermodular game, not only does a Nash equilibrium always exist, but a closed interval can be found whose endpoints are the smallest and largest serially undominated strategy profiles in the game, as well as the smallest and largest Nash equilibria in the game. Moreover, these endpoints represent monotone selections from the equilibrium correspondence. They also give comparative statics and welfare results for such games, as well as a large number of examples demonstrating how these theorems can be applied to yield interesting new results.

A common theme to most of this work, however, is the inherently cardinal nature of the underlying assumptions which drive the results, whereas the results themselves are inherently ordinal. Indeed, as related by Topkis (1978, pp. 310-11), Samuelson criticized such a definition of complementarity as applied to consumer decision making on the grounds that strictly monotone transformations in utility may not preserve the property of complementarity thus defined (Samuelson, 1947). Moreover, in many games, the issue is not so much the actual number which is the payoff to a player as it is the actions induced by comparisons of payoffs under different strategies, so that as in the consumer decision problem, it is really only the ordinal information contained in the payoff function which is of interest. This suggests that an ordinal theory of complementarity would not only be powerful, in that one could hope to recover all of the results known for supermodularity, but important as well, in that while encompassing supermodular games, potentially many more situations not previously covered could be analyzed using these powerful tools.

This paper develops an ordinal theory of strategic complementarity for which all of the existing results for supermodular games carry over, and discusses several methods of characterizing this class of games. Examples of games in this class are developed, including certain Bertrand oligopoly models with convex cost functions, and a pure exchange general equilibrium model with gross substitutes.

The paper proceeds as follows: section 2 details some preliminary results, including most of the relevant definitions and some results on optimization of functions with these ordinal concepts of complementarity; section 3 develops the theory of games which exhibit this type of complementarity; section 4 discusses methods of characterizing this class of games; section 5 presents several examples, and section 6 concludes.

## 2. Preliminaries

Before turning to the ordinal theory of games with strategic complementarity, some definitions and preliminary results are required. Let $X$ be a partially ordered set, with the transitive, reflexive, antisymmetric order relation $\geq$. Then recall the following order theoretic notions:

$x \in X$ is a **maximal element** of $X$ if $\nexists y \in X$ such that $y > x$.

$x \in X$ is the **largest element** of $X$ if $x \geq y \;\; \forall y \in X$.

**Minimal element** and **smallest element** are of course defined analogously.

Given $x, y \in X$, denote by $x \vee y$ the least upper bound of $x$ and $y$ (if it exists), and by $x \wedge y$ the greatest lower bound of $x$ and $y$ (if it exists). $X$ is then a **lattice** if $\forall x, y \in X,\ x \vee y \in X$ and $x \wedge y \in X$; thus $X$ is a lattice iff $\forall x, y \in X,\ x \vee y$ and $x \wedge y$ exist and are elements of $X$.

A lattice $X$ is **complete** if for every nonempty subset $Z \subset X$, $inf(Z) \in X$ and $sup(Z) \in X$.[2] A subset $Z$ of $X$ which is closed under the operations $\vee$ and $\wedge$ is a **sublattice.** A subset $Z$ of $X$ is a **quasisublattice** if $\forall x, y \in Z$, either $x \vee y \in Z$ or $x \wedge y \in Z$. Perhaps the simplest example of a complete lattice is $S = [0,1] \times [0,1]$; $\{0,1\} \times \{0,1\}$ is a complete sublattice of $S$, and $(0,0) \cup (0,1) \cup (1,0)$ is a quasisublattice of $S$ which is not a sublattice.

With these basic notions of the order structure of the spaces in question, some notions of the behavior of functions on such spaces are possible. Let $C \subset X$ be a chain, and let $c^* = sup(C)$, $c_* = inf(C)$. Then if $X$ is a complete lattice, $f : X \rightarrow \mathbf{R}$ is **order continuous** iff $\forall$ chain $C \subset X$,

$$lim_{x \in C,\ x \downarrow c_*} f(x) = f(c_*)$$

[2] Completeness is also equivalent to the lattice being compact in the interval topology; see Birkhoff (1967) or Frink (1942).

and

$$lim_{x \in C,\ x \uparrow c^*} f(x) = f(c^*).$$

$f$ is **order upper semi-continuous** iff $\forall$ chain $C \subset X$,

$$lim \quad sup_{x \in C,\ x \downarrow c_*} f(x) \leq f(c_*)$$

and

$$lim \quad sup_{x \in C,\ x \uparrow c^*} f(x) \leq f(c^*)$$

Finally, let $S$ be a lattice. Then $f : S \rightarrow \mathbf{R}$ is **supermodular** if $\forall x, y \in S$,

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

This property has been studied under various pseudonyms by Veinott (1989), Topkis (1978, 1979), Milgrom and Roberts (1989b), and others.

As discussed above, one of the main goals of this work is to explore the ordinal underpinnings of the theory of supermodular games and supermodular optimization, with the hope of developing a more general theory which encompasses the inherently ordinal results already in the literature. With this goal in mind, then, a closer examination of the ordinal implications of the cardinal property of supermodularity is in order.

Suppose $f : S \rightarrow \mathbf{R}$ is supermodular. Then if

$$f(x) \geq f(x \vee y),$$

by supermodularity it must necessarily be the case that

$$f(x \wedge y) \geq f(y).$$

This is clearly an ordinal implication of supermodularity; however, it is not sufficient to guarantee one of the main results on monotonicity (Proposition 1), as will be seen below. The remaining ordinal information contained in supermodularity is analogous:

$$\begin{aligned} & f(y) \geq f(x \wedge y) \\ \Rightarrow\ & f(x \vee y) \geq f(x) \end{aligned}$$

or, reformulated using the contrapositive,

$$\begin{aligned} & f(x) > f(x \vee y) \\ \Rightarrow\ & f(x \wedge y) > f(y). \end{aligned}$$

Then $f : S \to \mathbf{R}$ will be said to be **quasisupermodular (qsm)** iff $\forall x, y \in S$,

$$\begin{aligned} & f(x) \geq f(x \vee y) \\ \Rightarrow & f(x \wedge y) \geq f(y) \end{aligned}$$

and

$$\begin{aligned} & f(x) > f(x \vee y) \\ \Rightarrow & f(x \wedge y) > f(y). \end{aligned}$$

A similar cardinal condition which is important both in the existing literature on supermodular games (see Topkis, 1978, and Milgrom and Roberts, 1989b, for example) and in the present work, is increasing differences. Let $S = S_1 \times S_2$, where $S_1, S_2$ are lattices and $S$ is endowed with the product or component-wise ordering. Then $f : S \to \mathbf{R}$ exhibits **increasing** (or **isotone** ) **differences** in $(x, y)$ iff $\forall x' \geq x$, $f(x', y) - f(x, y)$ is nondecreasing in $y$. Note that this definition is symmetric in $(x, y)$; that is, this is equivalent to the condition that $f(x, y') - f(x, y)$ be nondecreasing in $x$.

