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Quadratic Concavity and Determinacy of Equilibrium*

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Abstract

One of the central features of classical models of competitive markets is the generic determinacy of competitive equilibria. For smooth economies with a finite number of commodities and a finite number of consumers, almost all initial endowments admit only a finite number of competitive equilibria, and these equilibria vary (locally) smoothly with endowments; thus equilibrium comparative statics are locally determinate. This paper establishes parallel results for economies with finitely many consumers and infinitely many commodities. The most important new condition we introduce, quadratic concavity, rules out preferences in which goods are perfect substitutes globally, locally, or asymptotically. Our framework is sufficiently general to encompass many of the models that have proved important in the study of continuous-time trading in financial markets, trading over an infinite time horizon, and trading of finely differentiated commodities.

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1 Introduction

One of the central features of classical models of competitive markets is the generic determinacy of competitive equilibria. For smooth economies with a finite number of commodities and a finite number of consumers, almost all initial endowments yield an economy that admits only a finite number of competitive equilibria, and these equilibria vary (locally) smoothly with endowments. These results, based on Debreu's (1970, 1972) seminal work, guarantee that equilibrium and local comparative statics in the Arrow-Debreu model are meaningful.

In this paper we establish parallel results for economies with finitely many consumers and infinitely many commodities. Our framework is sufficiently general to encompass many of the models that have proved important in the study of continuous-time trading in financial markets, trading over an infinite time horizon, and trading of finely differentiated commodities.

While our results parallel familiar results for economies with finitely many commodities, our methods are, of necessity, quite distinct. Debreu's methods, which build on the now standard techniques he introduced for the study of smooth economies, rely on the characterization of equilibrium as a zero of aggregate excess demand. If preferences are differentiably strictly convex and satisfy a standard boundary condition – that the closures of indifference sets at interior consumptions remain interior – then individual demands and hence aggregate excess demand are smooth. If the economy is regular, in the sense that the Jacobian of aggregate excess demand is invertible at every equilibrium, then the implicit function theorem guarantees that each equilibrium is locally unique and that local comparative statics are smooth and determinate. Finally, the transversality theorem guarantees that the set of endowment profiles that correspond to regular economies has full measure.

For economies with infinitely many commodities, it is by now well-known that no straightforward extension of Debreu's methodology is possible. The most obvious difficulty is that individual budget sets may be unbounded for many prices, whence individual demand may be undefined. Even restricting attention to prices at which demand is defined, that is, to candidate equilibrium prices, it is unclear whether there are many commodity spaces and preferences for which demand is a smooth function of prices.[1] Moreover, for most infinite-dimensional models the price space and the commodity space are different, so that even if demand were a smooth function of prices, it is unclear whether any statements about regular economies would be meaningful, much less valid.

To circumvent these problems, we follow much of the literature by using Negishi's argument to characterize equilibrium as a zero of the excess spending mapping. We cannot simply adapt Debreu's techniques to the excess spending mapping, however. As Shannon (1998a) stresses, the restrictions on preferences that are needed to ensure the validity of the second welfare theorem and the existence of equilibrium in many infinite-dimensional models involve bounds on consumers' marginal rates of substitution, and thus allow for boundary consumptions. Boundary consumptions may lead to "kinks" in the solution to the planner's problem and hence in the excess spending mapping. Even in those instances for which consumptions are not on the boundary, however, the absence of a suitable form of the implicit function theorem prevents us from concluding that the solution to the planner's problem, and hence the excess spending mapping, is smooth. Instead, we introduce a set of simple and natural restrictions on preferences that allow us to show that the excess spending function is Lipschitz continuous. The most important of these restrictions is a condition we call *quadratic concavity*, which requires that near any feasible bundle, utility differs from the linear approximation by an amount that is at least quadratic in the distance to the given bundle. Quadratic concavity implies that distinct commodities are not perfect substitutes — globally, locally, or asymptotically. Given the Lipschitz nature of the excess spending mapping, we build on the framework developed by Shannon (1998a,b) and the notion of genericity developed by Christensen (1974), Hunt, Sauer, and Yorke (1992) and Anderson and Zame (1997) to obtain our generic determinacy results.

This paper is part of a relatively small body of work on determinacy with infinitely many commodities. Much of this existing literature, beginning with the work of Kehoe and Levine (1985) on discrete-time infinite horizon models (with commodity space ℓ_∞), assumes that utilities are additively separable; see also Kehoe, Levine, and Romer (1990), Balasko (1997),

[1]See Araujo (1987).

and Chichilnisky and Zhou (1998). Additive separability is an economically restrictive assumption, but it is crucial to that work because it implies that the planner's problem can be decomposed into a sequence of independent finite-dimensional problems. Kehoe, Levine, Mas-Colell, and Zame (1989) take a different path, assuming that the commodity space is a Hilbert space, specifying a consumer by a smooth demand function rather than by a preference relation or utility function, and assuming that prices and consumptions lie in open sets. Because the positive cones of their consumption and price spaces have empty interior, however, the last assumption means that they allow for negative consumptions and negative prices, which are difficult to interpret economically.

Our paper is most closely related to Shannon (1998a), which has two parts. The first part shows that Lipschitz continuity of the excess spending mapping is enough to guarantee generic determinacy of equilibrium; we build directly on this framework. The second part gives conditions on preferences sufficient to guarantee Lipschitz continuity of the excess spending mapping, and hence generic determinacy, in models with countably many commodities, such as arise in considering trade over countably many dates or states of nature. The arguments establishing the latter results rely crucially on the fact that there is a natural way to approximate the solution to the planner's problem with a countable number of commodities by the solution to the planner's problem for a sufficiently large truncated finite set of commodities. It is unclear how such arguments could be extended to environments with a continuum of commodities, such as would arise in considering trade in continuous time, or over a continuum of states of nature, or in differentiated commodities.

By contrast, we analyze the planner's problem directly, rather than by approximation, in a manner that is independent of the number of commodities. We use a simple geometric argument to show that the solution to the planner's problem is Lipschitz. A parallel analysis of supporting prices establishes that the excess spending mapping is Lipschitz. Because the zeroes of the excess spending mapping characterize equilibrium prices and allocations, this allows us to obtain generic determinacy by applying the methods of Shannon (1998a), Anderson and Zame (1997) and Shannon (1998b).

Because our approach does not depend on the number of commodities, our results apply equally well to all commodity spaces, regardless of whether they have a finite, countably infinite, or uncountably infinite number of commodities. For spaces with a finite number of commodities, our results encompass those of Debreu (1970, 1972) and Shannon (1994). For the commodity space ℓ_2, our results encompass those of Shannon (1998a), while for the commodity space ℓ_∞, our results are not comparable to those in Shannon (1998a), although in spirit both our assumptions and our conclusions are weaker.[2]

Our paper proceeds as follows. In Section 2 we detail the basic assumptions maintained throughout. In Section 3 we introduce the notion of quadratic concavity. In Section 4 we characterize equilibrium in terms of welfare weights as the zeroes of the excess spending mapping. In Section 5 we study the social planner's problem characterizing Pareto optimal allocations, and in Section 6 we show that the excess spending map is Lipschitz. We use these results in Section 7 to show that equilibria are generically determinate in our economies. In Section 8 we present several illustrative examples, including models of continuous-time trading, trading in differentiated commodities, and trading over an infinite horizon.

[2] In particular, Shannon (1998a) uses a stronger notion of differentiability and a different notion of genericity, but obtains determinacy with respect to the ℓ_∞ norm, while we obtain determinacy with respect to the Mackey topology.

2 The Economy

In this section we lay out the basic assumptions that we will maintain throughout the paper.

We consider an exchange economy $\mathcal{E}$ with m consumers. Throughout we maintain the following quite standard assumptions on the commodity and price spaces and on consumer characteristics:

A1 the commodity space X is a vector lattice endowed with a Hausdorff, locally convex topology τ [3]

A2 the price space X^* is the topological dual of X and is a sublattice of the order dual of X [4]

A3 order intervals in X are weakly compact

A4 each consumer's consumption set is the positive cone X_+

A5 each individual endowment e_i is positive and the social endowment $\bar{e} = \sum e_i$ is strictly positive [5]

A6 each consumer's utility function $U_i : X_+ \to \mathbf{R}$ is τ-continuous, strictly monotone, and strictly concave

We view the social endowment as fixed and treat the distribution of endowments as parameters. Let $P(\bar{e}) \subset X^m$ denote the set of feasible Pareto optimal allocations of the social endowment $\bar{e}$ and $P^0(\bar{e}) \subset P(\bar{e})$ the subset of allocations $(x_1, \ldots, x_m)$ for which each $x_i \neq 0$. Let $P_i(\bar{e})$ and $P_i^0(\bar{e})$ denote

[3] Following Mas-Colell and Richard (1991), we do *not* assume X is a topological vector lattice, so the lattice operations may not be continuous.

[4] In particular, prices are τ-continuous and the supremum and infimum of prices in X^* are again in X^*.

[5] Recall that $\bar{e} \in X_+$ is *strictly positive* if the order ideal

$$X(\bar{e}) \equiv \{x \in X : |x| \leq k\bar{e} \text{ for some } k > 0\}$$

is weakly dense in X. If X is a topological vector lattice, this is equivalent to the more familiar requirement that $p \cdot \bar{e} > 0$ for every $p \in X_+^* \setminus \{0\}$.

the projections of $P(\bar{e})$ and $P^0(\bar{e})$ onto the i-th coordinate. In addition to the above we assume:

A7 for each i, U_i is Gateaux differentiable at each $x \in P_i^0(\bar{e})$ and the Gateaux derivative $DU_i(x) \in X^*_{++}$ [6]

We call an economy satisfying assumptions A1-A7 a *basic economy.*

These seven assumptions are standard conditions in equilibrium analysis with infinitely many commodities needed to ensure the existence of equilibria. The assumption that consumers' utilities are Gateaux differentiable plays the role of uniform properness here in ensuring the existence of prices supporting each Pareto optimal allocation. While it might seem strange to require differentiability only on the Pareto set, rather than on the entire consumption set, our weaker requirement allows us to include preferences satisfying Inada conditions, which might otherwise be excluded.[7] Of course differentiability on the entire consumption set or on the order interval $[0, \bar{e}]$ would suffice.

[6]Recall that U_i is *Gateaux differentiable* at $x \in X_+$ if there is a continuous linear functional $DU_i(x)$ such that

$$\left[\lim_{h \to 0^+} \frac{U_i(x + hy) - U_i(x)}{h} - DU_i(x) \cdot y\right] = 0$$

for each $y \in X$ having the property that $x + hy \in X_+$ for h sufficiently small.

[7]For more on this point see Duffie and Zame (1989) and Araujo and Monteiro (1991).

3 Quadratic Concavity

To motivate the central new notions we use, consider the simplest examples of robust indeterminacy of equilibrium in a two person, two commodity Edgeworth square: equilibria will be generically indeterminate if both consumers find the two commodities to be perfect complements or if both consumers find the two commodities to be perfect substitutes. Requiring utility functions to be smooth rules out perfect complements, while requiring utility functions to be differentiably strictly concave rules out perfect substitutes. Our assumptions are intended to have the same effect, but our infinite-dimensional setting requires some care in formulating them.

To understand the assumptions we use, let $U : \mathbf{R}^n_+ \to \mathbf{R}$ be a utility function that is twice continuously differentiable and differentiably strictly concave. For our purposes, these assumptions have three important implications. First, continuity of the derivative implies that it is bounded on compact sets, thus:

- there is a constant B such that

  $$|DU(x) \cdot y| \leq B \|y\|$$

  for each $x \in [0, \bar{e}]$ and $y \in \mathbf{R}^n$.

Moreover, continuity of the second derivative implies that the gradient mapping $x \mapsto DU(x)$ is Lipschitz on $[0, \bar{e}]$. That is, there is a constant c such that

$$\|DU(x) - DU(y)\| \leq c \|x - y\|$$

for all $x, y \in [0, \bar{e}]$.[8] In particular, for $z \in [0, \bar{e}]$,

$$|DU(x) \cdot z - DU(y) \cdot z| \leq \|DU(x) - DU(y)\| \|z\| \leq c \|z\| \|x - y\|$$

Because $[0, \bar{e}]$ is a bounded set, we conclude:

[8] To see this, apply Taylor's theorem to the first derivative. Given $x, y \in [0, \bar{e}]$, there is some $\tilde{x}$ on the line segment from x to y such that $DU(x) - DU(y) = D^2U(\tilde{x})(x - y)$. Hence $\|DU(x) - DU(y)\| \leq \|D^2U(\tilde{x})\| \|x - y\|$. Because $\tilde{x} \mapsto D^2U(\tilde{x})$ is continuous and $[0, \bar{e}]$ is compact, there is a constant c such that $\|DU(x) - DU(y)\| \leq c \|x - y\|$.

- there is a constant C such that

$$|DU(x) \cdot z - DU(y) \cdot z| \leq C \|x - y\|$$

  for each $x, y \in [0, \bar{e}]$ and $z \in [0, \bar{e}]$.

In other words, the evaluation map $x \mapsto DU(x) \cdot z$ is Lipschitz on $[0, \bar{e}]$, uniformly for $z \in [0, \bar{e}]$.

