1 Binary relations

Definition 1.1. $R \subseteq X \times Y$ is a *binary relation* from X to Y. We write "xRy" if $(x, y) \in R$ and "not xRy" if $(x, y) \notin R$.

When $X = Y$ and $R \subseteq X \times X$, we write R is a binary relation on X.

Exercise 1.2. Suppose R, Q are two binary relations on X. Prove that, given our notation, the following are equivalent:

1. $R \subseteq Q$
2. For all $x, y \in X$, $xRy \implies xQy$.

Example 1.3. Suppose $f : X \to Y$ is a function from X to Y. Then the binary relation $R \subseteq X \times Y$ defined by

$$xRy \iff f(x) = y$$

is exactly the graph of f. In fact, one way to think of a function more generally is as a binary relation R from X to Y such that: for each $x \in X$, there exists exactly one $y \in Y$ such that $(x, y) \in R$.

Example 1.4. Suppose $X = \{1, 2, 3\}$ and consider the following binary relation $R \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$, $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. In other words, R is the binary relation "is weakly greater than," or $\geq$. We can represent it graphically as:

3			$\bullet$
2		$\bullet$	$\bullet$
1	$\bullet$	$\bullet$	$\bullet$
$\geq$	1	2	3

The following graphically represents the binary relation $=$, or "is equal to:"

3			$\bullet$
2		$\bullet$	
1	$\bullet$		
$=$	1	2	3

The following would represent the binary relation $<$ or "is strictly less than:"

3	$\bullet$	$\bullet$	
2	$\bullet$		
1			
$<$	1	2	3

Definition 1.5. The *dual* R' of a binary relation R is defined by $xR'y$ if and only if yRx. The *asymmetric component* P of a binary relation R is defined by xPy if and only if xRy and not yRx. The *symmetric component* I of a binary relation R is defined by xIy if and only if xRy and yRx.

Example 1.6. Suppose $X = \mathbb{R}$ and R is the binary relation of $\geq$, or "weakly greater than." The dual R' is $\leq$ or "weakly less than," because $x \geq y$ if and only if $y \leq x$. The asymmetric component P is $>$ or "strictly greater than," because $x > y$ if and only if $[x \geq y$ and not $y \geq x]$. (Verify this). The symmetric component of I is $=$ or "is equal to," because $x = y$ if and only if $x \geq y$ and $y \geq x$.

Example 1.7. Suppose $X = \{1, 2, 3\}$ and consider the following binary relation R:

3		•	
2	•	•	•
1	•		•
R	1	2	3

Then the following represent R', P, and I:

3	•	•	
2		•	•
1	•	•	
R'	1	2	3

3			
2	•		
1			•
P	1	2	3

3		•	
2		•	•
1	•		
I	1	2	3

Definition 1.8. A binary relation R on X is:

- *reflexive* if, for all $x \in X$, xRx;
- *complete* if, for all $x, y \in X$, xRy or yRx;
- *irreflexive* if, for all $x, y \in X$, not xRx;
- *symmetric* if, for all $x, y \in X$, xRy implies yRx;
- *asymmetric* if, for all $x, y \in X$, xRy implies not yRx;
- *antisymmetric* if, for all $x, y \in X$, xRy and yRx imply $x = y$;
- *transitive* if, for all $x, y, z \in X$, xRy and yRz imply xRz;
- *quasi-transitive* if, for all $x, y, z \in X$, xPy and yPz imply xPz;
- *acyclic* if, for all $x_1, x_2, \ldots, x_n \in X$, x_1Px_2, x_2Px_3, $\ldots$, and $x_{n-1}Px_n$ imply x_1Rx_n.
- *negatively transitive* if not xRy and not yRz imply not xRz.

Exercise 1.9. Prove the following: If R is transitive, then R is quasi-transitive. If R is quasi-transitive, then R is acyclic.

Example 1.10. Suppose $X = \mathbb{R}$. The binary relation $\geq$ is reflexive, complete, antisymmetric, transitive, and negatively transitive; $\geq$ is not asymmetric. The binary relation $>$ is irreflexive, asymmetric, antisymmetric, transitive, negatively transitive, quasi-transitive, and acyclic; $>$ is not reflexive, not complete, and not symmetric.

Definition 1.11. A binary relation R on X is an *equivalence relation* if R is reflexive, symmetric, and transitive. The *equivalence class* of $x \in X$ is $\{y \in X : xRy\}$. Let X/R denote the collection of all equivalence classes.

Example 1.12. The following is an equivalence relation on $X = \{a, b, c, d\}$:

R	a	b	c	d
a	•	•		
b	•	•		
c			•	•
d			•	•

The equivalence classes of R are $\{a, b\}$ and $\{c, d\}$, so the collection $X/R = \{\{a, b\}; \{c, d\}\}$.

Exercise 1.13. Let X be the set of all living people. Verify if the following relations on X are reflexive, symmetric, transitive, and/or complete:

- "is married to" (assuming monogamy)
- "is the son or daughter of"
- "is an ancestor or descendant of"
- "is taller than"

For those which are equivalence relations, interpret X/R.

Definition 1.14. A binary relation R on X is a *preorder* if R is reflexive and transitive.

Definition 1.15. A binary relation R on X is a *weak order* if R is complete and transitive.

Definition 1.16. A binary relation R on X is a *linear order* if R is complete, transitive, and antisymmetric.

Example 1.17. Define the binary relation $\geq$ on $\mathbb{R}^2$ by

$$(x_1, x_2) \geq (y_1, y_2) \iff x_1 \geq y_1 \text{ and } x_2 \geq y_2.$$

Verify that $\geq$ is a preorder on $\mathbb{R}^2$. Verify that $\geq$ is not a weak order on $\mathbb{R}^2$.

Definition 1.18. Given a binary relation R on X, the *upper contour set* of $x \in X$ is $\{y \in X : yRx\}$; the *lower contour set* of $x \in X$ is $\{y \in X : xRy\}$.