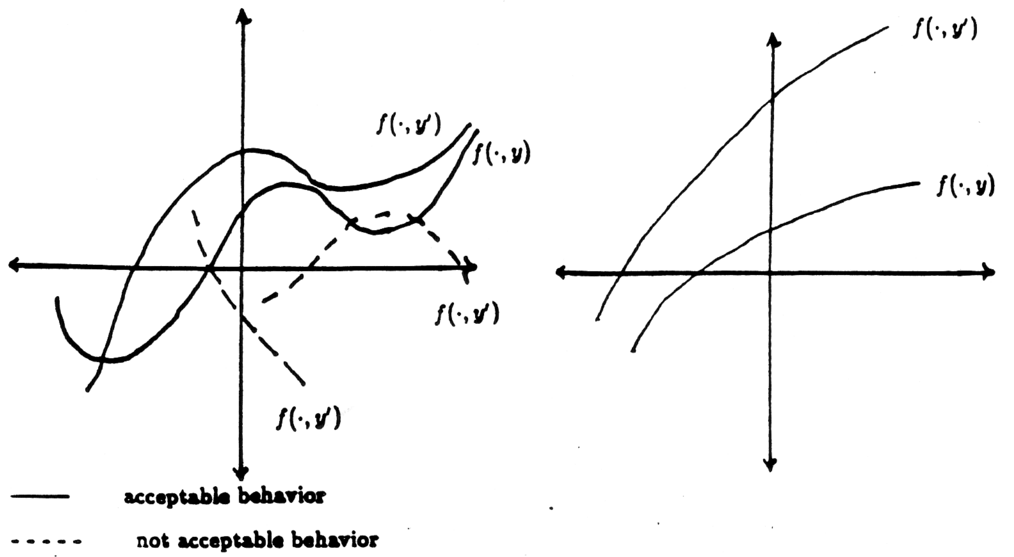
When $f$ if taken to be the payoff function of a player in a noncooperative game, $x$ that player's strategy variable, and $y$ the strategy variables of his opponents, increasing differences implies the existence of a type of strategic complementarity: facing an increase in the strategy variables of his opponents, the player finds a given increase in his strategy variable *more* profitable at the higher level chosen by his opponents than at the prior, lower level. In other words, the player finds that increasing his strategy variable is even more profitable to him if his opponents also increase their strategy variables.

The corresponding ordinal notion which is explored in this paper carries the (ordinal) interpretation that an increase in a player's strategy variable which is profitable at some level of the opponents' strategy variables is also profitable at any higher level chosen by the opponents. This is a weaker notion of complementarity than increasing differences, as increasing differences requires increases in a player's strategy variable to become more profitable when other players also increase, yet this ordinal notion, while requiring that any profitable increase *remain* profitable when other players increase, makes no statement about the degree of profitability. In the ordinal world, a player may find that the additional return from increasing his strategy variable, while remaining positive, could decrease under certain increases made by his opponents, yet this could never happen in a world of increasing differences.

Formally, $f : S \to \mathbf{R}$ has the **single crossing property (scp)** in $(x, y)$ iff

$$x' \geq x, \quad f(x', y) - f(x, y) \geq \ (>)\ 0$$
$$\Rightarrow f(x', y') - f(x, y') \geq \ (>)\ 0 \qquad \forall y' \geq y.$$

Some pictures might be useful at this point to clarify the concepts.




The picture on the left displays behavior which is acceptable for a function with the single crossing property as well as behavior which is not acceptable, explaining the name given to the condition. The picture on the right displays the behavior required of a function with increasing differences, and highlights the difference in character imposed by these two properties.

With these definitions in place, it is possible to begin establishing the properties of functions which are quasisupermodular, or have the single crossing property. First, such functions exhibit a type of monotonicity.

**Proposition 1.** *Let $S_1$ be a lattice, and $S_2$ be a partially ordered set. Let $f : S_1 \times S_2 \to \mathbf{R}$ be qsm in $x$ for $y$ fixed, and let $f$ have the scp in $(x, y)$. Let $y' \geq y$, and define $M \equiv argmax_{x \in S_1} f(x, y)$, $M' \equiv argmax_{x \in S_1} f(x, y')$. Let $x \in M$, $x' \in M'$. Then $x \wedge x' \in M$, and $x \vee x' \in M'$. Hence $M$ is a sublattice.*

**Proof:** Consider $x \vee x'$:

$x \in M \Rightarrow$

$$f(x, y) \geq f(x \wedge x', y)$$

$\Rightarrow$ by qsm

$$f(x \vee x', y) \geq f(x', y)$$

But $x \vee x' \geq x'$ and $y' \geq y$, so by the single crossing property,

$$f(x \vee x', y') \geq f(x', y').$$

$\Rightarrow x \vee x' \in M'$.

Consider $x \wedge x'$:

$x' \in M' \Rightarrow$

$$f(x', y') \geq f(x \vee x', y')$$

$x \vee x' \geq x'$, so by the single crossing property,

$$f(x \vee x', y) - f(x', y) \geq \ (>) \ 0$$
$$\Rightarrow f(x \vee x', y') - f(x', y') \geq \ (>) \ 0.$$

But

$$f(x \vee x', y') - f(x', y') \not> 0$$

Hence

$$f(x', y) - f(x \vee x', y) \geq 0$$

or

$$f(x', y) \geq f(x \vee x', y)$$

thus by qsm,

$$f(x \wedge x', y) \geq f(x, y)$$

$\Rightarrow x \wedge x' \in M$.

Letting $y = y'$ gives $\forall x, x' \in M$, $x \vee x' \in M$ and $x \wedge x' \in M$, i.e., $M$ is a sublattice. ∎

$M$ is then said to be **lower than** $M'$, denoted $M \leq_s M'$, a relation introduced by Veinott (see Topkis, 1978, Veinott, 1989). A result similar to Proposition 1 also holds for certain types of constrained optimization, as the following corollary shows.

**Corollary.** *Let $f$ be as in Proposition 1, and suppose $T(y) \leq_s T(y')$, for $y' \geq y$. Define $M \equiv argmax_{x \in T(y)} f(x,y)$, and $M' \equiv argmax_{x \in T(y')} f(x,y')$. Let $x \in M$ and $x' \in M'$. Then $x \wedge x' \in M$ and $x \vee x' \in M'$.*

The corollary is proved in the same manner as the proposition, using the relation $\leq_s$ to establish that if $x \in M$ and $x' \in M'$, then $x \wedge x' \in T(y)$ and $x \vee x' \in T(y')$.