Finally, Taylor's theorem implies that for $x, y \in [0, \bar{e}]$,

$$U(y) = U(x) + DU(x) \cdot (y - x) + \frac{1}{2}[D^2U(\hat{x})(y - x)] \cdot [y - x]$$

for some $\hat{x}$ on the line segment between x and y. Strict differential concavity together with continuity of the second derivative means that the second derivative matrix is strictly negative definite, uniformly on compact sets, so:

- there is a constant $K > 0$ such that

$$U(y) \leq U(x) + DU(x) \cdot (y - x) - K\|y - x\|^2$$

  for each $x, y \in [0, \bar{e}]$.

All of these statements are unambiguous in a finite-dimensional setting, both because there is a norm on $\mathbf{R}^n$ and because all norms on $\mathbf{R}^n$ are equivalent. In an infinite-dimensional setting, there may be no norm on X or, if there is, there may be many non-equivalent norms. While our key assumptions will abstract the above properties of differentiably strictly concave functions in $\mathbf{R}^n$, crucial to our approach — and to understanding these assumptions — is that we do not require that X be normed, or that the conditions above be satisfied with respect to the original norm of X even if X is normed. Rather, we require only that there be *some* norm with respect to which these conditions are satisfied, and which induces the given topology τ on the set $[0, \bar{e}]$ of feasible consumptions.

We begin by abstracting the first two conditions from the finite-dimensional setting.

Definition Let $U : X_+ \to \mathbf{R}$ be Gateaux differentiable on $Y \subset X_+$. We say the norm $\|\cdot\|$ is *adapted to* U *on* Y if the topology induced by $\|\cdot\|$ coincides with τ on the order interval $[0, \bar{e}]$,[9] and

- there is a constant B such that for each $y \in Y$ and $z \in X$

$$|DU(y) \cdot z| \leq B\|z\|$$

- there is a constant C such that for each $y, y' \in Y$ and $z \in [0, \bar{e}]$

$$|DU(y) \cdot z - DU(y') \cdot z| \leq C\|y - y'\|$$

  That is, the evaluation map $y \mapsto DU(y) \cdot z : Y \to \mathbf{R}$ is Lipschitz on Y, uniformly for $z \in [0, \bar{e}]$.

As in the finite-dimensional setting, the second assumption is implied by either of the simpler and more familiar conditions:

- the gradient map $y \mapsto DU(y) : X \to X^*$ is Lipschitz on Y
- U is twice continuously Gateaux differentiable on Y and $D^2U(y)$ is uniformly bounded with respect to $\|\cdot\|$ on Y

While it may seem that we are splitting hairs by insisting on the more complicated condition in the definition rather than either of these simpler conditions, the difference is real and important in a number of applications. Indeed, as the following example shows, the condition we use may be satisfied in an environment in which natural utility functions are never twice differentiable and gradient mappings are never Lipschitz.

Example 3.1 Let $X = L^1[0, 1], X^* = L^\infty[0, 1]$ and

$$U(x) = \int_0^1 u(x(t))\, dt$$

[9] For most of our purposes, it would suffice to assume that the topology induced by $\|\cdot\|$ is stronger than τ on the order interval $[0, \bar{e}]$.

where $u : [0, \infty) \to [0, \infty)$ is C^2. Then $DU(x)(t) = u'(x(t))$ for each $x \in X_+$. Set $x = \mathbf{1}$, the function that is identically 1, and for each $\varepsilon > 0$ define y_ε by

$$y_\varepsilon(t) = \begin{cases} 0 \text{ if } 0 \leq t \leq \varepsilon \\ 1 \text{ if } \varepsilon < t \leq 1 \end{cases}$$

Then $\|y_\varepsilon - x\| = \varepsilon$ but $\|DU(y) - DU(x)\| = u'(0) - u'(1)$. In particular, the map $x \mapsto DU(x)$ is not norm-to-norm continuous and not Lipschitz continuous, and U is not twice continuously differentiable.

On the other hand, fix $\bar{e} \in X_+$ and suppose that there exists $M > 0$ such that $\bar{e}(t) \leq M$ for all t. For each $x, y \in [0, \bar{e}]$ we have

$$\begin{aligned} |DU(y) \cdot z - DU(x) \cdot z| &= \left| \int_0^1 [u'(y(t)) - u'(x(t))] z(t) dt \right| \\ &\leq k \int_0^1 |y(t) - x(t)||z(t)| dt \\ &\leq k\|z\| \int_0^1 |y(t) - x(t)| dt \\ &\leq k \, \|\bar{e}\| \, \|y - x\|. \end{aligned}$$

Thus the evaluation map $x \mapsto DU(x) \cdot z$ is uniformly Lipschitz for $z \in [0, \bar{e}]$.

In economies with finitely many goods, a sufficient condition for generic determinacy is the additional condition that goods are never perfect substitutes, even locally. This idea is typically formalized by the assumption of strict differential concavity. Here we formalize the idea that goods are not perfect substitutes with a simpler version of this condition relying only on directional derivatives, which allows us to highlight the intuition underlying generic determinacy results and give a unified treatment of many of the most important equilibrium models that is independent of the number of commodities in the economy.

Definition Let $U : X_+ \to \mathbf{R}$ be a concave function and let $\|\cdot\|$ be a norm on X. We say U is *quadratically concave* on $Y \subset X_+$ with respect to $\|\cdot\|$ if U is Gateaux differentiable on Y and there is a constant $K > 0$ such that for each $x, y \in Y$

$$U(y) \leq U(x) + DU(x) \cdot (y - x) - K \, \|y - x\|^2$$

To understand this condition, recall that a differentiable concave function is bounded above by the linear approximation given by the gradient, i.e, for all $y, x \in X_+$,

$$U(y) \leq U(x) + DU(x) \cdot (y - x)$$

Quadratic concavity simply adds to this requirement the stipulation that the error in this linear approximation on Y increases at least quadratically, and at a rate independent of the direction or points in question. As a simple illustration, suppose $U : X_+ \to \mathbf{R}$ is twice continuously Gateaux differentiable on X_+ and differentiably strictly concave on a convex set $Y \subset X_+$. That is, suppose there exists $K > 0$ such that $z \cdot D^2U(y)z \leq -K\|z\|^2$ for all $z \in X$ and $y \in Y$. Then, as we argued above, U is quadratically concave on Y by Taylor's theorem.

While quadratic concavity includes the standard finite-dimensional version of strict differential concavity and its natural infinite-dimensional counterpart, the additional generality we get by stating our condition only in terms of the directional derivatives and the first-order approximation error is very useful. In some of the most basic models with infinitely many commodities, natural utility functions are quadratically concave but fail to satisfy the stronger condition of strict differential concavity. One example is given by the separable utility function we analyzed in Example 1, which is not twice continuously Gateaux differentiable. We discuss this point in more detail in the context of several other examples in Section 8.

Our key new assumptions will then be that each consumer's utility function is quadratically concave on weakly compact subsets of $P_i^0(\bar{e})$ with respect to some norm $\|\cdot\|_i$ that is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$. The flexibility both to choose a norm different from the underlying norm when X is a normed space, and to choose a different norm for each consumer, will be important in a number of applications, as the following example illustrates.

Example 3.2 Let $X = \ell_\infty$, the space of bounded real sequences, with the Mackey topology. Let $u : \mathbf{R}_+ \to \mathbf{R}$ be twice continuously differentiable and differentiably strictly concave, and define

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$$

for some discount factor $\beta < 1$. We claim that U is not quadratically concave with respect to the ℓ_∞ norm $\|\cdot\|_\infty$ on $[0, \bar{e}]$ for any positive social endowment $\bar{e}$.

Intuitively, this is not surprising. The natural assumption that consumers choosing over an infinite horizon discount future consumption relative to current consumption means that big changes in consumption may have a small effect on marginal utility provided these changes occur far enough in the future. Discounting generates the same insensitivity of marginal utility to changes in consumption typically associated with goods with a high degree of substitutability, and hence suggests the potential for robust indeterminacies.

For a Gateaux differentiable function U, however, quadratic concavity on some set Y requires that there exist some $K > 0$ such that for each $x, y \in Y$

$$U(x) \leq U(y) + DU(y) \cdot (x - y) - K\|x - y\|^2$$

and

$$U(y) \leq U(x) + DU(x) \cdot (y - x) - K\|y - x\|^2$$

Combining these and simplifying, quadratically concave utilities must satisfy the following inequality for some $K > 0$ and all $x, y \in Y$:

$$(DU(y) - DU(x)) \cdot (x - y) \geq 2K\|x - y\|^2$$

Now consider the commodity space ℓ_∞ equipped with the supremum norm. When a consumer is impatient, the change in the directional derivative corresponding to a change in consumption may be small even when the change in consumption is large in the supremum norm if this change in consumption occurs sufficiently far in the future. That is, the left hand side of this inequality may be quite small even when the right hand side is large in the supremum norm, provided the change in consumption occurs in the distant future.

To see this formally, set $x = (1, 1, \ldots)$. For each T, let $\chi^T \in \ell_\infty$ be the sequence which has 1 in the T-th coordinate and 0 elsewhere. Let $y^T = x + \chi^T$, and note that $\|y^T - x\| = 1$ for each T. Applying Taylor's theorem and noting that $DU(x) \cdot (y^T - x) = \beta^T u'(1)$, we conclude that there exists $\zeta \in (1, 2)$ such that

$$U(y^T) \;=\; U(x) + \beta^T(u(2) - u(1))$$

$$
\begin{aligned}
&= U(x) + \beta^T u'(1) + \frac{1}{2}\beta^T u''(\zeta) \\
&= U(x) + DU(x)\cdot(y^T - x) - \frac{1}{2}\beta^T |u''(\zeta)| \|y^T - x\|_\infty^2
\end{aligned}
$$

Because u'' is bounded on the interval $[1, 2]$, $\beta^T u''(\zeta)\|y^T - x\|_\infty^2 \to 0$ as $T \to \infty$. In particular, U is certainly not quadratically concave with respect to the ℓ_∞ norm.

On the other hand, there is a weighted norm which is adapted to U and with respect to which U is quadratically concave.[10] To see this, for each $z \in \ell_\infty$ define the β weighted norm of z by

$$
\|z\|_\beta = \sum_{t=0}^{\infty} \beta^t |z_t|
$$

This norm reflects the same impatience as the utility function, measuring as close bundles that differ only in the distant future. Moreover, for any $\bar{e} \in \ell_{\infty+}$ the topology generated by this norm agrees with the Mackey topology on $[0, \bar{e}]$.

To see that $\|\cdot\|_\beta$ has the desired properties, fix the social endowment $\bar{e} \in \ell_{\infty+}$. Simple computations show that $\|\cdot\|_\beta$ is adapted to U on $[0, \bar{e}]$. Moreover, U is quadratically concave on $[0, \bar{e}]$ with respect to $\|\cdot\|_\beta$. To see this, fix $x, y \in [0, \bar{e}]$. Applying Taylor's theorem to utility in period t yields

$$
u(y_t) = u(x_t) + u'(x_t)(y_t - x_t) + \frac{1}{2}u''(z_t)(y_t - x_t)^2
$$

for some z_t between x_t and y_t. Because u is differentiably strictly concave, there is a constant $c > 0$ such that $u''(\zeta) < -c$ for $\zeta \leq \sup_t e_t$. Hence

$$
\begin{aligned}
U(y) - U(x) &= \sum \beta^t (u(y_t) - u(x_t)) \\
&= \sum \beta^t u'(x_t)(y_t - x_t) + \sum \beta^t \frac{1}{2} u''(z_t)(y_t - x_t)^2 \\
&= DU(x)\cdot(y - x) + \frac{1}{2}\sum \beta^t u''(z_t)(y_t - x_t)^2 \\
&\leq DU(x)\cdot(y - x) - \frac{c}{2}\sum \beta^t (y_t - x_t)^2
\end{aligned}
$$

[10]See also Example 8.3 in Section 8.

$$
\begin{aligned}
&\leq \quad DU(x) \cdot (y - x) - cb \left(\sum \beta^t |y_t - x_t| \right)^2 \\
&= \quad DU(x) \cdot (y - x) - cb \|y - x\|_\beta^2
\end{aligned}
$$

for some $b > 0$, where the second inequality follows from the fact that in a finite measure space, there exists $B > 0$ such that $\|f\|_2 \geq B\|f\|_1$ for all f, where $\|\cdot\|_p$ denotes the L_p norm for $1 \leq p \leq \infty$. Thus U is quadratically concave with respect to $\|\cdot\|_\beta$ on $[0, \bar{e}]$.

4 Equilibrium and the Excess Spending Map

Given a distribution $e = (e_1, \ldots, e_m)$ of the social endowment $\bar{e}$, an equilibrium can be characterized, using the welfare theorems, as a Pareto optimal allocation x and a supporting price p for which the budget equations

$$\begin{aligned} p \cdot (x_1 - e_1) &= 0 \\ &\vdots \\ p \cdot (x_m - e_m) &= 0 \end{aligned}$$

are satisfied. Of course since x is a feasible allocation one of these equations is redundant; we henceforth suppress the last equation. Central to our approach is a simplification of this characterization, following Negishi, through which both Pareto optimal allocations and supporting prices are indexed by "welfare weights".

The first step in this simplification, the characterization of Pareto optima as the solutions to a social planner's problem, is quite familiar. Given a social endowment bundle $\bar{e}$ and a vector of "welfare weights" $\lambda \in \mathbf{R}^m_+$ with $\sum \lambda_i = 1$, the social planner's problem is to choose a feasible allocation $x(\lambda) = (x_1(\lambda), \ldots, x_m(\lambda))$ to maximize the weighted sum of utilities $\sum \lambda_i U_i(x_i)$. The following result records several basic properties of the solution to the planner's problem under our assumptions; we omit the familiar proof. We write

$$\Lambda \equiv \{\lambda \in \mathbf{R}^m_+ : \sum \lambda_i = 1\}$$

for the set of welfare weights and

$$\Lambda^0 \equiv \{\lambda \in \Lambda : \lambda_i > 0 \text{ for all } i\}$$

for the set of strictly positive weights.