Exercise 1.19. Suppose R is a preorder on X. Prove that if xRy, then the lower contour set of y is a subset of the lower contour set of x, i.e. $\{z \in X : yRz\} \subseteq \{z \in X : xRz\}$.

Problem 1.20. (difficult) Construct a linear order on $\mathbb{R}^2$.

Problem 1.21. Suppose $R_1, R_2, \ldots, R_n$ are binary relations on X. Define the binary relation $\mathbf{R}$ by

$$x\mathbf{R}y \text{ if and only if } xR_iy, \forall i = 1, \ldots, n.$$

Prove or provide counterexamples to the following statements: If each R_i is a preorder, then $\mathbf{R}$ is a preorder. If each R_i is a weak order, then $\mathbf{R}$ is a weak order.

2 Preference and choice

Definition 2.1. A binary relation $\succsim$ on X is a *preference relation* if it is a weak order, i.e., complete and transitive. For any binary relation $\succsim$, let $\precsim$ denote the dual of $\succsim$, defined by

$$x \precsim y \Longleftrightarrow y \succsim x;$$

$\succ$ denote the asymmetric component of $\succsim$, defined by

$$x \succ y \Longleftrightarrow [x \succsim y \text{ and not } y \succsim x];$$

$\sim$ denote the symmetric component of $\succsim$, defined by

$$x \sim y \Longleftrightarrow [x \succsim y \text{ and } y \succsim x].$$

Roughly, we interpret $x \succsim y$ to mean the decision maker weakly prefers x to y; $x \succ y$ to mean the decision maker strictly prefers x to y; and $x \sim y$ to mean the decision maker is indifferent between x and y. The upper contour set of x, $\{y \in X : y \succsim x\}$ consists of the elements which the decision maker weakly prefers to x.

Completeness and transitivity are sometimes considered requirement for rationality. This is somewhat controversial. You should be able to think of examples where your own preferences are either incomplete or intransitive.

Exercise 2.2. Let $X = \{a, b, c\}$. Determine if the following binary relations are complete and/or transitive:

1. $\succsim = X \times X$;
2. $\succsim = \emptyset$;
3. $\succsim = \{(a,a),(b,b),(c,c),(a,b),(b,a),(a,c),(b,c)\}$;
4. $\succsim = \{(a,a),(b,b),(c,c),(a,b),(b,c),(c,a)\}$;
5. $\succsim = \{(a,b),(b,c),(a,c)\}$.

Exercise 2.3. Prove that if $\succsim$ is a preference relation, then:

1. $\precsim$ is a preference relation;
2. $\succ$ is asymmetric and transitive;
3. $\sim$ is an equivalence relation.

Exercise 2.4. Prove that if $\succsim$ is a preference relation, then:

1. $x \succsim y$ and $y \sim z$ imply $x \succsim z$;

2. $x \succsim y$ and $y \succ z$ imply $x \succ z$

Have you ever seen a preference? If you have, please take a picture of it for me. Since we like to assert that economics is an empirical science, we would like the theoretical primitives to be observable, at least in principle, otherwise we're only engaged in a very mathematically rigorous form of theology.

Definition 2.5. A *correspondence* f from X to Y is a function from X to 2^Y, the power set of Y. This is notated as $f : X \rightrightarrows Y$.[1]

Definition 2.6. A *choice rule* for X is a correspondence $\mathcal{C} : 2^X \setminus \{\emptyset\} \rightrightarrows X$ such that $\mathcal{C}(A) \subseteq A$ for all $A \subseteq X$.[2]

We can think of sets in $2^X \setminus \{\emptyset\}$ as menus or budget sets. Then a particular $A \in 2^X \setminus \{\emptyset\}$ is interpreted as a set of available options, and the decision maker will choose one of the options from A. Her choice set $\mathcal{C}(A)$ is the set of options she will choose. If $\mathcal{C}(A)$ has more than one element, this does not mean that she selects all of them at once, but rather that she could choose any of them. For example if $\mathcal{C}(\{a, b, c\}) = \{a, b\}$, this is interpreted as meaning the decision maker would choose either the apple or the banana from a basket containing an apple, a banana, and a carrot; it is *not* interpreted as meaning the decision maker will consume both the apple and the banana.

It's unclear whether even $\mathcal{C}(A)$ is actually observable. At best, we see the decision maker choose an element of $\mathcal{C}(A)$. We can therefore *include* elements in the choice set from observation, but we cannot *exclude* them without making additional assumptions on $\mathcal{C}(A)$.

Definition 2.7. Given a binary relation $\succsim$, the *induced choice rule* $\mathcal{C}_{\succsim}$ is defined by

$$\mathcal{C}_{\succsim}(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}.$$

This gives a natural method for constructing a choice rule from a binary relation. The decision maker will choose an element which she weakly prefers to all other available options. The direction of the definition seems backwards, because it starts with a presumably unobservable binary relation and derives a choice set. In a few pages, we will study the opposite direction and find conditions which guarantee that a choice rule is induced by a preference relation.

Example 2.8. Suppose $X = \{1, 2, 3, \ldots\}$. Then $\mathcal{C}_{\geq}(A) = \max A$ if A is finite and $\mathcal{C}_{\geq} = \emptyset$ if A is infinite. On the other hand, $\mathcal{C}_{\leq}(A) = \min A$ for all sets A. Finally, $\mathcal{C}_{>}(A) = \emptyset$ for all A.

Definition 2.9. The choice rule $\mathcal{C}$ is *nonempty* if $\mathcal{C}(A) \neq \emptyset$ for all nonempty $A \subseteq X$.

If $\succsim$ is a preference relation, its induced choice rule is nonempty on finite menus.

[1]The expression 2^X denotes the power set of X.

[2]Caution: This domain for choice rules is different than that in MWG p.10; $\mathcal{C}$ is defined on *all* nonempty subsets of X, not just any family of subsets. Consequently, the subsequent axioms have much more mathematical bite in this domain.