It is to ensure that this proposition and corollary hold that the full ordinal content of supermodularity is required in the definition of quasisupermodularity.[3] The above proposition then shows that if $f(\cdot, y)$ is qsm for a given $y$, the set of maximizers of $f(\cdot, y)$ is a sublattice. However, it does not guarantee that this is a nonempty sublattice. A logical question is then when is $M = argmax_{x \in S_1} f(\cdot, y)$ nonempty? Conditions under which $M$ is nonempty arise from the following theorem due to Veinott (1989), which requires an additional definition.

Let $L^\alpha = \{x \in S : f(x) \geq \alpha\}, \quad \alpha \in \mathbf{R}$. Then $f : S \to \mathbf{R}$ is **upper chain subcomplete** iff $\forall \alpha \in \mathbf{R}$, $L^\alpha$ is chain subcomplete, i.e., iff $\forall$ chain $C \subset L^\alpha$, $sup(C) \in L^\alpha$ and $inf(C) \in L^\alpha$.

**Theorem (Veinott).** *If $f : S \to \mathbf{R}$ is upper chain subcomplete, $S$ is a complete lattice, and $L^\alpha$ is a quasisublattice $\forall \alpha \in \mathbf{R}$, then $f$ attains its maximum on $S$.*

The existence result alluded to above then follows as a corollary of this theorem, by showing that requiring $f$ to be qsm and order upper semi- continuous implies that $f$ is upper chain subcomplete and that $L^\alpha$ is a quasisublattice $\forall \alpha \in \mathbf{R}$.

**Proposition 2.** *If $f : S \to \mathbf{R}$ is qsm and order upper semi- continuous, and $S$ is a complete lattice, then $f$ attains its maximum on $S$.*

**Proof:** First, $f$ qsm $\Rightarrow \forall \alpha \in \mathbf{R}$, $L^\alpha$ is a quasisublattice:
Let $\alpha \in \mathbf{R}$, and $x, y \in L^\alpha$. Then $f(x) \geq \alpha$, and $f(y) \geq \alpha$.
Suppose $x \vee y \notin L^\alpha$, so that $f(x \vee y) < \alpha$. Then

$$f(x) \geq \alpha > f(x \vee y)$$

[3] A function which only satisfies the first ordinal implication of supermodularity as detailed above may fail to have its set of maximizers be a sublattice: consider the lattice $[0,1] \times [0,1]$, and the function $f(x_1, x_2) = 1 - min(x_1, x_2)$.

$\Rightarrow$ by qsm

$$f(x \wedge y) \geq f(y) \geq \alpha$$

i.e., $x \wedge y \in L^{\alpha}$.

Thus $L^{\alpha}$ is a quasisublattice.

Furthermore, if $f$ is order upper semi-continuous, $f$ is upper chain subcomplete:

Let $C \subset L^{\alpha}$. Then $\forall x \in C, f(x) \geq \alpha$. Thus order upper semi-continuity $\Rightarrow$

$$\alpha \leq lim\ sup_{x \in C, x \downarrow c_*} f(x) \leq f(c_*).$$

$\Rightarrow c_* \in L^{\alpha}$.

Similarly,

$$\alpha \leq lim\ sup_{x \in C, x \uparrow c^*} f(x) \leq f(c^*).$$

$\Rightarrow c^* \in L^{\alpha}$.

Therefore by the theorem cited above, $f$ attains its maximum on $S$. ■

It should be noted that as pointed out by Milgrom and Roberts (1989b), it is certainly not true that every bounded, order upper semi-continuous function from a complete lattice to $\mathbf{R}$ attains its maximum. For a counterexample, see Milgrom and Roberts (1989b).

The preceding two propositions serve to establish that if $f : S \rightarrow \mathbf{R}$ is qsm and order upper semi-continuous, where $S$ is a complete lattice, $M \equiv argmax_{x \in S} f(x)$ is a nonempty sublattice of $S$. Moreover, $M$ is actually a complete sublattice of $S$, which is established by the next proposition.

**Proposition 3.** *Let $f$ be an order upper semi-continuous function from a complete lattice $S$ to $\mathbf{R}$. If $M$ is a nonempty sublattice of $S$, then $M$ is a complete sublattice of $S$.*

The proof of this proposition follows from the proof of Theorem 2, Milgrom and Roberts (1989b).

This proposition shows in particular, since $M$ is a nonempty subset of itself, that under certain conditions, there exist greatest and least elements of the set $M$ of maximizers of a qsm function $f$. If $f(x_n, x_{-n})$ is the payoff function of player $n$ in a game, and $f(\cdot, x_{-n})$ is qsm for each $x_{-n}$ and satisfies the conditions of Propositions 1-3, then the preceding three propositions tell us that the player's best reply correspondence is nonempty, complete sublattice valued. In particular, to each strategy profile $x_{-n}$ of the opponents, the player always has a greatest and a least best response among his strategies. Furthermore, these largest and smallest best responses move monotonicly in $x_{-n}$, so that if $x'_{-n} \geq x_{-n}$, then $x_n^*(x'_{-n}) \geq x_n^*(x_{-n})$, and similarly for $x_{n*}$.

These results on the nature of qsm functions and their sets of maximizers will provide the foundation for the ordinal theory of games which exhibit this type of strategic complementarity, developed in the next section.

## 3. Games With the Single Crossing Property

Consider the following general environment in which a certain class of games will be defined. A nonempty set $N$ indexes the players in the game, and each player $n \in N$ chooses his strategies from a set $S_n$, which is partially ordered by $\geq_n$. The space of strategy profiles is then $S \equiv \prod_{n=1}^{N} S_n$, partially ordered by the product or component-wise ordering. Each player $n \in N$ has a payoff function $f_n(x_n, x_{-n})$. The combination $\Gamma = \{N, (S_n, f_n)_{n \in N}, \geq\}$ is called a game in ordered normal form by Milgrom and Roberts (1989b). The game $\Gamma$ will be called a **game with the single crossing property (scp game)** if $\forall n \in N$:

(1) $S_n$ is a complete lattice.

(2) $f_n : S \to \mathbf{R} \cup \{-\infty\}$ is order upper semi- continuous in $x_n$ for $x_{-n}$ fixed, and order continuous in $x_{-n}$ for fixed $x_n$.

(3) $f_n$ is qsm in $x_n$ for $x_{-n}$ fixed.

(4) $f_n$ has the scp in $(x_n, x_{-n})$.