Lemma 4.1 *If $\mathcal{E}$ is a basic economy then*

(i) for each $\lambda \in \Lambda$ the planner's problem has a unique solution $x(\lambda) \in P(\bar{e})$

(ii) the mapping $x : \Lambda \to P(\bar{e})$ is continuous when $P(\bar{e})$ is equipped with the weak topology of X^m

(iii) $x(\Lambda) = P(\bar{e})$ *and* $x(\Lambda^0) = P^0(\bar{e})$

To characterize equilibrium in similar terms, we must also be able to characterize equilibrium prices using the welfare weights. Because preferences are smooth, each interior Pareto optimal allocation admits a unique supporting price, up to normalization. A Pareto optimal allocation on the boundary, however, may admit multiple supporting prices.[11] Fortunately, supporting prices can be characterized uniquely in terms of welfare weights; the following lemma is just what we need. Surprisingly, this result does not seem well-known, even in the finite-dimensional context.

Lemma 4.2 *If $\mathcal{E}$ is a basic economy, x is a feasible allocation for which $x_i \neq 0$ for each i, and $q \in X_+^*$ is a non-zero price, then the following statements are equivalent:*

(i) *x is a Pareto optimal allocation and q supports x*

(ii) *there is a vector of welfare weights $\lambda \in \Lambda^0$ and a constant $\beta > 0$ such that x solves the planner's problem for the weights λ and*

$$q = \beta \bigvee_i \lambda_i DU_i(x_i)$$

Proof: (i) $\Rightarrow$ (ii): Let x be a Pareto optimal allocation and let q be a supporting price. We first establish the desired representation of q.

For each i, set

$$\beta_i = \frac{q \cdot x_i}{DU_i(x_i) \cdot x_i}$$

The fact that utility functions are strictly monotone guarantees that the denominator is strictly positive. The fact that q is a supporting price guarantees that the numerator, and hence β_i, is strictly positive. Our goal is to show that

$$q \cdot y = \left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y$$

[11] Consider a corner optimum in an Edgeworth box, for instance.

for every $y \in X$, from which we will easily obtain the desired representation. We proceed by verifying this equality first when $0 \leq y \leq x_i$ for some i, then when $0 \leq y \leq \bar{e}$, then when y is in the order ideal generated by $\bar{e}$, and finally for arbitrary $y \in X$.

Fix a consumer i. Supporting prices equate marginal rates of substitution so if $y \in X_+$ then

$$\frac{DU(x_i) \cdot x_i}{q \cdot x_i} \geq \frac{DU(x_i) \cdot y}{q \cdot y}$$

with equality if $y \leq x_i$. Rearranging yields

$$q \cdot y \geq \left(\frac{q \cdot x_i}{DU_i(x_i) \cdot x_i} \right) DU_i(y) \cdot y$$

for every $y \in X_+$, with equality if $y \leq x_i$. Using the definition of β_i and substituting, we have

$$q \cdot y \geq \beta_i DU_i(x_i) \cdot y \quad \text{for all } y \in X_+, \text{with equality if } y \leq x_i \tag{1}$$

If $y \in X_+$ and $y \leq x_j$ for $i \neq j$, then two applications of (1) imply that

$$\beta_j DU_j(x_j) \cdot y = q \cdot y \geq \beta_i DU_i(x_i) \cdot y$$

In particular

$$\beta_j DU_j(x_j) \cdot y \geq \beta_i DU_i(x_i) \cdot y \quad \text{if } y \in X_+, y \leq x_j \tag{2}$$

For $y \in X_+$,

$$\left[\bigvee_i \beta_i DU_i(x_i) \right] \cdot y = \sup \left\{ \sum \beta_i DU_i(x_i) \cdot a_i : a_i \geq 0, \sum a_i = y \right\} \tag{3}$$

by the definition of the supremum of linear functionals. Thus

$$\left[\bigvee_i \beta_i DU_i(x_i) \right] \cdot y = \beta_j DU_j(x_j) \cdot y = q \cdot y \quad \text{if } y \in X_+, y \leq x_j \tag{4}$$

Next consider any $y \in X_+$ for which $0 \leq y \leq \bar{e}$. The Riesz Decomposition Property of vector lattices guarantees that we can find vectors $y_j \in X_+$ such that $y = \sum y_j$ and $0 \le y_j \le x_j$ for each j. Repeated applications of (4) yield

$$\begin{aligned}
\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y &= \left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot \left[\sum_j y_j\right] \\
&= \sum_j \left\{\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y_j\right\} \\
&= \sum_j \beta_j DU_j(x_j) \cdot y_j \\
&= \sum_j q \cdot y_j \\
&= q \cdot \sum_j y_j \\
&= q \cdot y
\end{aligned}$$

Now consider any y in the order ideal generated by $\bar{e}$; that is, $y \in X$ such that $|y| \le k\bar{e}$ for some $k > 0$. Write $z = (1/k)y$ and decompose $z = z^+ - z^-$ as the sum of positive and negative parts. Then $0 \le z^+ \le \bar{e}$ and $0 \le z^- \le \bar{e}$, so the previous paragraph implies that

$$\begin{aligned}
\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot z^+ &= q \cdot z^+ \\
\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot z^- &= q \cdot z^-
\end{aligned}$$

It follows from linearity that

$$\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y = q \cdot y$$

Strict positivity of the social endowment $\bar{e}$ means that the order ideal generated by $\bar{e}$ is dense in X, so continuity entails that

$$\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y = q \cdot y$$

for every $y \in X$, which was our goal.

Write $\beta = \sum \beta_i$ and $\lambda_i = \beta_i/\beta$ for each i, and note that $\lambda_i > 0$ because $\beta_i > 0$. Then

$$q = \beta \bigvee_i \lambda_i DU_i(x_i) \tag{5}$$

which is the desired representation of q.

It remains only to show that x solves the planner's problem for these weights λ. To see this, suppose that x' is an allocation. Then $x'_i \geq 0$ for each i and $\sum x'_i = \bar{e}$, so

$$q \cdot \bar{e} \geq \beta \sum \lambda_i DU_i(x_i) \cdot x'_i$$

Moreover, since utilities are concave,

$$\begin{aligned}
\sum \lambda_i U_i(x'_i) - \sum \lambda_i U_i(x_i) &= \sum \lambda_i \left[U_i(x'_i) - U_i(x_i)\right] \\
&\leq \sum \lambda_i \left[DU_i(x_i) \cdot (x'_i - x_i)\right]
\end{aligned}$$

Thus

$$\begin{aligned}
\sum \lambda_i U_i(x'_i) - \sum \lambda_i U_i(x_i) &\leq \sum \lambda_i \left[DU_i(x_i) \cdot (x'_i - x_i)\right] \\
&= \sum \lambda_i DU_i(x_i) \cdot x'_i - \sum \lambda_i DU_i(x_i) \cdot x_i \\
&= \sum \lambda_i DU_i(x_i) \cdot x'_i - \frac{1}{\beta} \sum q \cdot x_i \\
&\leq \frac{1}{\beta} q \cdot \bar{e} - \frac{1}{\beta} \sum q \cdot x_i \\
&= \frac{1}{\beta} q \cdot \bar{e} - \frac{1}{\beta} q \cdot \sum x_i \\
&= \frac{1}{\beta} q \cdot \bar{e} - \frac{1}{\beta} q \cdot \bar{e} \\
&= 0
\end{aligned}$$

Thus x solves the planner's problem for the weights λ.

(ii) $\Rightarrow$ (i): Solutions to the planner's problem are Pareto optima, so we need only show that $\bigvee_i \lambda_i DU_i(x_i)$ is a supporting price. Note first that for every i, j the first order condition for Pareto optimality implies

$$\lambda_i DU_i(x_i) \cdot (-y) + \lambda_j DU_j(x_j) \cdot y \leq 0$$

if $y \in X_+, y \leq x_i$. Rewriting yields that

$$\lambda_i DU_i(x_i) \cdot y \geq \lambda_j DU_j(x_j) \cdot y$$

if $y \in X_+, y \leq x_i$. It follows as above that

$$q \cdot z \geq \lambda_i DU_i(x_i) \cdot z$$

for $z \in X_+$ with equality if $z \leq x_i$.

Now fix i. To see that q supports U_i at x_i, let $z \geq 0$. Then

$$\begin{aligned} U_i(z) - U_i(x_i) &\leq DU_i(x_i) \cdot (z - x_i) \\ &= DU_i(x_i) \cdot z - DU_i(x_i) \cdot x_i \\ &= DU_i(x_i) \cdot z - \frac{1}{\lambda_i} q \cdot x_i \\ &\leq \frac{1}{\lambda_i} q \cdot z - \frac{1}{\lambda_i} q \cdot x_i \\ &= \frac{1}{\lambda_i} q \cdot (z - x_i) \end{aligned}$$

Thus if $z \geq 0$ and $U_i(z) \geq U_i(x_i)$, then $q \cdot z \geq q \cdot x_i$. It follows that q is a supporting price, so the proof is complete. □

Given a vector of welfare weights $\lambda \in \Lambda^0$, write

$$p(\lambda) \equiv \bigvee_i \lambda_i DU_i(x_i(\lambda))$$

In view of Lemma 4.2 and the redundancy of the budget equations, we obtain immediately the following characterization of equilibrium in terms of welfare weights.

Lemma 4.3 *Let $\mathcal{E}$ be a basic economy. The allocation x and the price p constitute an equilibrium if and only if there exists a vector of welfare weights $\lambda \in \Lambda^0$ such that*

(a) x solves the planner's problem with weights λ

(b) $p = cp(\lambda)$ for some constant $c > 0$

(c) the budget equations

$$\begin{aligned} p\cdot(x_1(\lambda)-e_1) &= 0 \\ &\vdots \\ p\cdot(x_{m-1}(\lambda)-e_{m-1}) &= 0 \end{aligned}$$

are satisfied

Given these results, we characterize equilibrium in terms of the zeroes of the excess spending mapping. Given the social endowment, write

$$D^0(\bar{e}) \equiv \left\{ e \in X_+^m : e_i \neq 0 \text{ for each } i \text{ and } \sum e_i = \bar{e} \right\}$$

for the set of distributions of the social endowment that give no consumer zero endowment. Define the *excess spending mapping*

$$S : \Lambda^0 \times D^0(\bar{e}) \to \mathbf{R}^{m-1}$$

by defining the i-th component to be

$$S_i(\lambda, e) = p(\lambda)\cdot(x_i(\lambda)-e_i)$$

If e is a distribution of the social endowment $\bar{e}$, write $\mathcal{E}(e)$ for the economy with endowment profile e. In view of the discussion above, we may identify an equilibrium of $\mathcal{E}(e)$ with a zero of $S(\cdot, e)$. In the following sections, we show that the planner's problem is Lipschitz and then that the excess spending mapping is Lipschitz; we then use versions of Sard's theorem and the transversality theorem for Lipschitz functions to obtain our generic determinacy results.

5 The Social Planner's Problem

In this section, we carry out the first step in our program, analyzing the solution to the social planner's problem. As we show below, under the additional assumption of quadratic concavity with respect to an adapted norm, the solution to the planner's problem is locally Lipschitz continuous. This result will become the key to all of our determinacy results.

Lemma 5.1 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then the solution $x(\cdot)$ to the planner's problem is locally Lipschitz on Λ^0 with respect to these norms and continuous with respect to the topology τ.

Proof: We first show that each U_i is Lipschitz on weakly compact subsets of $P_i^0(\bar{e})$. For $\delta > 0$ set

$$\Lambda^\delta \equiv \{\lambda \in \Lambda : \lambda_i \geq \delta \text{ for all } i\}$$

Let $P^\delta(\bar{e}) = x(\Lambda^\delta)$ and let $P_i^\delta(\bar{e})$ be the projection of $P^\delta(\bar{e})$ onto the i-th coordinate. Lemma 4.1 guarantees that $P_i^\delta(\bar{e})$ is weakly compact, so by adaptedness for each i there is a constant B_i such that

$$|DU_i(x) \cdot z| \leq B_i \|z\|_i$$

for each $x \in P_i^\delta(\bar{e})$ and $z \in X$. If $x, y \in P_i^\delta(\bar{e})$ then concavity of U_i guarantees that

$$U_i(y) - U_i(x) \leq DU_i(x) \cdot (y - x) \leq B_i \|y - x\|_i$$

Reversing the roles of x and y yields

$$|U_i(y) - U_i(x)| \leq \max\left(|DU_i(x) \cdot (y - x)|, |DU_i(y) \cdot (x - y)|\right) \leq B_i \|y - x\|_i$$

which is the desired Lipschitz estimate.