Proposition 2.10. *If $\succsim$ is a preference relation, then $\mathcal{C}_{\succsim}(A) \neq \emptyset$ whenever A is finite.*

Proof. The proof is by induction on the size of $|A|$. Base step: $|A| = 1$. Then if $x \in A$, $x \succsim x$ by completeness and the only element of A is x, so $\mathcal{C}_{\succsim}(A) = \{x\}$.

Inductive step. Suppose $\mathcal{C}_{\succsim}(B)$ is nonempty whenever $|B| = n$. Suppose $|A| = n + 1$. Pick $x \in A$. Let $B = A \setminus \{x\}$. Then $|B| = n$, so $\mathcal{C}_{\succsim}(B)$ is nonempty by our induction hypothesis. So there exists some element $y \in \mathcal{C}_{\succsim}(B)$. By completeness, either $y \succsim x$ or $x \succsim y$.

Case 1: $y \succsim x$. Since $y \in \mathcal{C}(B)$, by definition $y \succsim z$ for all $z \in B$. Then $y \succsim z$ for all $z \in \{x\} \cup B = A$. So $y \in \mathcal{C}_{\succsim}(A)$.

Case 2: $x \succsim y$. By completeness, $x \succsim x$. For all $z \in B$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$ by transitivity. So $x \succsim z$ for all $z \in \{x\} \cup B = A$. Then $x \in \mathcal{C}_{\succsim}(A)$. ☐

Problem 2.11. Recall binary relation $\succsim$ is acyclic if $x_1 \succ x_2$, $x_2 \succ x_3$, …, and $x_{n-1} \succ x_n$ imply $x_1 \succsim x_n$. Prove the following stronger result: $\succsim$ is complete and acyclic (but not necessarily transitive) if and only if $\mathcal{C}_{\succsim}(A) \neq \emptyset$ whenever A is finite.

Exercise 2.12. Let $X = \{a, b, c\}$ and let $\succsim = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}$. Verify that $\mathcal{C}_{\succsim}(\{a, b, c\}) = \emptyset$.

Definition 2.13. A choice rule $\mathcal{C}$ is *rationalized by* $\succsim$ if $\mathcal{C} = \mathcal{C}_{\succsim}$ and $\succsim$ is a preference relation. $\mathcal{C}$ is *rationalizable* if there exists some preference relation $\succsim$ such that $\mathcal{C} = \mathcal{C}_{\succsim}$.

So, a choice rule is "rational" if it behaves as if it is maximizing some complete and transitive preference relation among the available alternatives. This is not immediately a very useful definition, because there are many possible weak orders which might work.

Definition 2.14. Given a choice rule $\mathcal{C}$, its *revealed preference relation* $\succsim_{\mathcal{C}}$ is defined by $x \succsim_{\mathcal{C}} y$ if there exists some A such that $x, y \in A$ and $x \in \mathcal{C}(A)$.

This definition is what we wanted, starting with the observable choice rule and identifying a binary relation from those observations. If someone chooses x when y is available, then we say that x is preferred to y. In this interpretation, preferences *do not* necessarily reflect levels of happiness or well-being for the decision maker; they only reflect what she decided to do. If we observe someone order a salad at a restaurant, we say she prefers the salad to the prime rib, but that's not necessarily the same as saying she thinks the salad tastes better. Notice that the definition states there exists *some* menu including x and y where x is chosen, not that x is chosen in *all* menus including x and y.

Proposition 2.15. *If $\mathcal{C}$ is rationalized by $\succsim$, then $\succsim = \succsim_{\mathcal{C}}$.*

Proof. Let $\succsim$ be a preference relation which rationalizes the choice rule $\mathcal{C}$, i.e. $\mathcal{C}(A) = \mathcal{C}_{\succsim}(A)$ for all $A \subseteq X$. We need to show two inclusions.

Suppose $x \succsim y$. Since $\succsim$ is a preference relation, then $x \succsim x$. Thus $x \succsim z$ for all $z \in \{x, y\}$, so $x \in \mathcal{C}_{\succsim}(\{x, y\})$. Since $\mathcal{C} = \mathcal{C}_{\succsim}$, this implies $x \in \mathcal{C}(\{x, y\})$. Therefore x is revealed to be preferred to y and $x \succsim_C y$.

Suppose $x \succsim_{\mathcal{C}} y$. By definition, there exists some set A with $x, y \in A$ and $x \in \mathcal{C}(A)$. Then $x \in \mathcal{C}_{\succsim}(A)$ because $\mathcal{C} = \mathcal{C}_{\succsim}$. By definition, $x \succsim z$ for all $z \in A$. But $y \in A$, so $x \succsim y$. □

Proposition 2.15 is a *uniqueness* result: the only preference relation that can rationalize $\mathcal{C}$ is its revealed preference relation. So, when checking the rationality of a choice rule, it suffices to check whether it acts as if its maximizing its induced preference relation. Contrapositively, if $\succsim_{\mathcal{C}}$ does not rationalize $\mathcal{C}$, then no other preference relation will rationalize $\mathcal{C}$.

Example 2.16. Let $X = \{a, b, c\}$ and suppose $\mathcal{C}(\{a\}) = \{a\}$, $\mathcal{C}(\{b\}) = \{b\}$, $\mathcal{C}(\{c\}) = \{c\}$,

$$\begin{aligned} \mathcal{C}(\{a, b\}) &= \{a\} \\ \mathcal{C}(\{b, c\}) &= \{b\} \\ \mathcal{C}(\{a, c\}) &= \{a\} \end{aligned}$$