A player in a game with the single crossing property is then assumed to experience a weak sort of strategic complementarity, both within her own strategy variables, and with her opponents. The requirement that $f_n$ be qsm in $x_n$ for given $x_{-n}$ expresses this idea of complementarity between a player's own strategy variables, while the requirement that $f_n$ have the scp in $(x_n, x_{-n})$ expresses the complementarity between players, so that if a player ever finds an increase in strategy profitable, she will also find that increase to be profitable under an increase in the opponents' strategies. The requirement that $f_n$ be qsm in $x_n$ for fixed $x_{-n}$ is often not very restrictive, however. An examination of the inequalities defining qsm shows that any function whose domain is $\mathbf{R}$, or any chain $C$ for that matter, is automatically qsm, as the order notions $\vee$, $\wedge$ are then trivial. Hence whenever the strategy variable of each player is real-valued, or lies in some chain, (3) becomes vacuous.

Once the particular characteristics of a game, such as the player set and payoff functions, have been specified, natural questions arise concerning the existence and nature of various equilibrium notions. The potential power of the relatively weak order notions of order continuity, order upper semi-continuity, qsm, and the single crossing property is displayed in the next several propositions. If the game in question is a game with the single crossing property, then as the following proposition will establish, not only does a Nash equilibrium always exist in the game, but in fact there exists an interval $[x_*, x^*] \subset S$ such that $x_*$ and $x^*$ are the smallest and largest serially undominated strategy profiles. The interval $[x_*, x^*]$ then contains all of the serially undominated strategies; i.e., those strategies which remain after the iterated removal from the strategy space of each player all pure strategies which are strongly dominated. Moreover, both $x_*$ and $x^*$ are Nash equilibrium profiles, and hence they are the smallest and largest Nash equilibrium profiles. A game with the single crossing property generates a closed interval that not only strictly bounds the set of serially undominated strategies in the sense that the endpoints of the interval are themselves serially undominated strategy profiles, but also necessarily bounds both the set of possible pure and mixed strategy Nash equilibria, and the set of possible correlated equilibria.[4] Furthermore, these bounds coincide with the actual bounds on the set of Nash equilibria in the game. Formally, the result is given in the following proposition.

**Proposition 4.** *Let $\Gamma$ be a scp game. Then $\forall n \in N$, there exist strategies $x_{n*}$ and $x_n^*$ which are the smallest and largest serially undominated strategies for that player. Moreover, the pure strategy profiles $x_* \equiv (x_{n*}; n \in N)$ and $x^* \equiv (x_n^*; n \in N)$ are Nash equilibria.*

Before discussing the proof of this proposition, more notation and a lemma are required. Given $x \in S$, let $B_{n*}(x)$ denote the smallest best response of player $n$ to $x_{-n}$, and $B_n^*(x)$ denote the largest best response to $x_{-n}$ in a game with the single crossing property, which are well-defined by Proposition 3. Let $B_*(x) \equiv (B_{n*}(x); n \in N)$ and $B^*(x) \equiv (B_n^*(x); n \in N)$. For $T \subset S$, define

$$U_n(T) \equiv \{x_n \in S_n : \forall x_n' \in S_n, \quad \exists \hat{x} \in T \quad s.t. \quad f_n(x_n, \hat{x}_{-n}) \geq f_n(x_n', \hat{x}_{-n})\}$$

Then $U_n(T)$ represents the strategies of player $n$ which are not strongly dominated when the player faces strategies in $T$. Let $U(T) \equiv (U_n(T); n \in N)$, and $\bar{U}(T) \equiv [inf\{U(T)\}, sup\{U(T)\}]$.

[4] It is well known that pure and mixed strategy Nash equilibria must be serially undominated, and that only serially undominated strategies will be played with positive probability at any correlated equilibrium.

**Lemma.** *Under the conditions for a scp game, let* $z_*$, $z^* \in S$ *be such that* $z_* \leq z^*$. *Then* $sup\{U([z_*, z^*])\} = B^*(z^*)$ *and* $inf\{U([z_*, z^*])\} = B_*(z_*)$; *hence*

$$\bar{U}([z_*, z^*]) = [B_*(z_*), B^*(z^*)].$$

**Proof:** By definition, $B^*(z^*)$, $B_*(z_*) \in U[z_*, z^*]$, hence $[B_*(z_*), B^*(z^*)] \subset \bar{U}([z_*, z^*])$. Suppose $z \notin [B_*(z_*), B^*(z^*)]$. Then $z \notin U([z_*, z^*])$. To see this, suppose $z_n \not\leq z_n^* \equiv B_n^*(z^*)$. Then let $x \in [z_*, z^*]$. By the scp,

$$f_n(z_n, x_{-n}) - f_n(z_n \wedge z_n^*, x_{-n}) \geq 0$$
$$\Rightarrow f_n(z_n, z_{-n}^*) - f_n(z_n \wedge z_n^*, z_{-n}^*) \geq 0$$

By qsm, if the last line above holds, then

$$f_n(z_n \vee z_n^*, z_{-n}^*) - f_n(z_n^*, z_{-n}^*) \geq 0.$$

But $z_n \vee z_n^* > z_n^*$, so by the definition of $z_n^*$

$$f_n(z_n \vee z_n^*, z_{-n}^*) - f_n(z_n^*, z_{-n}^*) < 0.$$

Thus

$$f_n(z_n, x_{-n}) - f_n(z_n \wedge z_n^*, x_{-n}) < 0.$$

Thus $z_n \wedge z_n^*$ strongly dominates $z_n$ against every $x \in [z_*, z^*]$. Hence $z \notin U([z_*, z^*])$. A similar argument shows that if $z_n \not\geq z_{n*} \equiv B_{n*}(z_*)$ for some $n$, then $z_n \vee z_{n*}$ strongly dominates $z_n$ against strategies in $[z_*, z^*]$. Therefore $\bar{U}([z_*, z^*]) = [B_*(z_*), B^*(z^*)]$. ■

With the establishment of this lemma, the proof of the proposition follows exactly as the proof of Theorem 5, Milgrom and Roberts (1989b), relying solely on the preceding lemma, the monotonicity of the operator $U$, the definition of serially undominated strategies, and order continuity. It is interesting to note that this proof is actually constructive, generating an algorithm for finding $x_*$ and $x^*$, and hence for finding a pure strategy Nash equilibrium in the scp game $\Gamma$.

Although there are certainly cases where the result described by this proposition is vacuous or nearly so, as when $[x_*, x^*] = S$, there are many instances when the implications are quite striking. First, as stated above, this proposition implies that for any scp game $\Gamma$, there exists a pure strategy Nash equilibrium; indeed, there exists a largest and smallest one. If the game in question has a unique pure strategy Nash equilibrium, it is actually dominance solvable; that is, the iterated elimination of strongly dominated strategies leaves each player with a singleton, and the resulting strategy profile composed of these singletons is the unique pure strategy Nash equilibrium. Finally, if the scp game $\Gamma$ is symmetric in $N$, and has a unique symmetric pure strategy Nash equilibrium, then as noted by Milgrom and Roberts (1989b), it is dominance solvable as well. This follows from the line of reasoning used above by noting that if $\Gamma$ is symmetric in $n$, then $x_{n*} = x_{1*}$ $\forall n \in N$, and $x_n^* = x_1^*$ $\forall n \in N$, hence $x_*$ and $x^*$ are symmetric pure strategy Nash equilibria, which must by uniqueness be equal.