Now fix $\delta > 0$ let $\lambda, \lambda' \in \Lambda^\delta$. Let $x = x(\lambda)$ and $x' = x(\lambda')$. For each i, the quadratic concavity of U_i with respect to $\|\cdot\|_i$ on $P_i^\delta(\bar{e})$ means there is a constant $C_i > 0$ such that

$$U_i(x_i') \leq U_i(x_i) + DU_i(x_i) \cdot (x_i' - x_i) - C_i \|x_i' - x_i\|_i^2$$

Multiplying by λ_i and summing over i yields

$$\sum \lambda_i U_i(x_i') \leq \sum \lambda_i U_i(x_i) + \sum \lambda_i DU_i(x_i) \cdot (x_i' - x_i) - \sum C_i \lambda_i \|x_i' - x_i\|_i^2 \quad (6)$$

By assumption, x solves the social planner's problem for weights λ, so the first order conditions imply that $\sum \lambda_i DU_i(x_i) \cdot (x_i' - x_i) \leq 0$. Substituting into (6) gives

$$\sum \lambda_i U_i(x_i') \leq \sum \lambda_i U_i(x_i) - \sum C_i \lambda_i \|x_i' - x_i\|_i^2 \quad (7)$$

Because x' solves the social planner's problem for weights λ', weighted utility is no greater at x, so:

$$0 \leq \sum \lambda_i' U_i(x_i') - \sum \lambda_i' U_i(x_i) \quad (8)$$

Adding (7) and (8) yields:

$$\sum \lambda_i U_i(x_i') \;\; \leq \;\; \sum \lambda_i U_i(x_i) - \sum C_i \lambda_i \|x_i' - x_i\|_i^2 + \sum \lambda_i' U_i(x_i') - \sum \lambda_i' U_i(x_i)$$

Rearranging terms gives

$$\begin{aligned} \sum C_i \lambda_i \|x_i' - x_i\|_i^2 \;\; &\leq \;\; \sum (\lambda_i - \lambda_i')[U_i(x_i) - U_i(x_i')] \\ &\leq \;\; \sum |\lambda_i' - \lambda_i| \, |U_i(x_i') - U_i(x_i)| \qquad (9) \end{aligned}$$

Because utility functions are Lipschitz on $P_i^\delta(\bar{e}) = x(\Lambda^\delta)$, for each i there is a constant $K_i > 0$ such that $|U_i(x_i') - U_i(x_i)| \leq K_i \|x_i' - x_i\|_i$. Substituting into (9) yields

$$\sum C_i \lambda_i \|x_i' - x_i\|_i^2 \leq \sum K_i |\lambda_i' - \lambda_i| \, \|x_i' - x_i\|_i \quad (10)$$

Let $C = \min C_i$ and $K = \max K_i$. The left hand side of (10) is the summation of m positive terms, so is at least as large as any one of them. Because $\lambda, \lambda' \in \Lambda^\delta$, it follows that

$$C\delta \left(\max_i \|x_i' - x_i\|_i \right)^2 \leq K \max_i \|x_i' - x_i\|_i \left(\sum |\lambda_i' - \lambda_i| \right)$$

Rearranging terms yields

$$\max_i \|x_i' - x_i\|_i \leq \frac{K}{C\delta} \sum |\lambda_i' - \lambda_i|$$

which gives the desired Lipschitz estimate.

Finally, because the topology induced by each of the norms $\|\cdot\|_i$ coincides with the topology τ on the set $[0, \bar{e}]$ of feasible consumptions, the solution $x(\cdot)$ to the planner's problem is continuous in the topology τ as well. $\square$

6 Spending, Wealth and Excess Spending

In this section we turn to the second step in our program, demonstrating that the excess spending map is locally Lipschitz continuous. In light of the dependence of the excess spending map on the solution to the planner's problem, and the Lipschitz continuity of this solution we established in the previous section, this result is certainly intuitive. It is convenient to separate the argument into several lemmas, and to begin by showing that the spending map is locally Lipschitz.

Lemma 6.1 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then for each i the spending map

$$\lambda \mapsto p(\lambda) \cdot x_i(\lambda)$$

is locally Lipschitz on Λ^0.

Proof: Fix $\delta > 0$. Recall that $\Lambda^\delta = \{\lambda \in \Lambda : \lambda_i \geq \delta \text{ for each } i\}$ and $P^\delta(\bar{e}) = x(\Lambda^\delta)$. Because Λ^δ is a compact set and x is weakly continuous, $P_i^\delta(\bar{e})$ is a weakly compact subset of $P_i^0(\bar{e})$ for each i. Because each of the norms $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$, there are constants $B_i, C_i > 0$ such that

$$|DU_i(x_i) \cdot y| \leq B_i \|y\|_i \tag{11}$$

for all $x_i \in P_i^\delta(\bar{e})$ and $y \in X$, and

$$|DU_i(x_i) \cdot z - DU_i(x_i') \cdot z| \leq C_i \|x_i - x_i'\|_i \tag{12}$$

for all $x_i, x_i' \in P_i^\delta(\bar{e})$ and $z \in [0, \bar{e}]$. By Lemma 5.1, the solution to the planner's problem is Lipschitz on Λ^δ, so there is a constant $K_i > 0$ such that

$$\|x_i(\lambda) - x_i(\lambda')\|_i \leq K_i \sum |\lambda_i - \lambda_i'| \tag{13}$$

for $\lambda, \lambda' \in \Lambda^\delta$.

Now fix a consumer j and weights $\lambda, \lambda' \in \Lambda^\delta$. To simplify notation, let $x = x(\lambda), x' = x(\lambda')$, $p = p(\lambda), p' = p(\lambda')$, $p_i = \lambda_i DU_i(x_i)$ and $p'_i = \lambda'_i DU_i(x'_i)$ for each i.

If $0 \le z \le x_j(\lambda)$ then, as in the proof of Lemma 4.2, the first order conditions imply that

$$\lambda_j DU_j(x_j) \cdot z \ge \lambda_k DU_k(x_k) \cdot z$$

for each k. Hence $p(\lambda) \cdot x_j = \lambda_j DU_j(x_j) \cdot x_j = p_j \cdot x_j$. Similarly, $p(\lambda') \cdot x'_j = \lambda'_j DU_j(x'_j) \cdot x'_j = p'_j \cdot x'_j$. Thus

$$\begin{aligned} |p \cdot x_j - p' \cdot x'_j| &= |p_j \cdot x_j - p_j \cdot x'_j + p_j \cdot x'_j - p'_j \cdot x'_j| \\ &\le |p_j \cdot x_j - p_j \cdot x'_j| + |p_j \cdot x'_j - p'_j \cdot x'_j| \\ &= |p_j \cdot (x_j - x'_j)| + |(p_j - p'_j) \cdot x'_j| \end{aligned} \tag{14}$$

Because the norm $\|\cdot\|_j$ is adapted to U_j on $P_j^\delta(\bar{e})$ and the planner's problem is Lipschitz on Λ^δ, we conclude that

$$|p_j \cdot (x_j - x'_j)| \le B_j \|x_j - x'_j\|_j \le B_j K_j \sum |\lambda_i - \lambda'_i| \tag{15}$$

$$|(p_j - p'_j) \cdot x'_j| \le C_j \|x_j - x'_j\|_j \le C_j K_j \sum |\lambda_i - \lambda'_i| \tag{16}$$

Combining (14), (15), (16) yields the desired Lipschitz estimate for the spending map. □

Next we show that the wealth map is locally Lipschitz.

Lemma 6.2 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then the wealth map

$$\lambda \mapsto p(\lambda) \cdot w$$

is locally Lipschitz on Λ^0, uniformly for $w \in [0, \bar{e}]$.

Proof: Again fix $\delta > 0$ and let $\lambda, \lambda' \in \Lambda^\delta$. We again write $x = x(\lambda), x' = x(\lambda')$, $p = p(\lambda), p' = p(\lambda')$, $p_i = \lambda_i DU_i(x_i)$ and $p_i' = \lambda_i' DU_i(x_i')$ for each i. Fix an arbitrary $w \in [0, \bar{e}]$. By definition,

$$p \cdot w = \left(\bigvee p_i\right) \cdot w = \sup\left\{\sum p_i \cdot a_i : a_i \geq 0, \sum a_i = w\right\}.$$

Fix $\varepsilon > 0$ and choose (a_i) so that $\sum a_i = w$ and

$$p \cdot w \leq \varepsilon + \sum p_i \cdot a_i.$$

As in the proof of Lemma 6.1, we use quadratic concavity and adaptedness of the norm $\|\cdot\|_i$ to U_i on $P_i^\delta(\bar{e})$ to choose constants B_i, C_i, K_i so that (11)-(13) obtain. Thus

$$\begin{aligned}
p \cdot w - p' \cdot w &\leq \varepsilon + \sum p_i \cdot a_i - \sum p_i' \cdot a_i \\
&\leq \varepsilon + \sum (p_i - p_i') \cdot a_i \\
&\leq \varepsilon + \sum C_i \|x_i - x_i'\|_i \\
&\leq \varepsilon + \sum_i C_i K_i \left(\sum_k |\lambda_k - \lambda_k'|\right) \\
&= \varepsilon + \left(\sum_i C_i K_i\right)\left(\sum_k |\lambda_k - \lambda_k'|\right).
\end{aligned}$$

Reversing the roles of p, p' and keeping in mind that $\varepsilon > 0$ was arbitrary, we obtain

$$|p \cdot w - p' \cdot w| \leq \left(\sum_i C_i K_i\right)\left(\sum_k |\lambda_k - \lambda_k'|\right)$$

Because $w \in [0, \bar{e}]$ was arbitrary, this is the desired uniform Lipschitz estimate. □

The previous lemmas imply that the excess spending map is locally Lipschitz in λ and jointly continuous in (λ, e).

Lemma 6.3 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i *is quadratically concave with respect to* $\|\cdot\|_i$ *on weakly compact subsets of* $P_i^0(\bar{e})$

then for each i

(i) $S(\cdot, e)$ *is locally Lipschitz on* Λ^0*, uniformly for* $e \in D^0(\bar{e})$

(ii) $S_i(\cdot,\cdot)$ *is jointly continuous on* $\Lambda^0 \times D^0(\bar{e})$

Proof: The first assertion is immediate from Lemmas 6.1 and 6.2. To establish the second assertion, fix an arbitrary $(\lambda, e) \in \Lambda^0 \times D^0(\bar{e})$ and let $(\lambda', e') \in \Lambda^0 \times D^0(\bar{e})$ be another point. To see that S_i is continuous at (λ, e), note that

$$\begin{aligned} |S_i(\lambda,e) - S_i(\lambda',e')| &= |p(\lambda)\cdot x_i(\lambda) - p(\lambda')\cdot x_i(\lambda') - p(\lambda)\cdot e_i + p(\lambda')\cdot e_i'| \\ &\leq |p(\lambda)\cdot x_i(\lambda) - p(\lambda')\cdot x_i(\lambda')| \\ &\quad + |p(\lambda)\cdot e_i - p(\lambda)\cdot e_i' + p(\lambda)\cdot e_i' - p(\lambda')\cdot e_i'| \\ &\leq |p(\lambda)\cdot x_i(\lambda) - p(\lambda')\cdot x_i(\lambda')| \\ &\quad + |p(\lambda)\cdot e_i - p(\lambda)\cdot e_i'| + |p(\lambda)\cdot e_i' - p(\lambda')\cdot e_i'| \end{aligned}$$

Let $\varepsilon > 0$ be given. By Lemma 6.1 and Lemma 6.2, there is a neighborhood V of λ such that if $\lambda' \in V$ then

$$|p(\lambda)\cdot x_i(\lambda) - p(\lambda')\cdot x_i(\lambda')| < \varepsilon/3$$

and

$$|p(\lambda)\cdot e_i' - p(\lambda')\cdot e_i'| < \varepsilon/3$$

Continuity of the linear functional $p(\lambda)$ guarantees that there exists a neighborhood W of e such that if $e' \in W$ then

$$|p(\lambda)\cdot e_i - p(\lambda)\cdot e_i'| < \varepsilon/3$$

Thus if $(\lambda', e') \in V \times W$, then $|S_i(\lambda,e) - S_i(\lambda',e')| < \varepsilon$. It follows that S_i is continuous at (λ, e). Since (λ, e) is arbitrary, we conclude that S_i is continuous on $\Lambda^0 \times D^0(\bar{e})$, as asserted. □

7 Generic Determinacy

In this section we will use the results of Sections 5 and 6 to establish generic determinacy of equilibria. Here we consider the basic features of the economy $\mathcal{E}$ — the commodity space, price space, utility functions, and social endowment — as fixed, and consider variations in the initial endowment profile e over the set of all distributions of the social endowment $\bar{e}$. As before, we write

$$D^0(\bar{e}) \equiv \left\{ e \in X_+^m : e_i \neq 0 \text{ for each } i \text{ and } \sum e_i = \bar{e} \right\}$$

for the set of non-zero endowment distributions. For $e \in D^0(\bar{e})$, let $\mathcal{E}(e)$ denote the economy $\mathcal{E}$ with initial endowment profile e, and let $E(e)$ denote the set of equilibrium allocations of $\mathcal{E}(e)$.

As the arguments of the previous sections highlight, the Negishi method results in a characterization of equilibria in our basic economies that is independent of the dimensionality of the commodity space and formally identical to that arising in a standard Arrow-Debreu economy with a finite-dimensional commodity space. Equilibria are solutions to a finite system of equations in a finite number of variables, so a simple counting of equations and unknowns suggests that we might expect the qualitative features of equilibria in these economies to be similar to economies with a finite set of commodities. In particular, we might expect generic determinacy. By determinacy for the economy $\mathcal{E}(e)$ we mean finiteness of the number of equilibria and continuity of the equilibrium allocation correspondence. Formally:

Definition The economy $\mathcal{E}(e)$ is *determinate* if the number of equilibria is finite and the equilibrium allocation correspondence $E : D^0(\bar{e}) \to X_+^m$ is continuous at e.