Suppose $\mathcal{C}$ is rationalizable. Then $\mathcal{C}(\{a, b, c\}) = \{a\}$. To see this, observe that $a \succsim_{\mathcal{C}} a$, $a \succsim_{\mathcal{C}} b$, and $a \succsim_{\mathcal{C}} c$, so $a \in \mathcal{C}_{\succsim_C}(\{a, b, c\})$. Since $b \notin \mathcal{C}(\{a, b\}) = \mathcal{C}_{\succsim_C}(\{a, b\})$, b must fail to be preferred under $\succsim_C$ to some element of $\{a, b\}$, either not $b \succsim_C a$ or not $b \succsim_C b$. But we have $b \succsim_C b$ by completeness, forcing the former, not $b \succsim_C a$. Since we know not $b \succsim_{\mathcal{C}} a$ then $b \notin \mathcal{C}_{\succsim_C}(\{a, b, c\})$. Similarly, we know not $c \succsim_{\mathcal{C}} a$ because $c \notin \mathcal{C}(\{a, c\}) = \mathcal{C}_{\succsim_C}(\{a, c\})$, so $c \notin \mathcal{C}_{\succsim_C}(\{a, b, c\})$. Collecting these findings, $\mathcal{C}_{\succsim_C}(\{a, b, c\}) = \{a\}$. Since $\mathcal{C}$ is rationalizable, $\mathcal{C}(A) = \mathcal{C}_{\succsim_C}(A)$ for all $A \subseteq X$. Therefore $\mathcal{C}(\{a, b, c\}) = \{a\}$.

Axiom 2.17 (Houthakker's Axiom or Weak Axiom of Revealed Preference)**.** If $x, y \in A \cap B$, $x \in \mathcal{C}(A)$, and $y \in \mathcal{C}(B)$, then $x \in \mathcal{C}(B)$.

Exercise 2.18. Verify that Houthakker's Axiom is equivalent to the following: if $A \cap \mathcal{C}(B) \neq \emptyset$, then $\mathcal{C}(A) \cap B \subseteq \mathcal{C}(B)$.

Exercise 2.19. Prove that if $\mathcal{C}$ is rationalizable, then $\mathcal{C}$ satisfies Houthakker's Axiom.

Exercise 2.20. Suppose $X = \{a, b, c\}$ and assume $C(\{a, b\}) = \{a\}$, $C(\{b, c\}) = \{b\}$, and $C(\{a, c\}) = \{c\}$. Prove that if $\mathcal{C}$ is nonempty, then it must violate Houthakker's Axiom. [Hint: Is there any value for $\mathcal{C}(\{a, b, c\})$ which will work?]

Axiom 2.21 (Sen's α or Independence of Irrelevant Alternatives)**.** If $x \in B \subseteq A$ and $x \in \mathcal{C}(A)$, then $x \in \mathcal{C}(B)$.

One intuition for Sen's α is as follows: if Italy is one of the best soccer teams in the world, then Italy is one of the best soccer teams in Europe. Another is: if I would buy a Walter Mosley novel at Barnes and Noble, then I would also buy a Walter Mosley novel at my local mystery bookstore.

Example 2.22. Often, social choices fail to exhibit desirable features, even when the individuals in society are totally rational. For example, suppose we have three individuals $\{1, 2, 3\}$ who have complete and transitive preferences over $\{a, b, c\}$ as follows:

$$\begin{array}{l} a \succ_1 b \succ_1 c \\ b \succ_2 a \succ_2 c \\ c \succ_3 a \succ_3 b \end{array}.$$

Now suppose we let

$$\mathcal{C}(A) = \{x \in A : |\{i : x \in \mathcal{C}_{\succsim_i}(A)\}| \geq |\{i : y \in \mathcal{C}_{\succsim_i}(A)|, \text{ for all } y \in A\}.$$

In words, $x \in \mathcal{C}(A)$ if there is no alternative which would be chosen by strictly more individuals than x. Then $\mathcal{C}(\{a, b, c\}) = \{a, b, c\}$, but $\mathcal{C}(\{a, b\}) = \{a\}$. So b is chosen from the larger menu, but not from the smaller, violating Sen's α. For a less contrived example, many people argue that George Bush would have lost the 2000 U.S. presidential election if Ralph Nader had not run and taken votes away from Al Gore, which violates Sen's α.

Exercise 2.23. Verify that Sen's α is equivalent to the following: if $B \subseteq A$, then $\mathcal{C}(A) \cap B \subseteq \mathcal{C}(B)$.

Exercise 2.24. Suppose that $\mathcal{C}$ satisfies Sen's α. Verify that if $\mathcal{C}(B) = B$, then $\mathcal{C}(A) = A$ for all $A \subseteq B$.

Problem 2.25. Prove that if $\mathcal{C}$ is nonempty and satisfies Houthakker's Axiom, then $\mathcal{C}$ satisfies Sen's α.

Axiom 2.26 (Sen's β)**.** If $x, y \in \mathcal{C}(A)$, $A \subseteq B$, and $y \in \mathcal{C}(B)$, then $x \in \mathcal{C}(B)$.

One intuition of Sen's β is as follows: If Italy and France are the best soccer teams in Europe and Italy is one of the best soccer teams in the world, then France is also one of the best soccer teams in the world. Another example is: if I might buy either a Walter Mosley or Elmore Leonard novel at my local mystery bookstore and I might buy a Walter Mosley novel at Barnes and Noble, then I might also buy an Elmore Leonard novel at Barnes and Noble.

Exercise 2.27. Verify that Sen's β is equivalent to the following: if $A \subseteq B$ and $\mathcal{C}(A) \cap \mathcal{C}(B) \neq \emptyset$, then $\mathcal{C}(A) \subseteq \mathcal{C}(B)$. Sometimes Sen's β is called expansion consistency because of this expression.

Problem 2.28. Prove that if $\mathcal{C}$ is nonempty and satisfies Houthakker's Axiom, then $\mathcal{C}$ satisfies Sen's β.

Exercise 2.29. Suppose $X = \{a, b, c\}$. Construct a nonempty choice rule that satisfies Sen's α but violates Sen's β. Construct a nonempty choice rule that satisfies Sen's β but violates Sen's α.

The following theorem is the main result of this section. It starts with an observable choice rule $\mathcal{C}$ and provides necessary and sufficient conditions for that choice rule to look "as if" the decision maker is using a preference relation to generate her choice behavior.