The last result in this section concerning the theory of games with the single crossing property develops some of the properties of comparative statics in such games. More precisely, suppose the payoff functions of players are parameterized by $\tau \in T$, where $T$ is a partially ordered set. Under the additional assumption

(5) $\quad f_n(x_n, x_{-n}; \tau)$ has the scp in $(x_n, \tau)$ for $x_{-n}$ fixed,

the following proposition holds.

**Proposition 5.** *Let $\Gamma_\tau = \{N, S_n, f_n(x_n, x_{-n}, \tau), \geq\}$ be a family of games with the scp satisfying (5), where $\tau \in T$ and $T$ is a partially ordered set. Then $x_{n*}(\tau)$ and $x_n^*(\tau)$ are nondecreasing functions of $\tau$.*

Again, the proof follows as the proof of the analogous result in Milgrom and Roberts (1989b). The preceding results have established that the strong properties of supermodular games all carry over to the purely ordinal notion of games with the single crossing property. Several issues which remain to be addressed relate to the breadth and importance of this class of games. The following section deals with the question of breadth by considering the nature of quasisupermodularity and the single crossing property, and possible characterizations of these conditions.

## 4. Characterizing Quasisupermodularity and the Single Crossing Property

It is in searching for conditions which are necessary and/or sufficient to guarantee that a function $f : S \to \mathbf{R}$ be qsm that one comes to realize how much more general and encompassing quasisupermodularity is in relation to the cardinal notion of supermodularity. This increased generality is both a help and a hinderance, in that while increasing the scope of the results pertaining to these games, the difficulty of determining when games fall into the class of scp games may increase as well.

Before discussing various characterizations of qsm, however, it should first be demonstrated that qsm is not a trivial ordinal extension of supermodularity, in the sense that the class of qsm functions is much larger than the set of strictly monotone increasing transformations of supermodular functions. Indeed, it is clear from the definition of qsm that if $S$ is a lattice and $f : S \to \mathbf{R}$ is supermodular, then $g \circ f$ is qsm for any strictly monotone increasing function $g : \mathbf{R} \to \mathbf{R}$, fortunately. However, given a qsm function $h : S \to \mathbf{R}$, there does not always exist a strictly monotone increasing function $g : \mathbf{R} \to \mathbf{R}$ such that $g \circ h$ is supermodular, as the following example illustrates. Consider a function which takes values as indicated on the lattice $\{0, 1\} \times \{0, 1, 2, 3\}$ with the usual order on $\mathbf{R}$ in each chain.

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ 3 & 4 & 5 & 3 \\ & & & \\ \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 2 & 1 \end{array}$$

Then $h$ is qsm, yet if $g : \mathbf{R} \to \mathbf{R}$ is strictly monotone increasing and $g \circ f$ is supermodular, then

$$g(3) + g(2) \leq g(4) + g(1)$$
$$g(5) + g(1) \leq g(3) + g(2)$$

Hence $g(5) \leq g(4)$, which is a contradiction.

Furthermore, the above example also shows that given a qsm function $h$, there does not always exist a transformation $g : \mathbf{R} \times \mathrm{S} \to \mathbf{R}$, where $g$ is strictly increasing in the first argument, such that $g(h(x), x_{-n})$ is supermodular. To see this, note that on a product of discrete sets, such a transformation is simply a collection of row-specific, strictly monotone increasing transformations $g_1$ and $g_2$. Then supermodularity requires

$$g_1(2) + g_2(3) \leq g_1(1) + g_2(4)$$
$$g_1(1) + g_2(5) \leq g_1(2) + g_2(3)$$

But this implies $g_2(5) \leq g_2(4)$, which is a contradiction.

Therefore, at least in the sense given above, the class of qsm functions is not merely a trivial extension of the notion of supermodularity. In fact, the class of qsm functions is really quite broad. For example, any strictly monotone increasing or decreasing function $g : S \to \mathbf{R}$, where $S$ is a lattice, is qsm, whereas monotonicity is clearly not sufficient for supermodularity. From this observation one can construct many qsm functions. For example, let $g_i : S_i \to \mathbf{R}$ be nonnegative and monotone increasing, where $S_i$ is a lattice. Then $f : \prod S_i \to \mathbf{R}$ given by

$$f(x_1, ...x_k) = g_1(x_1) \cdots g_k(x_k)$$

is qsm, as is

$$f(x_1, ...x_k) = g_1(x_1) + ... + g_k(x_k).$$

The generality of qsm can also be seen by comparison with the lengths to which one must go to construct supermodular functions as demonstrated by Topkis (1978, p. 312).

Topkis also points out that supermodularity is closely related to concavity: both are cardinal, second-order properties which involve an infinite number of inequalities in general, yet for $C^2$ functions on $\mathbf{R}^k$, each has a simple characterization in terms of restrictions placed on second partial derivatives. Indeed, many properties of concave functions served as inspiration for the derivation of similar properties of supermodular functions (Topkis, 1978, p.319). Moreover, the relationship between supermodularity and quasisupermodularity is much like the relationship between concavity and quasiconcavity, with the analogy between quasiconcavity and quasisupermodularity exploited by the choice of name for this ordinal condition. Although the increased generality of quasiconcavity makes characterizing it more difficult than characterizing concavity, several sets of necessary and sufficient conditions do exist for quasiconcavity, chief among which are restrictions placed on the bordered Hessian matrix of first and second derivatives, and restrictions on the level sets.

One might hope that some sort of similar characterizations can be developed for quasisupermodular functions. Results of varying degrees of power along these lines are developed below by appealing to this hope.