In view of our discussion at the end of Section 4, we may identify an equilibrium of $\mathcal{E}(e)$ with a zero of $S(\cdot, e)$. It is convenient to define the equilibrium weight correspondence $E_\Lambda : D^0(\bar{e}) \to \Lambda$ by

$$E_\Lambda(e) \equiv \{\lambda \in \Lambda : S(\lambda, e) = 0\}$$

Lemma 5.1 guarantees that the solution to the planner's problem x is continuous with respect to the topology τ. It follows that $\mathcal{E}(e)$ is determinate if and only if $E_\Lambda(e)$ is finite and E_Λ is continuous at e.[12]

Our goal is to show that, given our assumptions, almost all endowment distributions lead to determinate economies. To make this statement precise, we need to explain what we mean by "almost all" endowment distributions. In a finite-dimensional setting, it is natural to interpret "almost all" to mean having full Lebesgue measure in the set of endowment distributions. In an infinite-dimensional setting, however, there is no natural measure on the set of endowment distributions. We provide two alternatives; the first makes use of a finite dimensional parameterization of endowment distributions, while the second uses an infinite-dimensional analogue of Lebesgue measure 0.

For our first determinacy result, fix a profile $e^* = (e_1^*, \ldots, e_m^*) \in X^m$ for which $\sum e_i^* = \bar{e}$ and a vector $v \in X_+ \setminus \{0\}$.[13] Set

$$A(e^*, v) = \left\{\alpha \in \mathbf{R}^m : e_i^* + \alpha_i v > 0 \text{ all } i \text{ and } \sum \alpha_i = 0\right\}$$

To each vector $\alpha \in A(e^*, v)$ corresponds an initial endowment distribution e^α of the social endowment $\bar{e}$ defined by $e_i^\alpha = e_i + \alpha_i v$ for each i. We view e^α as a perturbation of the initial profile e^*. Considering the family of such perturbations gives us a simple finite-dimensional parameterization of initial endowments indexed by $A(e^*, v)$. Our first determinacy result shows that those perturbations for which the economy $\mathcal{E}(e^\alpha)$ is determinate form a set of full $(m-1)$-dimensional Lebesgue measure.

Theorem 7.1 *Let $\mathcal{E}$ be a basic economy in which for each i there is a norm $\|\cdot\|_i$ such that*

[12]Note that we do not require continuity of the equilibrium price correspondence; as our first example of Section 3 suggests, continuity of the equilibrium price correspondence may be a delicate issue. However, for any fixed $z \in [0, \bar{e}]$, consider the equilibrium "evaluation" correspondence $P_z : D^0(\bar{e}) \to \mathbf{R}$ defined by

$$P_z(e) = \{p(\lambda) \cdot z : \lambda \in E_\Lambda(e)\}$$

Our results in Theorem 7.2, together with Lemma 6.2, show that P_z is continuous at e for each $z \in [0, \bar{e}]$ if the economy $\mathcal{E}(e)$ is determinate.

[13]For applicability in Theorem 7.2, we allow for the possibility that e_i^* is not positive. Moreover, note that for some choices of e^* and v, $A(e^*, v)$ may be empty, which we permit for use in Theorem 7.2 as well.

(a) $\|\cdot\|_i$ *is adapted on weakly compact subsets of* $P_i^0(\bar{e})$

(b) U_i *is quadratically concave with respect to* $\|\cdot\|_i$ *on weakly compact subsets of* $P_i^0(\bar{e})$

Then for each $e^* \in X^m$ *with* $\sum e_i^* = \bar{e}$ *and* $v \in X_+\backslash\{0\}$, *almost all parameters* $\alpha \in A(e^*, v)$ *lead to a determinate economy. That is,*

$$A_d(e^*, v) \equiv \{\alpha \in A(e^*, v) : \mathcal{E}(e^\alpha) \text{ is determinate } \}$$

is a set of full $(m-1)$*-dimensional Lebesgue measure in* $A(e^*, v)$.

Proof: First, normalize prices by defining

$$\widehat{p}(\lambda) = \left(\frac{1}{p(\lambda) \cdot v}\right) p(\lambda)$$

for each $\lambda \in \Lambda$. Let $\widehat{S} : \Lambda^0 \times D^0(\bar{e}) \to \mathbf{R}^{m-1}$ be the corresponding normalized excess spending mapping whose i-th coordinate is:

$$\widehat{S}_i(\lambda, e) \quad = \quad \widehat{p}(\lambda) \cdot [x_i(\lambda) - e_i]$$

Note that S and $\widehat{S}$ have the same zeroes. Define $\sigma : \Lambda^0 \to \mathbf{R}^{m-1}$ by

$$\sigma(\lambda) = \widehat{S}(\lambda, e^*)$$

In particular, note that

$$\widehat{S}_i(\lambda, e^\alpha) = \widehat{S}_i(\lambda, e^*) - \alpha_i = \sigma_i(\lambda) - \alpha_i$$

for each i. Equivalently, for each $\alpha \in A(e^*, v)$, let $\alpha_{-m} = (\alpha_1, \ldots, \alpha_{m-1}) \in \mathbf{R}^{m-1}$. Then

$$\widehat{S}(\lambda, e^\alpha) = \sigma(\lambda) - \alpha_{-m}$$

Lemma 6.3 guarantees that the excess spending map $S(\cdot, e)$ is locally Lipschitz on Λ^0 for each e, and Lemma 6.2 guarantees that $\lambda \mapsto p(\lambda) \cdot v$ is locally Lipschitz on Λ^0. It follows that $\widehat{S}(\cdot, e)$ is locally Lipschitz on Λ^0 for each e and thus that σ is locally Lipschitz on Λ^0.

If U is an open subset of $\mathbf{R}^{m-1}$ and $f : U \to \mathbf{R}^{m-1}$ is a mapping, recall that $\gamma \in \mathbf{R}^{m-1}$ is said to be a regular value of f if $Df(\zeta)$ exists and is nonsingular whenever $\zeta \in U$ and $f(\zeta) = \gamma$. Sard's theorem for locally Lipschitz functions (see Rader (1973) Lemma 2) guarantees that if f is locally Lipschitz then almost every element of $\mathbf{R}^{m-1}$ is a regular value of f. Because σ is locally Lipschitz, it follows that almost every $\gamma \in \mathbf{R}^{m-1}$ is a regular value of σ. It is evident that α_{-m} is a regular value of σ if and only if 0 is a regular value of $\widehat{S}(\cdot, e^\alpha)$, so the set

$$A_r(e^*, v) \equiv \left\{\alpha \in A(e^*, v) : 0 \text{ is a regular value of } \widehat{S}(\cdot, e^\alpha)\right\}$$

has full $(m-1)$-dimensional Lebesgue measure. To complete the proof it remains only to show that $A_r(e^*, v) \subset A_d(e^*, v)$, that is, if 0 is a regular value of $\widehat{S}(\cdot, e^\alpha)$ then $\mathcal{E}(e^\alpha)$ is a determinate economy.

To see this, fix an $\alpha \in A_r(e^*, v)$. We must show that $E_\Lambda(e^\alpha)$ is finite and that E_Λ is continuous at e^α.

To see that $\mathcal{E}(e^\alpha)$ has finitely many equilibria, note that by individual rationality, each equilibrium corresponds to a vector $\lambda \in \Lambda^o$. Each equilibrium vector of weights is locally unique because 0 is a regular value of $\widehat{S}(\cdot, e^\alpha)$ (see Shannon (1994)). Then to show there are only finitely many equilibria it suffices to show that

$$\Lambda^{IR} \equiv \{\lambda \in \Lambda : U_i(x_i(\lambda)) \geq U_i(e_i^\alpha) \text{ for all } i\}$$

is a compact subset of Λ^o. To that end, first note that because $x(\cdot)$ is weakly continuous on Λ and $U_i(\cdot)$ is weakly upper semi-continuous, Λ^{IR} is a compact set. Moreover, because utilities are strictly monotone, $x_i(\lambda) = 0$ if $\lambda_i = 0$. Since $e_i^\alpha > 0$ for each i, $U_i(e_i^\alpha) > U_i(0)$ for each i, again using the strict monotonicity of utilities, which implies that $\Lambda^{IR} \subset \Lambda^0$. Now each equilibrium vector of welfare weights is locally unique and contained in the compact set Λ^{IR}, so there are only finitely many equilibria in $\mathcal{E}(e^\alpha)$.

The upper hemi-continuity of E_Λ at e^α (and indeed, at every $e \in D^0(\bar{e})$) follows immediately from the joint continuity of S on $\Lambda^0 \times D^0(\bar{e})$. Although this is a standard argument, we repeat it here for completeness. Let $e^n \to e$ and $\lambda^n \in E_\Lambda(e^n)$ for each n. Since $U_i(e_i^n) \to U_i(e_i)$, there exists N such that for $n \geq N$, $U_i(e_i^n) \geq U_i(e_i/2)$ for each i. Thus for $n \geq N$,

$$\lambda^n \in \Lambda_e \equiv \{\lambda \in \Lambda : U_i(x_i(\lambda)) \geq U_i(e_i/2) \text{ for each } i\}$$

Moreover, $\Lambda_e \subset \Lambda^0$ and Λ_e is a compact set. Now choose a convergent subsequence of $\{\lambda^n\}$, and relabeling if necessary, choose $\lambda \in \Lambda_e$ such that $\lambda^n \to \lambda$. Since $\lambda^n \in E_\Lambda(e^n)$ for each n, $S(\lambda^n, e^n) = 0$ for each n. Using the joint continuity of S, we conclude that $S(\lambda, e) = \lim_n S(\lambda^n, e^n) = 0$, i.e., $\lambda \in E_\Lambda(e)$.

It remains only to show that E_Λ is lower hemi-continuous at e^α. Given $\lambda^* \in E_\Lambda(e^\alpha)$ and a neighborhood V^* of λ^* in Λ^0, we must find a neighborhood W of e^α in $D^0(\bar{e})$ such that if $e \in W$ then $\widehat{S}(\lambda, e) = 0$ for some $\lambda \in V^*$. To accomplish this, we use the invariance of Brouwer degree under small perturbations. If $N \subset \Lambda^0$ is an open set and $f : N \to \mathbf{R}^{m-1}$ is a continuous mapping, write $\deg(f, N, 0)$ for the Brouwer degree of f on N at 0.

Let $\lambda^* \in E_\Lambda(e^\alpha)$. Choose a neighborhood V of λ^* in Λ^0 such that $E_\Lambda(e^\alpha) \cap V = \{\lambda^*\}$. Because 0 is a regular value of $\widehat{S}(\cdot, e^\alpha)$, $|\deg(\widehat{S}(\cdot, e^\alpha), V', 0)| = 1$ for every neighborhood $V' \subset V$ of λ^* (see Shannon (1994), Theorem 9). Then for each such neighborhood $V' \subset V$ of λ^* there exists a neighborhood B' of graph $\widehat{S}(\cdot, e^\alpha)|_{V'}$ such that $|\deg(f, V', 0)| = 1$ for any continuous function $f : V' \to \mathbf{R}^{m-1}$ for which graph $f \subset B'$. In particular, for any such function f there exists $\lambda \in V'$ such that $f(\lambda) = 0$. To establish our result, it thus suffices to show that given a neighborhood B' of graph $\widehat{S}(\cdot, e^\alpha)|_{V'}$ there exists a neighborhood W of e^α in $D^0(\bar{e})$ such that graph $\widehat{S}(\cdot, e)|_{V'} \subset B'$ for each $e \in W$.

To see this, note that for any $e \in D^0(\bar{e})$ and $\lambda \in V'$,

$$\widehat{S}_i(\lambda, e) - \widehat{S}_i(\lambda, e^\alpha) = \hat{p}(\lambda) \cdot (e_i - e_i^\alpha)$$

Now let $\varepsilon > 0$ be given. The map $(\lambda, v) \mapsto p(\lambda) \cdot v$ is jointly continuous on $\Lambda^0 \times [0, \bar{e}]$ by Lemma 6.3. Thus for each $\lambda \in \overline{V}'$ there exists a neighborhood V_λ of λ and a neighborhood W_λ of e^α in $D^0(\bar{e})$ such that for each $e \in W_\lambda$ and $\tilde{\lambda} \in V_\lambda$

$$|\hat{p}(\tilde{\lambda}) \cdot (e_i - e_i^\alpha)| < \varepsilon$$

Since $\{V_\lambda\}$ is an open cover of $\overline{V}'$ and $\overline{V}'$ is compact, there is a finite subcover $\{V_{\lambda^1}, \ldots, V_{\lambda^N}\}$. Set $W = \cap W_{\lambda^j}$. Then for $e \in W$,

$$|\widehat{S}_i(\lambda, e) - \widehat{S}_i(\lambda, e^\alpha)| < \varepsilon$$

for each $\lambda \in V'$. Thus given a neighborhood B' of graph $\widehat{S}(\cdot, e^\alpha)|_{V'}$ there exists a neighborhood W of e^α in $D^0(\bar{e})$ such that graph $\widehat{S}(\cdot, e)|_{V'} \subset B'$ for each $e \in W$. Hence for each $e \in W$ there exists $\lambda \in V'$ such that $\widehat{S}(\lambda, e) = 0$, that is, such that $\lambda \in E_\Lambda(e)$. We conclude that E_Λ is lower-hemi-continuous at e^α, so the proof is complete. $\square$

Because of the finite-dimensional nature of this parameterization, this result is not entirely satisfactory. To obtain a more satisfactory result, we need a notion of "almost all" economies that is suitable in an infinite-dimensional setting. Unfortunately, there is no natural analogue of Lebesgue measure in an infinite-dimensional space, and topological notions of "almost all" are not entirely satisfactory, particularly in problems like ours in which "almost all" is interpreted in a probabilistic sense as a statement about the likelihood of an event occurring.[14] However, Christensen (1974) and Hunt, Sauer, and Yorke (1992) have developed measure-theoretic analogues of Lebesgue measure 0 and full Lebesgue measure for infinite-dimensional spaces, called shyness and prevalence. In many applications, particularly in economics, the parameters are drawn not from the whole space but from some subset, such as a convex cone or an order interval, that may be a very small subset of the ambient space. To address this problem, Anderson and Zame (1997) have extended the work of Hunt, Sauer and Yorke and Christensen by defining prevalence and shyness relative to a convex subset that may be a small subset of the ambient space. Their notion of relative prevalence, given below, is the most appropriate concept of "almost all" here.