Theorem 2.30. *Suppose $\mathcal{C}$ is nonempty. Then the following are equivalent:*

1. *$\mathcal{C}$ meets Sen's α and β;*

2. *$\mathcal{C}$ meets Houthakker's axiom;*

3. *$\mathcal{C}$ is rationalizable.*

Proof. We will show (1) implies (2) implies (3) implies (1).

Step 1: Sen's α and β imply Houthakker's Axiom. Suppose $\mathcal{C}$ meets Sen's α and β. Assume $x, y \in A \cap B$, $x \in \mathcal{C}(A)$, and $y \in \mathcal{C}(B)$. Applying Sen's α to $A \cap B \subseteq A$, $x \in \mathcal{C}(A)$ implies $x \in \mathcal{C}(A \cap B)$. Similarly, applying Sen's α to $A \cap B \subseteq B$, $y \in \mathcal{C}(B)$ implies $y \in \mathcal{C}(A \cap B)$. Since $x, y \in \mathcal{C}(A \cap B)$ and $y \in \mathcal{C}(B)$, we can invoke Sen's β to obtain $x \in \mathcal{C}(B)$.

Step 2: If $\mathcal{C}$ is nonempty and satisfies Houthakker's Axiom, then $\mathcal{C}$ is rationalizable. We will show that $\succsim_{\mathcal{C}}$ rationalizes $\mathcal{C}$, which is the obvious candidate relation. First, we need to prove that $\succsim_{\mathcal{C}}$ is a preference relation, i.e. complete and transitive.

Let $x, y \in X$. Since $\mathcal{C}$ is nonempty, either $x \in \mathcal{C}(\{x, y\})$ or $y \in \mathcal{C}(\{x, y\})$. Then either $x \succsim_{\mathcal{C}} y$ or $y \succsim_{\mathcal{C}} x$. This proves $\succsim_{\mathcal{C}}$ is complete.

For transitivity, suppose $x \succsim_{\mathcal{C}} y$ and $y \succsim_{\mathcal{C}} z$. Then there exist a menu A_{xy} such that $x, y \in A_{xy}$ and $x \in \mathcal{C}(A_{xy})$. There also exists a menu A_{yz} with $y, z \in A_{yz}$ and $y \in \mathcal{C}(A_{yz})$.

$\mathcal{C}(\{x, y, z\})$ is nonempty, so we proceed by cases.

- Case 1: $x \in \mathcal{C}(\{x, y, z\})$. Then we've immediately satisfied $x \succsim_{\mathcal{C}} z$.

- Case 2: $y \in \mathcal{C}(\{x, y, z\})$. Observe $x, y \in \{x, y, z\} \cap A_{xy}$, $x \in \mathcal{C}(A_{xy})$, and $y \in \mathcal{C}(\{x, y, z\})$. Then Houthakker's axiom implies $x \in \mathcal{C}(\{x, y, z\})$.

- Case 3: $z \in \mathcal{C}(\{x, y, z\})$. Observe $y, z \in A_{yz} \cap \{x, y, z\}$, $y \in \mathcal{C}(A_{yz})$ and $z \in \mathcal{C}(\{x, y, z\})$. Then Houthakker's Axiom implies $y \in \mathcal{C}(\{x, y, z\})$. Then apply Case 2.

In all cases, we conclude $x \succsim_{\mathcal{C}} z$. Thus $\succsim_{\mathcal{C}}$ is transitive

Next, we need to show that $\succsim_{\mathcal{C}}$ actually rationalizes $\mathcal{C}$, i.e. $\mathcal{C}(A) = \mathcal{C}_{\succsim_{\mathcal{C}}}(A)$. This involves demonstrating two set inclusions. Suppose $x \in \mathcal{C}(A)$. Then for any $y \in A$, $x \succsim_{\mathcal{C}} y$, since $x, y \in A$. So $\mathcal{C}(A) \subseteq \mathcal{C}_{\succsim_{\mathcal{C}}}(A)$.

Now, suppose $x \in \mathcal{C}_{\succsim_{\mathcal{C}}}(A)$, i.e. for any $y \in A$, there exists some B_{xy} such that $x \in \mathcal{C}(B_{xy})$. Since $\mathcal{C}$ is nonempty, fix some $z \in \mathcal{C}(A)$. Houthakker's axiom applied to $x, z \in B_{xz} \cap A$, $x \in \mathcal{C}(B_{xz})$, and $z \in \mathcal{C}(A)$ delivers $x \in \mathcal{C}(A)$. So $\mathcal{C}_{\succsim_{\mathcal{C}}}(A) \subseteq \mathcal{C}(A)$.

Step 3: If $\mathcal{C}$ is rationalizable, then $\mathcal{C}$ satisfies Sen's α and β. Since $\mathcal{C} = \mathcal{C}_{\succsim}$, we can assume $\succsim = \succsim_{\mathcal{C}}$ by Proposition 2.15. First, let $x \in B \subseteq A$ and $x \in \mathcal{C}(A)$. Then for all $y \in B$, $x, y \in A$ and $x \in \mathcal{C}(A)$, i.e. $x \succsim_{\mathcal{C}} y$. Thus $x \in \mathcal{C}_{\succsim_{\mathcal{C}}}(B) = \mathcal{C}(B)$. This proves Sen's α holds. Now let $x, y \in \mathcal{C}(A)$, $A \subseteq B$, and $y \in \mathcal{C}(B)$. For all $z \in B$, $y \succsim_{\mathcal{C}} z$. Since $y \in \mathcal{C}(A) \subseteq A$ and $x \in \mathcal{C}(A)$, $x \succsim_{\mathcal{C}} y$. By transitivity, for all $z \in B$, $x \succsim_{\mathcal{C}} z$. Thus $x \in C_{\succsim_{\mathcal{C}}}(B) = C(B)$. This proves Sen's β holds. □

Exercise 2.31. Suppose $\mathcal{C}(A) \neq \emptyset$ for all $A \subseteq X$ and $\mathcal{C}$ satisfies Houthakker's axiom. You are given: $\mathcal{C}(\{a, b\}) = \{a, b\}$ and $\mathcal{C}(\{a, b, c\}) = \{c\}$. Use Theorem 2.30 to find $\mathcal{C}(\{a, c\})$.