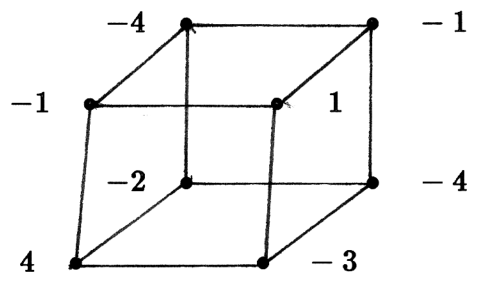
The result alluded to above regarding the characterization of $C^2$ supermodular functions is given in Topkis (1978), and follows from an important characterization also given by Topkis (1978) concerning functions whose domains are finite products of chains:

**Theorem (Topkis).** *If $S_i$ is a chain for $i = 1, \ldots, n$, then $f$ has (strictly) increasing differences in $(x_i, x_j)$ $\forall i \neq j$ on $\prod_{i=1}^{n} S_i$ iff $f$ is (strictly) supermodular on $\prod_{i=1}^{n} S_i$.*

This theorem implies that when the function under consideration has domain equal to a finite product of chains, the question of whether the function is supermodular reduces to the question of whether the function has increasing differences in every pair of variables. When $S_i = \mathbf{R}$ for every $i$, so that $\prod_{i=1}^{n} S_i = \mathbf{R}^n$, then $f$ is supermodular iff for every $i$, $f(x + \epsilon e_i) - f(x)$ is nondecreasing in $x_j$ for every $j \neq i$ and for every $\epsilon > 0$, where $e_i = (0, \ldots, 1, \ldots, 0)$ is the $i^{th}$ standard basis vector in $\mathbf{R}^n$. Hence if $f \in C^1$, then $f$ is supermodular iff $\partial f / \partial x_i$ is nondecreasing in $x_j$ for every $i \neq j$ and for every $x \in \mathbf{R}$. For $f \in C^2$, $f$ is then supermodular iff $\partial^2 f / \partial x_i \partial x_j \geq 0$, for every $i \neq j$, and for every $x \in \mathbf{R}$.

The equivalence of supermodularity and increasing differences for functions whose domain is $\mathbf{R}^n$, combined with the equivalent differential conditions for $C^1$ and $C^2$ functions, is crucial, as many economic examples will involve individual strategy spaces which are at least chains, if not subsets of $\mathbf{R}$, and payoff functions which are $C^k$ for some $k \geq 2$. Furthermore, although supermodularity is often easier to work with as a technical condition, increasing differences carries the interpretation of strategic complementarity which is often found in economic examples; thus their equivalence for functions whose domain is a finite product of chains is quite helpful in identifying supermodular games.

However, with the increased generality of qsm and the single crossing property, some of these useful results are lost, in turn complicating the job of identifying scp games. For example, it is no longer true that for a function on a finite product of chains, qsm is equivalent to the single crossing property in every pair $(x_i, x_j)$, $i \neq j$. Indeed, this equivalence breaks down even on a relatively simple product of discrete sets. For example, consider a function $f$ which takes values on $\{0,1\} \times \{0,1\} \times \{0,1\}$, as depicted below, where $0 < 1$ in every chain.




Then $f$ has the single crossing property on the given lattice, but $f$ is not qsm there, as $-1 = f(1,0,1) > f(0,0,0) = -2$, but $1 = f(1,1,1) \not> f(0,1,0) = 4$. However, the converse is true.

**Proposition 6.** *If $S_i$ is a lattice for $i = 1,\ldots m$, $S \subset \prod_{i=1}^m S_i$ is a sublattice, and $f : S \to \mathbf{R}$ is qsm, then $f$ has the single crossing property in $(x_i, x_j)$ $\forall i \neq j$ (and hence in $(x_n, x_{-n})$ $\forall n$).*

The proof is clear from the relevant definitions.

Results similar to the differential characterization of supermodularity, concavity, and quasiconcavity do exist for qsm and the single crossing property, as illustrated in the following proposition, yet the conditions derived are merely necessary and not sufficient.

**Proposition 7.** *Let $f : U \to \mathbf{R}$ have the single crossing property in $(x_i, x_j)$ $\forall i \neq j$, where $U \subset \mathbf{R}^m$ is open. Then*
*(1) If $f \in C^1$, then $\forall i$,*

$$D_i f(x_i, x_{-i}) > 0 \Rightarrow D_i f(x_i, x'_{-i}) \geq 0 \qquad \forall x'_{-i} \geq x_{-i}.$$

*(2) If $f \in C^2$, then $\forall i$,*

$$D_i f(x_i, x_{-i}) = 0 \Rightarrow D_{ij} f(x_i, x_{-i}) \geq 0 \qquad \forall i \neq j.$$

**Proof:** Both (1) and (2) follow from the observation that $f$ has the single crossing property in $(x_i, x_j)$ $\forall i \neq j \iff \forall i,\ \ \forall \epsilon > 0$,

$$\begin{aligned} &f(x_i + \epsilon, x_{-i}) - f(x_i, x_{-i}) \geq (>)0 \\ \Rightarrow\ &f(x_i + \epsilon, x'_{-i}) - f(x_i, x'_{-i}) \geq (>)0 \end{aligned}$$

$\forall x'_{-i} \geq x_{-i}$.

Then if $f \in C^1$, and the above holds, so that $f$ has the scp, $D_i f(x_i, x_{-i}) > 0 \Rightarrow$

$$lim_{h\to 0}\frac{f(x_i + h, x_{-i}) - f(x_i, x_{-i})}{h} > 0 \tag{$\dagger$}$$

($\dagger$) holds $\iff \exists \bar{h} \quad s.t. \quad h < \bar{h} \Rightarrow$

$$f(x_i + h, x_{-i}) - f(x_i, x_{-i}) > 0 \tag{$\ddagger$}$$

Then by the scp, ($\ddagger$) $\Rightarrow \forall x'_{-i} \geq x_{-i}$,

$$f(x_i + h, x'_{-i}) - f(x_i, x'_{-i}) > 0 \qquad \forall h < \bar{h}.$$

$\Rightarrow D_i f(x_i, x'_i) \geq 0 \qquad \forall x'_{-i} \geq x_{-i}$.
Then if $f \in C^2$, (2) follows from (1). ■

The same proof can be used to establish a slightly more general result. Let $f$ be qsm and real-valued on an open set $U \subset \mathbf{R}^n$, and $f \in C^1$. Then if $v \in \mathbf{R}^n$, let $D_v f(x)$ denote the directional derivative of $f$ in the direction $v$. One can show that if $v$ is nonnegative, $D_v f(x) > 0 \Rightarrow D_v f(x') \geq 0$ for every $x' \geq x$. Similarly, if $v$ is nonpositive, then $D_v f(x') < 0 \Rightarrow D_v f(x) \leq 0$ for every $x \leq x'$.

Disappointingly, although these conditions are necessary for a function to have the single crossing property, and hence to be qsm, they are not sufficient. It is not very difficult to discover examples of functions which satisfy (1) and (2) as above, but which fail to have the single crossing property. For example, let $g(x, y) = y \cdot sin^2(x)$, where $y \in \mathbf{R}$, and $x \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)$. Then (1) and (2) will hold, but clearly $\exists x', x$ in this range such that $x' \geq x$ and $sin^2(x') - sin^2(x) < 0$, so that $g(x', y) - g(x, y) > 0$ when $y < 0$, yet $g(x', y) - g(x, y) < 0$ when $y > 0$.