Definition Let Y be a topological vector space and let $C \subset Y$ be a convex Borel subset which is completely metrizable in the relative topology. Let $c \in C$. A universally measurable subset $E \subset Y$ is *shy in C at c* if for each $\delta > 0$ and each neighborhood W of 0 in Y, there is a regular Borel probability measure μ on Y with compact support such that $\operatorname{supp}\mu \subset (\delta(C - c) + c) \cap (W + c)$ and $\mu(E + y) = 0$ for every $y \in Y$.[15] E is *shy in C* if it is shy at each point $c \in C$. A (not necessarily universally measurable) subset $F \subset C$ is *shy in C* if it is contained in a shy universally measurable set. A subset $E \subset C$ is *prevalent in C* if its complement $C \setminus E$ is shy in C.

[14]For example, open and dense sets in $\mathbf{R}^n$ can have arbitrarily small measure, and residual sets can have measure 0.

[15]Recall that a set $E \subset Y$ is universally measurable if for every Borel measure η on Y, E belongs to the completion with respect to η of the sigma algebra of Borel sets.

Anderson and Zame (1997) show that relative shyness and prevalence have the properties we ought to require of measure-theoretic notions of "smallness" and "largeness." In particular, the countable union of shy sets is shy, no relatively open subset is shy, and a subset of $\mathbf{R}^n$ is shy in $\mathbf{R}^n$ if and only if it has Lebesgue measure 0. Hunt, Sauer, and Yorke (1992) and Anderson and Zame (1997) also provide simple sufficient conditions for their notions of shyness and prevalence. Again the relative version from Anderson and Zame (1997) is the appropriate concept for our problem.

Definition Let Y be a topological vector space and let $C \subset Y$ be a convex Borel subset which is completely metrizable in the relative topology. A universally measurable set $E \subset C$ is *finitely shy in C* if there is a finite dimensional subspace $V \subset Y$ such that $(E+y) \cap V$ has Lebesgue measure 0 in V for every $y \in Y$. A universally measurable set $E \subset C$ is *finitely prevalent in C* if its complement $C \setminus E$ is finitely shy.

Anderson and Zame (1997) show that sets that are finitely shy are shy, hence sets that are finitely prevalent are prevalent. Using this fact leads to a more satisfactory determinacy result.

Theorem 7.2 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then almost all endowment distributions lead to a determinate economy. That is

$$D_d^0(\bar{e}) = \left\{ e \in D^0(\bar{e}) : \mathcal{E}(e) \text{ is determinate } \right\}$$

is prevalent in $D^0(\bar{e})$.

Proof: We will show that $D_d^0(\bar{e})$ is finitely prevalent in $D^0(\bar{e})$. As before, we use the fact that $\mathcal{E}(e)$ is determinate exactly if $E_\Lambda(e)$ is finite and E_Λ is continuous at e.

It is evident that $D^0(\bar{e})$ is a Borel set. To see that it is completely metrizable, define a norm on X^m by

$$\|(x_1, \ldots, x_m)\| = \max_i \|x_i\|_i$$

Adaptedness of $\|\cdot\|_i$ implies that the topology induced by $\|\cdot\|_i$ agrees with the topology τ on the order interval $[0, \bar{e}]$, so the topology induced by $\|\cdot\|$ agrees with the product topology τ^m on the set $D(\bar{e}) = \{e \in X^m_+ : \sum e_i = \bar{e}\}$. Because order intervals are weakly compact in X, $D(\bar{e})$ is weakly compact in X^m. It follows that $D(\bar{e})$ is complete in the metric induced by the norm $\|\cdot\|_{max}$.[16] Because $D^0(\bar{e})$ is a relatively open subset of $D(\bar{e})$, there is a complete metric on $D^0(\bar{e})$ having the property that the metric topology coincides with the topology τ^m.

We next show that $D^0_d(\bar{e})$ is a Borel set. Toward this end, write $D^0_f(\bar{e})$ for the endowment distributions e for which $E(e)$ is finite (equivalently, for which $E_\Lambda(e)$ is finite), and $D^0_c(\bar{e})$ for the endowment distributions e at which the equilibrium correspondence E is continuous (equivalently, at which the equilibrium weight correspondence E_Λ is continuous). As $D^0_d(\bar{e}) = D^0_f(\bar{e}) \cap D^0_c(\bar{e})$, it suffices to show that these are Borel sets.

To see that $D^0_f(\bar{e})$ is a Borel set, write $\mathbf{Q}_+$ for the set of strictly positive rational numbers. For each positive integer n, let $\mathcal{R}^n = (\mathbf{Q}^m_+ \cap \Lambda^0)^n$ be the set of n-tuples of points in Λ^0 with rational coordinates. For $r_j \in \mathbf{Q}^m_+ \cap \Lambda^0$ and $\beta_j \in \mathbf{Q}_+$, let $B(r_j, \beta_j)$ be the open ball in $\mathbf{R}^m$ with center r_j and radius β_j. An endowment distribution $e \in D^0(\bar{e})$ leads to an economy with at most n equilibria exactly if the set of equilibrium weights is contained in the union of n balls with rational centers and arbitrarily small rational radii. Hence

$$D^0_f(\bar{e}) = \bigcup_{n=1}^{\infty} \bigcap_{r \in \mathcal{R}^n} \bigcup_{\beta \in \mathbf{Q}^n_+} \left\{ e \in D^0(\bar{e}) : E_\Lambda(e) \subset \bigcup_{j=1}^{n} B(r_j, \beta_j) \right\}$$

Because E_Λ is upper hemi-continuous (see the proof of Theorem 7.1), each of the sets in curly brackets is is open, so $D^0_f(\bar{e})$ is a Borel set.

[16]Let $\widetilde{X}^m$ be the completion of X^m with respect to the topology τ. Note that X^m and $\widetilde{X}^m$ have the same dual spaces. Hence $D(\bar{e})$ is weakly closed in $\widetilde{X}^m$. Since τ is a stronger topology, $D(\bar{e})$ is also τ-closed in $\widetilde{X}^m$. Now because $\widetilde{X}^m$ is complete, $D(\bar{e})$ is τ-complete as well.

To see that $D^0_c(\bar{e})$ is a Borel set, let h denote the Hausdorff distance between compact subsets of Λ^0. The correspondence E_Λ is continuous at e exactly if for each integer n there is a neighborhood W of e with the property that $h(E_\Lambda(e'), E_\Lambda(e'')) < 1/n$ for $e', e'' \in W$. Hence

$$\begin{aligned} D^0_c(\bar{e}) \quad &= \quad \bigcap_{n=1}^{\infty} \{e \in D^0(\bar{e}) : \exists \text{ open } W \subset D^0(\bar{e}) \text{ s.t. } e \in W \\ &\qquad\qquad \text{and } h(E_\Lambda(e'), E_\Lambda(e'')) < 1/n \text{ for all } e', e'' \in W\} \end{aligned}$$

Thus, $D^0_c(\bar{e})$ is the countable intersection of open sets, and in particular is a Borel set.

Now let $D^0_{nd}(\bar{e}) = D^0(\bar{e}) \setminus D^0_d(\bar{e})$. To show that $D^0_{nd}(\bar{e})$ is finitely shy, set $v = \frac{1}{m}\bar{e}$, and let $V \subset X^m$ be the $(m-1)$-dimensional subspace

$$V = \{(\alpha_1 v, \ldots, \alpha_m v) : \sum \alpha_i = 0\}$$

If $\eta^* = (v, \ldots, v)$ then

$$V \cap [D^0(\bar{e}) - \eta^*] = \{(\alpha_1 v, \ldots, \alpha_m v) : \sum \alpha_i = 0 \text{ and } \alpha_i > -1 \text{ all } i\,\}$$

so $V \cap [D^0(\bar{e}) - \eta^*]$ certainly has positive measure in V. Now let $\eta \in X^m$ and consider $V \cap [D^0_{nd}(\bar{e}) - \eta]$. If $y \in V \cap [D^0_{nd}(\bar{e}) - \eta]$, then there exists $e \in D^0_{nd}(\bar{e})$ and α such that $\sum \alpha_i = 0$ for which $y_i = \alpha_i v = e_i - \eta_i$ for each i. In particular, $e_i = \eta_i + \alpha_i v > 0$ for each i and $\mathcal{E}(e)$ is not determinate. Thus $\alpha \in A_{nd}(\eta, v)$, which has $(m-1)$-dimensional Lebesgue measure 0 by Theorem 7.1. Thus

$$V \cap [D^0(\bar{e}) - \eta^*] = \{(\alpha_1 v, \ldots, \alpha_m v) : \alpha \in A_{nd}(\eta, v)\}$$

has $(m-1)$-dimensional measure 0. We conclude that $D^0_{nd}(\bar{e})$ is finitely shy, and thus that $D^0_d(\bar{e})$ is finitely prevalent, in $D^0(\bar{e})$ as asserted. $\square$

Stronger assumptions on consumers' utility functions lead to a stronger conclusion about local comparative statics. To make this statement precise we need two additional notions.

Definition The economy $\mathcal{E}(e)$ is *Lipschitz determinate* with respect to $\|\cdot\|$ if it is determinate and for every equilibrium $x \in E(e)$ there exist neighborhoods O of x and W of e such that every selection from $O \cap E$ is locally Lipschitz on W with respect to $\|\cdot\|$.

For this result we will need to specify a single norm on the commodity space. For each $x \in X$ define

$$\|x\|_{max} = \max_i \|x\|_i.$$

If each individual norm $\|\cdot\|_i$ is *absolute*, that is if $\||x|\| = \|x\|$ for every $x \in X$, then Lipschitz determinacy with respect to $\|\cdot\|_{max}$ holds for a prevalent set of endowment distributions.[17]

Theorem 7.3 *If $\mathcal{E}$ is a basic economy and for each i there is an absolute norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then almost all endowment distributions lead to an economy that is Lipschitz determinate with respect to the norm $\|\cdot\|_{max}$ on X. That is,

$$D^0_{ld}(\bar{e}) = \left\{ e \in D^0(\bar{e}) : \mathcal{E}(e) \textit{ is Lipschitz determinate with respect to } \|\cdot\|_{max} \right\}$$

is prevalent in $D^0(\bar{e})$.

[17] Requiring that a norm be absolute is not innocuous. For instance, there are many norms on $M[0,1]$ (the space of signed measures on the unit interval) for which the topology induced by the norm coincides with the weak star topology (viewing $M[0,1]$ as the dual of the space $C[0,1]$ of continuous functions) on order intervals, but there is no absolute norm with this property. To see this, let μ be Lebesgue measure on $[0,1]$. For each n let

$$\begin{aligned} E^n &= \bigcup_{k=0}^{n-1} [2k/2n, (2k+1)/2n) \\ F^n &= [0,1] \setminus E^n \end{aligned}$$

and let $\nu^n = \mu_{|E^n} - \mu_{|F^n}$. It is easily checked that $\nu^n \to 0$ in the weak star topology but that $|\nu^n| = \mu$ for each n. Hence for any norm $\|\cdot\|$ on $M[0,1]$ that induces the weak star topology on the order interval $[0,\mu]$, $\|\nu^n\| \to 0$ while $\||\nu^n|\| = \|\mu\|$ for each n.

Proof: Fix a compact subset $\Lambda^* \subset \Lambda^0$. We first show that, for $\lambda \in \Lambda^*$, supporting prices $p(\lambda)$ are uniformly bounded in the $\|\cdot\|_{max}$ norm. To this end, note that $\|\cdot\|_{max}$ is an absolute norm, as

$$\||x|\|_{max} = \max_i \||x|\|_i = \max_i \|x\|_i = \|x\|_{max}$$

Moreover, by definition,

$$\|p(\lambda)\|_{max} = \sup_{\|z\|_{max} \leq 1} |p(\lambda) \cdot z|$$

Because $p(\lambda)$ is positive and $\|\cdot\|_{max}$ is absolute,

$$\|p(\lambda)\|_{max} = \sup_{\|z\|_{max} \leq 1} |p(\lambda) \cdot z| = \sup_{\substack{\|z\|_{max} \leq 1 \\ z \in X_+}} p(\lambda) \cdot z$$

By definition, $p(\lambda) = \bigvee \lambda_i DU_i(x_i(\lambda))$ and each $\lambda_i DU_i(x_i(\lambda))$ is a positive linear functional, so for $z \in X_+$ we have

$$0 \leq p(\lambda) \cdot z \leq \left[\sum \lambda_i DU_i(x_i(\lambda))\right] \cdot z = \sum \left[\lambda_i DU_i(x_i(\lambda)) \cdot z\right]$$

Using the adaptedness of $\|\cdot\|_i$ on $x(\Lambda^*)$ and the definition of $\|\cdot\|_{max}$ we obtain

$$\begin{aligned}
\|p(\lambda)\|_{max} &= \sup_{\substack{\|z\|_{max} \leq 1 \\ z \in X_+}} p(\lambda) \cdot z \\
&\leq \sup_{\substack{\|z\|_{max} \leq 1 \\ z \in X_+}} \sum \left[\lambda_i DU_i(x_i(\lambda)) \cdot z\right] \\
&\leq \sum \lambda_i B_i \|z\|_i \\
&\leq \sum \lambda_i B_i \\
&\leq \sum B_i
\end{aligned}$$

for some constants $B_i > 0$.