Theorem 2.30 shows that we can check the rationality of $\mathcal{C}$ by verifying certain axioms.

We now will apply the theory of choice to a specific application, namely the direct study of consumer choice. The classic study of demand, which we turn to next, analyzes the constrained maximization of utility functions. Some of the infrastructure is unnecessary, and we can develop some meaningful results by directly studying the observed consumption behavior.

Definition 2.32. A *Walrasian demand function* is a function from price-wage pairs to chosen consumption: $x^* : \mathbb{R}^n_{++} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that $p \cdot x^*(p, w) \leq w$.

If we interpret the implied choice set at a price-wage pair as $\{x : p \cdot x \leq w\}$, then we have less than the complete choice behavior available to us, we only have the choice rule from very specific looking menus, namely the budget sets $B_{p,w} = \{x : p \cdot x \leq w \text{ and } x_i \geq 0\}$, and then $x^*(p, w) = C_{\succsim}(B_{p,w})$. But, in classic demand theory, these are the only menus in the economy. Also, we are making the assumption that the choice rule is single-valued.

Definition 2.33. A Walrasian demand function is *homogeneous of degree zero* if $x^*(\alpha p, \alpha w) = x^*(p, w)$ for all $\alpha > 0$.

This condition just means that nominal price changes have no effect on consumption.

Definition 2.34. A Walrasian demand function satisfies *Walras's Law* if $p \cdot x^*(p, w) = w$.

Walras's Law implies that the consumer spends all of her income.

Definition 2.35. A Walrasian demand function satisfies the *weak axiom of revealed preference* if

$$[p' \cdot x^*(p, w) \leq w' \text{ and } x^*(p, w) \neq x^*(p', w')] \Longrightarrow p \cdot x^*(p', w') > w$$

for all pairs (p, w) and (p', w').

The intuition for this condition should sound familiar. Suppose prices and wage change from (p, w) to (p', w'). The old consumption bundle $x^*(p, w)$ is still affordable at the new pair (p', w') $(p' \cdot x^*(p, w) \leq w')$, but the consumer still changes her choice $(x^*(p, w) \neq x^*(p', w'))$. Then the new choice could not have been affordable at the old price-wage situation, because otherwise she would have chosen $x^*(p', w')$ while facing (p, w).

Exercise 2.36. Verify that Houthakker's axiom, imposed on the limited choice data included in x^*, is equivalent to the weak axiom of revealed preference.

Definition 2.37. The pairs (p, w) and (p', w') are a *compensated price change* if $p' \cdot x^*(p, w) = w'$.

A compensated price change from (p, w) to (p', w') provides the consumer just enough new income w' to still have the option, at the new price p', to consume whatever bundle $x^*(p, w)$ she chose at the old price-wage pair. The following result shows that if we can pass the weak axiom of revealed preference for these special price changes, then we can pass it for all price changes.

Proposition 2.38. *Suppose x^* is homogeneous of degree zero and satisfies Walras's Law.*[3] *If x^* satisfies the weak axiom of revealed preference for all compensated price changes, i.e.*

$$[p' \cdot x^*(p,w) = w' \text{ and } x^*(p,w) \neq x^*(p',w')] \Longrightarrow p \cdot x^*(p',w') > w,$$

then it satisfies the weak axiom of revealed preference for all price changes.

The following proof will not be covered in lecture and should be studied independently.

Proof. We prove this by contraposition. Suppose we have a general violation of the weak axiom of revealed preference: there exists an arbitrary price change, (q,v) and (q',v'), such that $q' \cdot x^*(q,v) \leq v'$ and $x^*(q,v) \neq x^*(q',v')$, but $q \cdot x^*(q',v') \leq v$. Now, if either $q' \cdot x^*(q,v) = v'$ or $q \cdot x^*(q',v') = v$, or both, we have a compensated price change and we're done. So, now suppose $q \cdot x^*(q',v') < v$ and $q' \cdot x^*(q,v) < v'$. Then, by Walras's Law:

$$\begin{aligned} q \cdot x^*(q',v') &< v & \qquad q' \cdot x^*(q,v) &< v' \\ q \cdot x^*(q',v') &< q \cdot x^*(q,v) & q' \cdot x^*(q,v) &< q' \cdot x^*(q',v') \\ q \cdot x^*(q,v) - q \cdot x^*(q',v') &> 0 & q' \cdot x^*(q,v) - q' \cdot x^*(q',v') &< 0. \end{aligned}$$

So, there exists some strictly convex combination of the left hand and right hand quantities which sums to zero, i.e. there exists some $\alpha \in (0,1)$ such that

$$\begin{aligned} \alpha[q \cdot x^*(q,v) - q \cdot x^*(q',v')] + (1-\alpha)[q' \cdot x^*(q,v) - q' \cdot x^*(q',v')] &= 0 \\ [\alpha q + (1-\alpha)q'] \cdot x^*(q,v) - [\alpha q + (1-\alpha)q'] \cdot x^*(q',v') &= 0. \end{aligned}$$

Letting $p = \alpha q + (1-\alpha)q'$ and $w = p \cdot x^*(q,v)$, we have $p \cdot x^*(q,v) = w = p \cdot x^*(q',v')$. So,

$$\begin{aligned} \alpha v + (1-\alpha)v' &= \alpha q \cdot x^*(q,v) + (1-\alpha)q' \cdot x^*(q',v'), \text{ by Walras's Law} \\ &= p \cdot x^*(q,v) + (1-\alpha)q' \cdot (x^*(q',v') - x^*(q,v)) \\ &> w + 0 \\ &= p \cdot x^*(p,w) \\ &= \alpha q \cdot x^*(p,w) + (1-\alpha)q' \cdot x^*(p,w) \end{aligned}$$

This forces either $v > q \cdot x^*(p,w)$ or $v' > q' \cdot x^*(p,w)$. Suppose the former inequality, the latter case is identical. Then $q \cdot x^*(p,w) \neq v = q \cdot x^*(q,v)$, so $x^*(p,w) \neq x^*(q,v)$. Also, $p \cdot x^*(q,v) = w$ and $q \cdot x^*(p,w) < v$, i.e. a violation of the weak axiom of revealed preference for a compensate price change. □

Proposition 2.39. *Suppose $x^*(p,w)$ is homogeneous of degree zero and satisfies Walras's Law.*[4]

[3] Homogeneity of degree zero is unnecessary, but included to maintain consistency with the textbook.