Another method which is slightly more fruitful when attempting to characterize qsm functions is to consider the level sets of such functions, and examine the restrictions on such sets which are implied by or imply qsm. As discussed above, this approach is suggested by the analogy between concavity and quasiconcavity on one hand, and supermodularity and quasisupermodularity on the other hand, together with the characterization of quasiconcave functions as precisely those having concave level sets. The lattice structure of the level sets of qsm functions cannot be quite so precisely determined, yet more can be said about these sets than could be determined about the differential nature of such functions. Using the same notation as above, where $L^\alpha = \{x : f(x) \geq \alpha\}$, it can be shown that

$$L^\alpha \text{ is a sublattice } \forall\alpha \Rightarrow f \text{ is qsm} \Rightarrow L^\alpha \text{ is a quasisublattice } \forall\alpha.$$

Of course the second implication has already been demonstrated in the proof of Proposition 2.[5]

This result follows from the next proposition and work done by Veinott (1989) concerning another class of functions which he refers to as supermeet and superjoin. Formally, a function $f$ is called **(strictly) supermeet** if $f(x \vee y) \wedge f(x \wedge y) \geq \ (>) f(x) \wedge f(y)$. Superjoin is defined analogously, with the join relation in place of the meet relation. It is interesting to note first that supermeet and superjoin are ordinal conditions, and moreover, a function $f$ is supermeet iff $L^\alpha$ is a sublattice for every $\alpha \in \mathbf{R}$. Furthermore, supermeet and superjoin are related to the notion of qsm, as the next proposition demonstrates.

**Proposition 8.** *If $S$ is a lattice and $f : S \to \mathbf{R}$ is either strictly supermeet or strictly superjoin, then $f$ is qsm. Moreover, if $f$ is qsm, then $\forall x, y \in S$, $f$ is either supermeet or superjoin at $x, y$.*

**Proof:** Suppose $f$ is strictly superjoin on $S$. Then suppose $f(x) \geq f(x \vee y)$. Since $f$ is strictly superjoin, $f(x \vee y) \neq f(x \vee y) \vee f(x \wedge y)$. Then $f(x \wedge y) = f(x \vee y) \vee f(x \wedge y)$. Hence $f(x \wedge y) > f(x) \vee f(y) \geq f(y)$. Therefore $f$ is qsm.

Suppose $f$ is strictly supermeet, and suppose $f(x) \geq f(x \vee y)$. Then either

(1) $f(x \vee y) = f(x \vee y) \wedge f(x \wedge y)$ and hence $f(x) \neq f(x) \wedge f(y)$

or

(2) $f(x \vee y) \neq f(x \vee y) \wedge f(x \wedge y)$.

(1) $\Rightarrow f(x \wedge y) \geq f(x \vee y) > f(x) \wedge f(y) = f(y)$.

(2) $\Rightarrow f(x) = f(x) \vee f(y)$, as $f(x) \geq f(x \vee y) \geq f(x \wedge y) > f(x) \wedge f(y)$. Hence $f(x \wedge y) > f(y)$. Therefore, $f$ is qsm.

Finally, suppose $f$ is qsm, and suppose $\exists x, y \in S$ such that $f$ is neither superjoin nor supermeet at $x, y$. Then

$$f(x) \wedge f(y) > f(x \vee y) \wedge f(x \wedge y)$$
$$f(x) \vee f(y) > f(x \vee y) \vee f(x \wedge y).$$

[5] Having quasisublattice-valued level sets is not sufficient for a function to be qsm, however, and counterexamples are easy to come by.

However, this contradicts the quasisupermodularity of $f$. ■

Then this proposition, combined with the result quoted above about the level sets of supermeet functions, yields the first implication, that if $L^{\alpha}$ is a sublattice for every $\alpha$ in $\mathbf{R}$, then $f$ is qsm.

Veinott (1989) has developed further necessary and sufficient conditions for a function to be supermeet in the special case that $S = \prod_{i=1}^{m} S_i$, where each $S_i$ is a chain, and $f : S \to T$, where $T$ is a complete chain, as detailed in the following theorem.

**Theorem (Veinott).** *With the above conditions, $f$ is supermeet iff $f = \wedge f_{ij}$, where $i,j \in \{1,...m\}$, $f_{ij} : S \to T$, and $f_{ij}$ is a function only of $x_i, x_j$ which is increasing in $x_i$ and decreasing in $x_j$.*

Combining the above results yields a broad, ordinal class of functions which is a subset of the class of qsm functions, yet for which relatively simple necessary and sufficient conditions can be found. One very interesting consequence of the fact that strictly supermeet functions are a subset of qsm functions is striking confirmation of the size of the extension from supermodular to quasisupermodular functions: in general, strictly supermeet is a concept which is disjoint from supermodularity. Supermeet does not imply supermodularity, which is not surprising, and moreover supermodularity does not imply supermeet. Counterexamples can be readily constructed, and several simple ones are given below for illustration.

supermeet $\not\Rightarrow$ supermodular:

| | |
|---|---|
| • | • |
| 10 | 5 |
| • | • |
| 4 | 0 |

supermodular $\not\Rightarrow$ supermeet:

| | |
|---|---|
| • | • |
| 3 | 5 |
| • | • |
| 0 | 1 |

If $f(x_n, x_{-n})$ is the payoff function of a player in a game in ordered normal form, to say that $f$ is supermeet assumes that a certain amount of return to coordination exists in the game. For suppose $x'_n \geq x_n$, and $x'_{-n} \geq x_{-n}$; then let $y = (x'_n, x_{-n})$; $z = (x_n, x'_{-n})$. The assertion that $f$ is strictly supermeet requires that $f(x'_n, x'_{-n}) \wedge f(x_n, x_{-n}) > f(x'_n, x_{-n}) \wedge f(x_n, x'_{-n})$. In a worst case scenario, coordination pays off more than disparity, as the minimum payoff to the player if everyone chooses either higher levels or lower levels is greater than the minimum payoff if only some players choose higher or some players choose lower levels. Such a situation is related to the coordination failure models discussed by authors such as Cooper and John (1988) and Heller (1986). For example, Heller (1986, pp.157-158) discusses a model of markets for two complementary goods in which demand in each industry is conditional on the equilibrium outcome in the other industry. A low-level and a high-level equilibrium exist in the economy, hence coordination can move the economy out of the low-level equilibrium and into the high-level equilibrium. However, without coordination, expansion by just one producer in an industry may just increase demand for other producers' goods in the industry, without increasing demand for his.

Several examples of games which display such complementarity, or the more general single crossing property, are discussed in the next section.