We now show that the excess spending map S is jointly locally Lipschitz on $\Lambda^0 \times D^0(\bar{e})$. Fix a consumer i. For $\lambda, \lambda' \in \Lambda^0$ and $e, e' \in D^0(\bar{e})$

$$\begin{aligned}
|S_i(\lambda, e) - S_i(\lambda', e')| &= |p(\lambda) \cdot [x_i(\lambda) - e_i] - p(\lambda') \cdot [x_i(\lambda') - e'_i]| \\
&\leq |p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| + |p(\lambda') \cdot e'_i - p(\lambda) \cdot e_i|
\end{aligned}$$

$$
\begin{aligned}
&\leq |p(\lambda)\cdot x_i(\lambda) - p(\lambda')\cdot x_i(\lambda')| \\
&\quad + |p(\lambda')\cdot e_i' - p(\lambda')\cdot e_i| \\
&\quad + |p(\lambda')\cdot e_i - p(\lambda)\cdot e_i| \\
&\leq |p(\lambda)\cdot x_i(\lambda) - p(\lambda')\cdot x_i(\lambda')| \\
&\quad + \|p(\lambda')\|_{max}\|e_i' - e_i\|_{max} \\
&\quad + |[p(\lambda') - p(\lambda)]\cdot e_i|
\end{aligned}
$$

Consider the last three terms. Lemma 6.1 guarantees that there is a constant C_1 such that

$$|p(\lambda)\cdot x_i(\lambda) - p(\lambda')\cdot x_i(\lambda')| \leq C_1|\lambda - \lambda'|$$

The bound obtained in the previous paragraph guarantees that

$$\|p(\lambda')\|_{max}\|e_i' - e_i\|_{max} \leq \sum B_i\|e_i' - e_i\|_{max}$$

Lemma 6.2 guarantees there is a constant C_2 such that

$$|[p(\lambda') - p(\lambda)]\cdot e_i| \leq C_2|\lambda - \lambda'|$$

Putting these together, we conclude that S is Lipschitz on $\Lambda^* \times D^0(\bar{e})$, and in particular, is locally Lipschitz on $\Lambda^0 \times D^0(\bar{e})$, as asserted.

The result now follows from the transversality results in Theorem 2.2 and Theorem 3.7 in Shannon (1998b). □

8 Examples

In this section we develop several examples illustrating our results in a variety of different settings. We show that our results can be applied to canonical preferences in each of the central infinite-dimensional models, continuous time trading in financial markets, commodity differentiation, and trade over an infinite horizon.

Example 8.1 Continuous-Time Trading in Financial Markets. The standard model[18] of continuous time trading begins with a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t : 0 \le t \le T\}$ of sub-sigma-algebras of $\mathcal{F}$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_t \subset \mathcal{F}_{t'}$ if $t < t'$. The filtration $\{\mathcal{F}_t\}$ represents revelation of information over the time interval $[0, T]$, as $\mathcal{F}_t$ is the sigma-algebra of events observable at time t. Commodity bundles are square integrable predictable stochastic processes, leading to the commodity space $L^2(\Omega \times [0, T], \mathcal{P}, \nu)$, where $\mathcal{P}$ is the predictable sigma algebra and $\nu = P \times \mu$ is the product of P with (normalized) Lebesgue measure on the time interval $[0, T]$.

Each consumer i is characterized by an initial endowment e_i and a utility function, usually assumed to have an expected utility representation of the form

$$U^i(x) = E\left[\int_0^T u^i(x_t, t)dt\right] = \int_\Omega \left[\int_0^T u^i(x_t(\omega), t)dt\right] dP(\omega)$$

where $u^i(\cdot, t) : \mathbf{R}_+ \to \mathbf{R}$ is strictly increasing and strictly concave for each t. It is usually assumed that utility functions satisfy Inada conditions, so that the partial derivatives $u^i_c(x, t) \to \infty$ as $x \to 0$, uniformly in t, and that the social endowment $\bar{e}$ is bounded above and uniformly bounded away from 0.

Under these assumptions for each consumer, it is easily verified that every Pareto optimal allocation in $P^0(\bar{e})$ is uniformly bounded away from 0. A straightforward computation then shows that for each i the L^2 norm is adapted to U_i on weakly compact subsets of $P^0_i(\bar{e})$.

If in addition $u^i(\cdot, t)$ is C^2 and differentiably strictly concave, uniformly in t, then U^i is quadratically concave on weakly compact subsets of $P^0_i(\bar{e})$

[18] See Duffie and Zame (1989) or Breeden (1979) for instance.

with respect to the L^2 norm. To see this, note that for each $b^* \in \mathbf{R}_+$ there is a constant $K_i > 0$ such that

$$u^i(b,t) \leq u^i(a,t) - u^i_c(a,t)(b-a) - K_i\|b-a\|^2$$

for each $t \in [0,T]$ and $a, b \in [0, b^*]$.[19] Then for $x, y \in [0, \bar{e}]$

$$\begin{aligned}
U^i(y) - U^i(x) &= E\left[\int_0^T u^i(y_t,t)dt\right] - E\left[\int_0^T u^i(x_t,t)dt\right] \\
&= E\left[\int_0^T [u^i(y_t,t) - u^i(x_t,t)]dt\right] \\
&\leq E\left[\int_0^T [u^i_c(x_t,t)(y_t - x_t) - K_i|y_t - x_t|^2]dt\right] \\
&= E\left[\int_0^T u^i_c(x_t,t)(y_t - x_t)dt\right] - K_i E\left[\int_0^T |y_t - x_t|^2 dt\right] \\
&= DU^i(x)\cdot(y-x) - K_i\|y-x\|^2
\end{aligned}$$

which is the required inequality.

Because the L^2 norm is absolute, Theorem 7.3 guarantees that almost all endowment distributions lead to economies which are Lipschitz determinate with respect to the L^2 norm.

In the framework above, a commodity bundle x represents a rate of consumption. Hindy, Huang, and Kreps (1992) (see also Hindy and Huang (1992, 1993)) argue that intertemporal consumption patterns should admit the possibility of consumption in discrete lumps, which they term "gulps", as well as in rates, which they term "sips". For consumption over the time interval $[0,1]$, they suggest that commodities should be represented by positive, increasing, right continuous functions $\varphi : [0,1] \to \mathbf{R}_+$, where $\varphi(t)$ gives total consumption at or before time t. In this formulation, consumption occurs in "sips" at points of continuity of φ and in "gulps" at points where φ has an upward jump. For our purposes it is convenient to adopt an equivalent formulation in which commodity bundles are non-negative measures x on

[19] Indeed, it would suffice to assume that $u^i(\cdot,t)$ is C^1 and appropriately quadratically concave.

$[0,1]$, so that $x[0,t]$ represents total consumption on the interval $[0,t]$. In our formulation, consumption occurs in "sips" at points where x has no mass and in "gulps" at atoms of x. Equivalently, note that the functions that represent commodity bundles in the Hindy, Huang and Kreps formulation are just the cumulative distribution functions of the measures that represent commodity bundles in our formulation. This alternative formulation leads to the commodity space $M[0,1]$, the space of signed measures on $[0,1]$. As the following example shows, models such as those developed by Hindy, Huang and Kreps also satisfy our requirements.

Example 8.2 Lumpy Consumption. The commodity space is $M[0,1]$, endowed with the weak star topology when viewed as the dual of the space of continuous functions $C[0,1]$. To capture the idea that consumptions at nearby dates should be nearly perfect substitutes at the margin, Hindy, Huang, and Kreps (1992) assume that preferences are continuous in the weak star topology and uniformly proper with respect to one of a particular family of norms of the form

$$\|x\|_p = \left[\int_0^1 |x[0,t]|^p dt + |x[0,1]|^p\right]^{1/p}$$

for $p \geq 1$.

A typical utility function satisfying their assumptions is one of the form:

$$U(x) = \int_0^1 u(x[0,t],t)dt + v(x[0,1])$$

where $u(\cdot,t) : \mathbf{R}_+ \to \mathbf{R}$ is C^2, strictly increasing and strictly concave for each t and $v : \mathbf{R}_+ \to \mathbf{R}$ is C^2, strictly increasing, and strictly concave. Suppose in addition that $v''(c) < 0$ for each c and that $u_{cc}(c,t) < 0$ for each c,t.[20] We assert that the norm $\|\cdot\|_1$ is adapted to U on every order interval $[0,\bar{e}]$, and that U is quadratically concave on every order interval $[0,\bar{e}]$ with respect to this norm.[21]

[20] Again, it would suffice to assume that v and u are C^1 and appropriately quadratically concave.

[21] Note that $\|\cdot\|_1$ is not an absolute norm: if $x \in M[0,1]$ then $\|x\|_1 \geq \||x|\|_1$, but equality holds exactly when $x \geq 0$ or $x \leq 0$. Indeed, as we have noted in footnote 17, no absolute

Hindy, Huang, and Kreps (1992) show that the topology induced by $\|\cdot\|_1$ (or indeed any of their norms) coincides with the weak star topology on order intervals. To verify that the norm $\|\cdot\|_1$ is adapted to U, therefore, we must only verify the relevant properties of derivatives. To this end, note that our assumptions provide a constant C such that $v'(c) \leq C$ and $u_c(\cdot, t) \leq C$ for every $c \leq \bar{e}[0,1]$. Thus

$$\begin{aligned} |DU(x) \cdot y| &= \left| \int_0^1 u_c(x[0,t],t) y[0,t] dt + v'(x[0,1]) y[0,1] \right| \\ &= \leq \int_0^1 C|y[0,t]| dt + C|y[0,1]| \\ &= C\|y\|_1 \end{aligned}$$

for $x \in [0, \bar{e}]$ and $y \in M[0,1]$. Similarly, there exists $C' > 0$ such that

$$\begin{aligned} |DU(y) \cdot z - DU(y') \cdot z| &\leq \left| \int_0^1 \left[u_c(y[0,t],t) - u_c(y'[0,t],t)\right] z[0,t] dt \right| \\ &\quad + |[v'(y[0,1]) - v'(y'[0,1])] z[0,1]| \\ &\leq C'\|y - y'\|_1 \end{aligned}$$

for $y, y', z \in [0, \bar{e}]$, so $\|\cdot\|_1$ is adapted to U on $[0, \bar{e}]$.

To see that U is quadratically concave, fix $x, y \in [0, \bar{e}]$. Differential strict concavity of v and u provides a constant $C'' > 0$ such that:

$$\begin{aligned} u(c',t) - u(c,t) &\leq u_c(c,t)(c' - c) - C''|c' - c|^2 \\ v(c') - v(c) &\leq v'(c)(c' - c) - C''|c' - c|^2 \end{aligned}$$

for each $c, c' \leq \bar{e}[0,1]$ and for each t. Hence

$$\begin{aligned} U(y) - U(x) &= \int_0^1 \left[u(y[0,t],t) - u(x[0,t],t)\right] dt \ + \ v(y[0,1]) - v(x[0,1]) \\ &\leq \int_0^1 \left[u_c(x[0,t],t)(y[0,t] - x[0,t]) \ - \ C''|y[0,t] - x[0,t]|^2\right] dt \\ &\quad + v'(x[0,1])(y[0,1] - x[0,1]) \ - \ C''|y[0,1] - x[0,1]|^2 \end{aligned}$$

norm on $M[0,1]$ has the property that the norm topology coincides with the weak star topology on order intervals. Of course the total variation norm is an absolute norm on $M[0,1]$ — but the utility functions considered above are not quadratically concave with respect to the total variation norm.

$$
\begin{aligned}
&= DU(x) \cdot (y - x) \\
&\quad - C'' \left[\int_0^1 |y[0,t] - x[0,t]|^2 dt + |y[0,1] - x[0,1]|^2 \right] \\
&\leq DU(x) \cdot (y - x) - C'' \|y - x\|_1^2
\end{aligned}
$$

where the last inequality is a consequence of Jensen's inequality.

Infinite horizon economies are perhaps the most familiar examples of models with infinitely many commodities. As we have noted in Example 3.2, the standard assumption that individuals discount future consumption entails that utility functions will not be quadratically concave with respect to the ℓ_∞ norm, but only with respect to some weighted norm. The following example makes the same point for a more general specification of utilities.

Example 8.3 Infinite-Horizon Economies. Consider a discrete time infinite horizon economy in which the commodity space is ℓ_∞ endowed with the Mackey topology. Consumer i is characterized by an endowment e_i and a utility function that displays habit formation of the form:

$$U_i(x) = v_i(x_0) + \sum_{t=1}^{\infty} \beta_i^t u_i(x_{t-1}, x_t)$$

where $\beta_i \in (0,1)$ is a discount factor. We assume that $v_i : \mathbf{R}_+ \to \mathbf{R}$ and $u_i : \mathbf{R}_+^2 \to \mathbf{R}$ are C^2, strictly increasing and differentiably strictly concave.