[4] As in the previous proposition, homogeneity of degree zero is unnecessary, but included to maintain consistency with the textbook.

Then the weak axiom of revealed preferences is satisfied if and only if, for any compensated price change (p, w) to (p', w') with $w' = p' \cdot x^*(p, w)$,

$$(p' - p) \cdot [x^*(p', w') - x^*(p, w)] \leq 0,$$

with strict inequality if $x^*(p, w) \neq x^*(p', w')$.

Proof. [$\Rightarrow$]. Suppose the weak axiom of revealed preference is satisfied. The desired inequality is immediate if the demands are equal, so without loss assume $x^*(p, w) \neq x^*(p', w')$. Since $p' \cdot x^*(p, w) = w'$ and $w' = p' \cdot x^*(p', w')$ by Walras's Law, we have

$$p' \cdot [x^*(p', w') - x^*(p, w)] = 0.$$

Since $p' \cdot x^*(p, w) \leq w'$, that is, $x^*(p, w)$ was affordable at (p', w'), the weak axiom implies $p \cdot x^*(p', w') > w$. Thus:

$$p \cdot [x^*(p', w') - x^*(p, w)] > 0.$$

So:

$$(p' - p) \cdot [x^*(p', w') - x^*(p, w)] = \underbrace{p' \cdot [x^*(p', w') - x^*(p, w)]}_{=0} - \underbrace{p \cdot [x^*(p', w') - x^*(p, w)]}_{>0} < 0.$$

[$\Leftarrow$]. By Proposition 2.38, it suffices to verify the weak axiom of revealed preference for compensated price changes. So, suppose $p' \cdot x^*(p, w) = w'$ and $x^*(p, w) \neq x^*(p', w')$. Then

$$\begin{aligned} 0 &> (p' - p) \cdot [x^*(p', w') - x^*(p, w)] \\ &= \underbrace{p' \cdot x^*(p', w')}_{=w'} - \underbrace{p' \cdot x^*(p, w)}_{=w'} - p \cdot x^*(p', w') + \underbrace{p \cdot x^*(p, w)}_{=w} \\ &= w - p \cdot x^*(p', w'). \end{aligned}$$

□

The inequality in Proposition 2.39 is also known as the *law of compensated demand*. Roughly, the negative cross product means that price and demand move in opposite directions: if price goes up, demand goes down. But, this is true only for compensated demand, not for uncompensated demand. Later, we will prove a differential version of this result, the negative semidefiniteness of the Slutsky matrix, using calculus on utility functions. The point of this exercise is to show that some of that infrastructure is extraneous, and that the law of compensated demand is a result of homogeneity of degree zero, Walras's Law, and rationality as defined by the weak axiom of revealed preference suffices.

The following material is optional.

Definition 2.40. Suppose $\mathcal{C}$ is a choice rule. Define the *base relation* $\succsim^b_{\mathcal{C}}$ by $x \succsim^b_{\mathcal{C}} y$ if and only if $x \in \mathcal{C}(\{x, y\})$.

Problem 2.41. A binary relation $\succsim$ is quasi-transitive if $x \succ y$ and $y \succ z$ imply $x \succ z$. A choice rule $\mathcal{C}$ on X satisfies *path independence* if $\mathcal{C}(A \cup B) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B))$ for all $A, B \subseteq X$. Prove that if $\mathcal{C}$ satisfies path independence and $\mathcal{C}(A) \neq \emptyset$ for all finite $A \subseteq X$, then its base relation $\succsim^b_{\mathcal{C}}$ is quasi-transitive. (You actually don't have to use finite nonemptiness of C here, but there is one proof where it is useful.)

Problem 2.42. A binary relation $\succsim$ is acyclic if $x_1 \succ x_2$, $x_2 \succ x_2$, ..., and $x_{n-1} \succ x_n$ imply $x_1 \succsim x_n$. Prove that if a choice rule C for X is nonempty for all finite sets, i.e. $C(A) \neq \emptyset$ for all finite $A \subseteq X$, and meets Sen's α, then its base relation $\succsim^b_C$ is acyclic.

Problem 2.43. Suppose X is finite. Prove that if $C(A) \neq \emptyset$ for all $A \subseteq X$ and C satisfies Sen's α, then there exists a preference relation $\succsim$ on X such that $C_{\succsim}(A) \subseteq C(A)$ for all $A \subseteq X$. (Hint: The proof is tricky. Neither the base relation nor the revealed preference relation will work.)

Problem 2.44. Suppose X is finite and the decision maker has a complete and transitive preference relation $\succsim$ on X. She "satisfices" in the following sense: she selects an alternative which is at least as good as the "median" option. Formally, her choice set from A is $S_{\succsim}(A)$, defined as follows:

$$S_{\succsim}(A) = \{x \in A : |\{y \in A : x \succsim y\}| \geq |A|/2\}.$$

Does $S_{\succsim}$ satisfy Sen's α and β? Does the revealed preference $\succsim_{S_{\succsim}}$ replicate $\succsim$? Does the base relation $\succsim_{S_{\succsim}}$ replicate $\succsim$?

3 Preference and ordinal utility

Definition 3.1. A *utility function* on X is a function $u : X \to \mathbb{R}$.

Definition 3.2. The utility function $u : X \to \mathbb{R}$ *represents* the binary relation $\succsim$ on X if

$$x \succsim y \iff u(x) \geq u(y).$$

Exercise 3.3. Suppose $\succsim$ is a preference relation. Prove that $u : X \to \mathbb{R}$ represents $\succsim$ if and only if:

1. $x \succsim y \Rightarrow u(x) \geq u(y)$; and
2. $x \succ y \Rightarrow u(x) > u(y)$.