## 5. Examples

The first example of a game with the single crossing property is the class of Bertrand oligopoly games with differentiated products in which each firm faces a convex cost function. Milgrom and Roberts (1989b) have analyzed the class of Bertrand games where each firm faces a constant unit cost, so that firm $n$ faces the profit function

$$\pi_n = (p_n - c_n) D_n(p_n, p_{-n}).$$

$D_n(p_n, p_{-n})$ is the demand function which firm $n$ faces for its product, and Milgrom and Roberts (1989b) have shown that if $log(D_n)$ is supermodular for every $n$, and firms are restricted to choose prices in some closed interval $[0, \bar{P}_n]$, then the corresponding game is supermodular as well. There are many demand functions for which $log(D)$ is supermodular, including several which arise often in applications. For example, linear, CES, logit, and translog demand functions are all included. Milgrom and Roberts (1989b) were further able to show that any such Bertrand game has a unique Nash equilibrium, and thus is dominance solvable.

Milgrom and Shannon (1990) show that given any such Bertrand game with unit costs for which each firm's profit function has the single crossing property in $(p_n, p_{-n})$ for every choice of the constant unit cost level $c_n$, the corresponding game in which constant unit cost functions are replaced by convex cost functions is a game with the single crossing property, under the assumption that $D_n(p_n, p_{-n})$ is nondecreasing in $p_{-n}$.

The results of Section 3 imply that for any such Bertrand game, there exists an interval of price vectors $[p_*, p^*]$ such that $p_*$ and $p^*$ are the smallest and largest serially undominated strategy profiles, as well as the smallest and largest Nash equilibrium profiles. Also, for every player $n \in N$, $p_{n*}(p_{-n})$ and $p_n^*(p_{-n})$ are monotone increasing in $p_{-n}$. If there were a unique pure strategy Nash equilibrium, the corresponding game would be dominance solvable; however, uniqueness has not yet been determined.

A second example of a game with the single crossing property, also detailed by Milgrom and Shannon (1990), deals with a model about which quite a bit is already known: an exchange economy in a general equilibrium setting with gross substitutes.

Consider an Arrow-Debreu economy with $l$ goods and a finite number of consumers, and let $Z_i(p)$ denote the aggregate excess demand function for good $i$, $i = 1, ..., l$. Demand for each good is assumed to exhibit gross substitutability, so that $Z_i(p)$ is decreasing in $p_i$ and increasing in $p_j$ for every $j \neq i$. This model can be converted into a game with the single crossing property by creating a player for each market whose duty it is to set a price $p_i \in [0, \infty]^*$ for good $i$, where $[0, \infty]^*$ is the one point compactification of $[0, \infty)$, and whose payoff function is $-|Z_i(p)|$. Each payoff function is then of the form $g \circ f$, where $g : \mathbf{R} \to \mathbf{R}$ is strictly quasiconcave, and $f$ is a function which is increasing in one argument and decreasing in all other arguments. Such a function has the single crossing property, as shown in Milgrom and Shannon (1990).

Under the condition of gross substitutes, there exists a unique competitive equilibrium price vector for the economy. Clearly the only pure strategy Nash equilibria in the game correspond to players announcing competitive equilibrium prices; hence there is a unique Nash equilibrium in the game. Then it is dominance solvable, and moreover, utilizing the results of Milgrom and Roberts (1990), this equilibrium is stable under a wide class of learning processes other than tatonnemont.

An area in which one might think ordinal restrictions on payoff functions would be crucial as well as applicable is consumer games, such as models of externalities, where the players' payoff functions would usually be utility functions. That these models are not compatible with single crossing property games indicates some of the difficulties with quasisupermodularity and the single crossing property. Any sort of condition relating to these payoff functions, such as qsm, or increasing differences, should be invariant to monotone transformations of the payoff functions in accordance with standard ordinal utility theory. Unfortunately, problems arise when one attempts to reconcile usual assumptions about consumer behavior with the notions of single crossing games. Suppose player $i$ has utility function or payoff function $U_i(x_i, x_{-i})$, so that his utility level depends both on his consumption vector $x_i$, as well as the consumption vectors of the other $N-1$ consumers in the economy, denoted by $x_{-i}$. As applied to this function, the single crossing property would require that if $x_i' \geq x_i$,

$$\begin{aligned} &U_i(x_i', x_{-i}) - U_i(x_i, x_{-i}) \geq 0 \\ \Rightarrow\ &U_i(x_i', x_{-i}') - U_i(x_i, x_{-i}') \geq 0 \end{aligned}$$

for every vector $x_{-i}' \geq x_{-i}$. Yet a standard assumption regarding consumer behavior is monotonicity in own consumption, so that with $x_{-i}$ fixed, $x_i' \geq x_i$ implies that $U_i(x_i', x_{-i}) \geq U_i(x_i', x_{-i})$. Making this monotonicity assumption first guarantees that the player's payoff function will be quasisupermodular in $x_i$ for fixed $x_{-i}$, and strong monotonicity (i.e., $x_i' \geq x_i,\ x_i' \neq x_i \Rightarrow U_i(x_i', x_{-i}) > U_i(x_i, x_{-i})$) will guarantee that the payoff function has the single crossing property with respect to $(x_i, x_{-i})$.

However, the consumer does not carry out an unconstrained maximization of his utility function, so $U_i(\cdot, \cdot)$ is not the relevant payoff function for the player. Instead, the consumer's problem when faced with prices $p$ and initial endowment $\omega_i$ is $max_{x_i}\ U_i(x_i, x_{-i})$ subject to $p \cdot x_i \leq p \cdot \omega_i$. However, the budget set will *not* be a sublattice in general with strictly positive prices, hence the results derived for constrained optimization in section 2 are not applicable. The relevant payoff function if this model is viewed as a game is then the Lagrangian $\mathcal{L} = U_i(x_i, x_{-i}) + \lambda_i(p \cdot \omega_i - p \cdot x_i)$. With this payoff function, however, satisfaction of the single crossing property will be in general incompatible with an ordinal theory of utility, for consider the basic item of concern in single crossing:

$$\mathcal{L}(x_i', \lambda_i, x_{-i}) - \mathcal{L}(x_i, \lambda_i, x_{-i}) = U_i(x_i', x_{-i}) - U_i(x_i, x_{-i}) - \lambda_i(p \cdot x_i' - p \cdot x_i).$$

The nature of this quantity, and the similar quantity with $x_{-i}'$ in place of $x_{-i}$, while invariant to monotone increasing transformations of the function $\mathcal{L}$, will no longer be invariant to monotone increasing transformations of the utility function $U_i$.

## 6. Conclusions

This paper has sought to develop a theory of complementarity in economic behavior which is purely ordinal in nature, and to demonstrate both the wide range of results which pertain to such situations, as well as the wide range of situations included in the theory. Many questions remain unanswered, however. Simple and powerful necessary and sufficient conditions for quasisupermodularity and the single crossing property are important to further develop the theory of such games, and to aid in the determination of other examples of such games and the extension of some of the examples given. Moreover, future research might attempt to develop a theory of strategic substitutability analogous to the theory of strategic complementarity.

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