As in Example 3.2, it is easily seen that such utility functions are not quadratically concave on any bounded set with respect to the ℓ_∞ norm, but only with respect to the weighted norm:

$$\|x\|_{\beta_i} = \sum_{t=1}^{\infty} \beta_i^t |x_t|$$

To see this, let $\bar{e} \in \ell_{\infty+}$ be given and let $x, y \in [0, \bar{e}]$. To simplify notation, for each $z \in \ell_\infty$ and for each t let $z(t) = (z_{t-1}, z_t)$. Then

$$U_i(y) - U_i(x) \quad = \quad v_i(y_0) - v_i(x_0) + \sum_{t=1}^{\infty} \beta_i^t \left[u_i(y(t)) - u_i(x(t)) \right]$$

$$
\begin{aligned}
&\leq \quad v_i'(x_0)(y_0 - x_0) - c|y_0 - x_0|^2 + \sum_{t=1}^{\infty} \beta_i^t Du_i(x(t)) \cdot [y(t) - x(t)] \\
&\quad\quad -c \sum_{t=1}^{\infty} \beta_i^t \|y(t) - x(t)\|^2 \\
&= \quad DU_i(x) \cdot (y - x) - c \left[|y_0 - x_0|^2 + \sum_{t=1}^{\infty} \beta_i^t \|y(t) - x(t)\|^2 \right] \\
&\leq \quad DU_i(x) \cdot (y - x) - c(1 + \beta_i) \sum_{t=0}^{\infty} \beta_i^t |y_t - x_t|^2 \\
&\leq \quad DU_i(x) \cdot (y - x) - c(1 + \beta_i) b \left(\sum_{t=0}^{\infty} \beta_i^t |y_t - x_t| \right)^2 \\
&= \quad DU_i(x) \cdot (y - x) - c(1 + \beta_i) b \|y - x\|_{\beta_i}^2
\end{aligned}
$$

for some $c, b > 0$, where the first inequality follows from the quadratic concavity of u and the last from the fact that in a finite measure space, there exists $B > 0$ such that $\|f\|_2 \geq B\|f\|_1$ for all f, where $\|\cdot\|_p$ denotes the L_p norm for $1 \leq p \leq \infty$.

Similarly, it is straightforward to verify that this weighted norm is adapted to the utility function U_i on bounded sets. However, note that if different individuals discount future consumption at different rates then we must use different norms for each consumer.

Because each of these weighted norms is absolute, Theorem 7.3 guarantees that almost all endowment distributions lead to Lipschitz determinate economies.

Another important context in which infinite-dimensional models are natural is the choice of product characteristics or spatial location, as in the classic work of Hotelling and Lancaster. As the following example, based on Jones (1984) shows, some of the utility functions which are most natural in this setting also lead to economies that are generically determinate.

Example 8.4 Differentiated Commodities. Let Ω be a compact Hausdorff space representing the characteristics goods may possess. A commodity bundle is a non-negative measure on Ω and the commodity space is thus $M(\Omega)$, the space of finite signed measures on Ω. For simplicity, take $\Omega = [0,1]$. The commodity space is then $M[0,1]$, endowed with the weak star topology when viewed as the dual of the space of continuous functions $C[0,1]$. We interpret a positive measure as a description of the characteristics of a commodity bundle. Jones (1984) argues that nearby commodity characteristics should be nearly perfect substitutes, and gives a class of preferences satisfying such substitutability in characteristics. These preferences are represented by utility functions constructed via "rolling averages", that is, of the form

$$U(x) = \int_0^1 u(x[t-\alpha, t+\alpha])dt$$

where $u : \mathbf{R}_+ \to \mathbf{R}$ is C^2, strictly increasing and strictly concave, and α is a parameter with $0 < \alpha < 1/2$. Here and in what follows we interpret addition, subtraction and multiplication modulo 1, unless explicitly noted otherwise.

Rolling averages utility functions may model distinct commodities as perfect substitutes, however: if the parameter $\alpha = k/\ell$ is rational, such utility functions are not even strictly concave. To see this, let μ be Lebesgue measure on $[0,1]$. Set

$$\begin{aligned} E &= \bigcup_{j=0}^{\ell-1} [2j/2\ell, (2j+1)/2\ell) \\ F &= \bigcup_{j=0}^{\ell-1} [(2j+1)/2\ell, (2j+2)/2\ell) \end{aligned}$$

Let $x = \mu_{|E}$ and $y = \mu_{|F}$. The measures x and y, and all convex combinations of x and y, assign equal mass to every interval of length $2\alpha = 2k/\ell$. Since a consumer having such a utility function cares only about the total mass of each interval of length 2α, such a consumer is indifferent between x and y and any convex combination of x and y. Using this fact, it is easy to construct economies in which consumer utility functions are of this form and equilibria display robust indeterminacies.

However, if the parameter α is irrational, then these utility functions are quadratically concave with respect to an appropriate adapted norm. To see this, fix an irrational α. For each $x \in M[0,1]$, define

$$\|x\|_\alpha = \int_0^1 |x[t-\alpha, t+\alpha]|dt$$

We assert that $\|\cdot\|_\alpha$ is a norm on $M[0,1]$,[22] that $\|\cdot\|_\alpha$ is adapted to U on order intervals, and that U is quadratically concave with respect to this norm on order intervals. These facts, which we establish in detail below, and Theorem 7.2 imply that almost all endowment distributions lead to determinate economies under such preferences. As in Example 8.3, note that if α differs across consumers, so that consumers average over intervals of different lengths, then we must use a different adapted norm for each consumer.

To verify that $\|\cdot\|_\alpha$ is a norm, we must show first that if $\|x\|_\alpha = 0$ then x is the 0 measure. Suppose therefore that $\|x\|_\alpha = 0$. Let $A \subset [0,1]$ be the (possibly empty) set of atoms of x, and define

$$A^* = \{t \in [0,1] : \exists \text{ integer } n \text{ s.t. } t + n\alpha \in A\}$$

Note that A and A^* are both countable, although possibly both empty.

Suppose $a \in [0,1] \setminus A^*$ and $x[a, a+2\alpha] \neq 0$. For $0 < \varepsilon < \frac{1}{2} - \alpha$, observe that

$$x[a+\varepsilon, a+2\alpha+\varepsilon] - x[a, a+2\alpha] = -x[a, a+\varepsilon) + x(a+2\alpha, a+2\alpha+\varepsilon]$$

For ε sufficiently small, both terms on the right hand side are small because $a \notin A^*$, so if $x[a, a+2\alpha] \neq 0$ then $x[a+\varepsilon, a+2\alpha+\varepsilon] \neq 0$ for all ε sufficiently small. But then

$$\|x\|_\alpha = \int_0^1 |x[t-\alpha, t+\alpha]| dt \geq \int_{a-\alpha}^{a-\alpha+\varepsilon} |x[t-\alpha, t+\alpha]| dt > 0$$

which is a contradiction. We conclude that $x[a, a+2\alpha] = 0$ for every $a \in [0,1] \setminus A^*$.

Now fix $a, b \in [0,1] \setminus A^*$ with $b > a$. If $0 < \varepsilon < b - a$, irrationality of α implies that we can find infinitely many positive integers k, ℓ such that $b - \varepsilon + \ell < a + k(2\alpha) < b + \ell$, where here we are departing from our convention of interpreting quantities mod 1. Reading modulo 1 again, we conclude that given any $\varepsilon > 0$, we can find infinitely many positive integers k such that $b - \varepsilon < (a + k(2\alpha)) \bmod 1 < b$. Because $a \in [0,1] \setminus A^*$, we conclude from the previous paragraph that

$$x[a + n(2\alpha), a + (n+1)2\alpha] = 0$$

[22] Note that $\|\cdot\|_\alpha$ is *not* an absolute norm; see the discussion in footnote 21.

for every integer $n \geq 0$. Hence for integers k, ℓ as above,

$$\begin{aligned} 0 &= \sum_{n=0}^{k} x[a + n(2\alpha), a + (n+1)2\alpha] \\ &= \sum_{n=0}^{k} \int_{a+n(2\alpha)}^{a+(n+1)2\alpha} 1\,dx \\ &= (\ell - 1) \int_0^1 dx + \int_a^{a+k(2\alpha)} 1\,dx \\ &= (\ell - 1)x[0,1] + x[a, a + k(2\alpha)] \end{aligned}$$

Thus for each k, ℓ as above, $(\ell - 1)x[0,1] = -x[a, a + k(2\alpha)]$. For each k, $|x[a, a + k(2\alpha)]| \leq \|x\|$, hence $(\ell - 1)x[0,1]$ must be bounded for each ℓ. On the other hand, if $k \to \infty$ then $\ell \to \infty$, so $(\ell - 1)x[0,1]$ can be bounded only if $x[0,1] = 0$. Then for each k,

$$0 = (\ell - 1)x[0,1] + x[a, a + k(2\alpha)] = x[a, a + k(2\alpha)]$$

Because $\varepsilon > 0$ is arbitrary and a, b are not atoms of x, we can choose k as large as we like and still arrange that $x[a, a + k(2\alpha)]$ is as close as we like to $x[a, b]$. It follows that $x[a, b] = 0$ for $a, b \notin A^*$. Because A^* is countable, every interval $I \subset [0,1]$ is the descending intersection of countably many intervals whose endpoints do not lie in A^*, and hence $x(I) = 0$ for every interval $I \subset [0,1]$. Hence $x(E) = 0$ for every Borel set, whence x is the 0 measure.

Verification of the other requirements for $\|\cdot\|_\alpha$ to be a norm is straightforward and left to the reader.

To see that the topology induced by $\|\cdot\|_\alpha$ coincides with the weak star topology on bounded sets, recall that weak star closed, bounded sets are weak star compact and metrizable. It therefore suffices to show that if (x^n) is a bounded sequence converging weak star to 0 then $\|x^n\|_\alpha \to 0$. To this end, fix ε with $0 < \varepsilon < \frac{1}{2} - \alpha$. Define $\varphi : [0,1] \to [0,1]$ by

$$\varphi(t) = \begin{cases} \frac{1}{\varepsilon}t & \text{if} \quad 0 \leq t \leq \varepsilon \\ 1 & \text{if} \quad \varepsilon \leq t \leq 2\alpha + \varepsilon \\ 1 - \frac{1}{\varepsilon}(t - 2\alpha - 2\varepsilon) & \text{if} \quad 2\alpha + \varepsilon \leq t \leq 2\alpha + 2\varepsilon \\ 0 & \text{if} \quad 2\alpha + 2\varepsilon \leq t \end{cases}$$

Define $F : [0,1] \times [0,1] \to [0,1]$ by

$$F(t,s) = \varphi(s + \alpha + \varepsilon - t)$$

F is a continuous function, and for each $t \in [0,1]$, $F(t,\cdot)$ is identically 1 on the interval $[t-\alpha, t+\alpha]$ and identically 0 on the complement of the ε-neighborhood of $[t-\alpha, t+\alpha]$. We obtain

$$\begin{aligned}
\int_0^1 |x^n[t-\alpha,t+\alpha]|\,dt &= \int_0^1 \left| \int_0^1 \chi_{[t-\alpha,t+\alpha]}(s) dx^n(s) \right| dt \\
&= \int_0^1 \left| \int_0^1 \left\{ F(t,s) + \chi_{[t-\alpha,t+\alpha]}(s) - F(t,s) \right\} dx^n(s) \right| dt \\
&\le \int_0^1 \left| \int_0^1 F(t,s) dx^n(s) \right| dt \\
&\quad + \int_0^1 \left| \int_0^1 \left\{ \chi_{[t-\alpha,t+\alpha]}(s) - F(t,s) \right\} dx^n(s) \right| dt \\
&\le \int_0^1 \left| \int_0^1 F(t,s) dx^n(s) \right| dt \\
&\quad + \int_0^1 \int_0^1 \chi_{[t-\alpha-\varepsilon,t-\alpha]}(s) d|x^n|(s) dt \\
&\quad + \int_0^1 \int_0^1 \chi_{[t+\alpha,t+\alpha+\varepsilon]}(s) d|x^n|(s) dt \\
&\le \int_0^1 \left| \int_0^1 F(t,s) dx^n(s) \right| dt \\
&\quad + \int_0^1 \chi_{[t-\alpha-\varepsilon,t-\alpha]}(s) dt\, d|x^n|(s) \\
&\quad + \int_0^1 \chi_{[t+\alpha,t+\alpha+\varepsilon]}(s) dt\, d|x^n|(s) \\
&\le \int_0^1 \left| \int_0^1 F(t,s) dx^n(s) \right| dt + \int_0^1 2\varepsilon d|x^n|(s)
\end{aligned}$$

Because $x^n \to 0$ in the weak star topology, $\int_0^1 F(t,s) dx^n(s) \to 0$ for each t, so the bounded convergence theorem entails that

$$\int_0^1 \left| \int_0^1 F(t,s) dx^n(s) \right| dt \to 0$$

Because the sequence (x^n) is bounded, $\int_0^1 2\varepsilon d|x^n|(s)$ can be made arbitrarily small by choosing ε small. We conclude that $\|x^n\|_\alpha \to 0$, as desired.

Simple calculations similar to those in Example 8.2 show that $\|\cdot\|_\alpha$ is adapted to U on order intervals, and that U is quadratically concave with respect to $\|\cdot\|_\alpha$ on order intervals.

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