Exercise 3.4. Prove that if $\succsim$ admits a utility representation, then $\succsim$ is a preference relation, i.e. complete and transitive.

Proposition 3.5. *Suppose $u : X \to \mathbb{R}$ represents $\succsim$. Then $v : X \to \mathbb{R}$ represents $\succsim$ if and only if there exists a strictly increasing function $h : u(X) \to \mathbb{R}$ such that $v = h \circ u$.*[5]

Proof. [$\Leftarrow$] Suppose $v = h \circ u$. If $x \succsim y$, then $u(x) \geq u(y)$. Since h is increasing, $h(u(x)) \geq h(u(y))$. If $y \succ x$, then $u(y) > u(x)$. Since h is strictly increasing, $h(u(y)) > h(u(x))$.

[$\Rightarrow$] (optional) Suppose u and v both represent $\succsim$. For any real number in the image of u, $r \in u(X)$, there exists some some element $x \in X$ which is mapped by u to r: $x \in u^{-1}(r)$ or $u(x) = r$. Define a function $w : u(X) \to X$ by choosing $w(r) \in u^{-1}(r)$ for each $r \in u(X)$.[6] This function w is sort of like an "inverse" of u, because by construction, $u(w(u(x))) = u(x)$. Then $w(u(x)) \sim x$, since u represents $\succsim$. Let $h : u(X) \to \mathbb{R}$ be defined by $h = v \circ w$, i.e. $h(r) = v(w(r))$. For any $x \in X$, $h(u(x)) = v(w(u(x)) = v(x)$, because $w(u(x)) \sim x$ and v represents $\succsim$. It remains to show that h is strictly increasing, so suppose $r, s \in u(X)$ and $r > s$. Then $u(w(r)) > u(w(s))$, so $w(r) \succ w(s)$. Since v represents $\succsim$, $v(w(r)) > v(w(s))$. By construction, $h(r) > h(s)$. Thus h is increasing on $u(X)$. □

The following optional exercise demonstrates that the transformation h only has to be monotone on the range of the utility function $u(X)$, and not on the entire real line.

Exercise 3.6. Suppose $X = [0, 1] \cup (2, 3]$. Let $u(x) = x$ and

$$v(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x - 1 & \text{if } x \in (2,3] \end{cases}.$$

Prove that u and v represent the same preference relation on X (namely $\succsim = \geq$). Find the strictly increasing transformation $h : u(X) \to \mathbb{R}$ such that $v = h \circ u$. A bit trickier: prove that there exists no strictly increasing extension of h to the entire real line, i.e. if $h' : \mathbb{R} \to \mathbb{R}$ satisfied $h'(x) = h(x)$ for all $x \in u(X)$, then h' cannot be strictly increasing on the entire real line. (Hint: First prove that if h' is strictly increasing, then we must have $h'(2) \leq 1$. But $h'(1) = 1$, so $h'(1) \geq h'(2)$, a contradiction.)

[5] Suppose $Z \subseteq \mathbb{R}$. The function $h : Z \to \mathbb{R}$ is *strictly increasing* if $h(z) > h(z')$ for all $z, z' \in Z$ such that $z > z'$.

[6] The whole proof is much cleaner if we define $w : u(X) \to X/\sim$, if this makes sense.

Proposition 3.7. *Define the binary relation $\asymp$ on $\mathbb{R}^X$ by $u \asymp v$ if there exists a strictly increasing $h : u(X) \to \mathbb{R}$ such that $v = h \circ u$. Then $\asymp$ is an equivalence relation on $\mathbb{R}^X$.*

Proof. First note $u = I \circ u$, where I is the identity function, so $\asymp$ is reflexive. Suppose $v = h \circ u$. Since h is strictly increasing on $u(X)$ and maps to $v(X)$, it has a strictly increasing inverse on $v(X)$. Then $h^{-1} \circ v = h^{-1} \circ h \circ u = u$. So $\asymp$ is symmetric. If $v = h \circ u$ and $u = g \circ w$, then $v = h \circ g \circ w$. Since $h \circ g$ is also strictly increasing, $\asymp$ is transitive. □

The previous two results are a kind of uniqueness result. The uniqueness here is weaker than that in Proposition 2.15, because there are many utility functions, rather than a single utility function, in $\mathbb{R}^X$ that represent $\succsim$. Instead, there is a single equivalence class, under $\asymp$, of utility functions that represents $\succsim$. The representation is therefore unique in $\mathbb{R}^X / \asymp$.

Theorem 3.8. *Suppose X is countable. Then $\succsim$ is a preference relation if and only if there exists some utility function $u : X \to \mathbb{R}$ that represents $\succsim$.*

Proof. We will prove that if $\succsim$ is a preference relation, then there exists a utility representation; the other direction is implied by Exercise 3.4. For expositional purposes, we first start with the special finite case, $|X| < \infty$. Let $u(x) = |\{z \in X : x \succsim z\}|$. Since X is finite, $u(x)$ is finite. Suppose $x \succsim y$. Suppose $z \in \{z \in X : y \succsim z\}$, i.e. $y \succsim z$. We supposed $x \succsim y$, so by transitivity, $x \succsim z$, i.e. $z \in \{z \in X : x \succsim z\}$. Thus $\{z \in X : y \succsim z\} \subseteq \{z \in X : x \succsim z\}$. Therefore $|\{z : y \succsim z\}| \leq |\{z : x \succsim z\}|$. By construction, $u(y) \leq u(x)$.

Now suppose $x \succ y$. Since $x \succ y$ implies $x \succsim y$, the same transitivity argument we used above implies $\{z : y \succsim z\} \subseteq \{z : x \succsim z\}$. We know $x \succsim x$ by completeness, so $x \in \{z : x \succsim z\}$. Also, the definition of $\succ$ implies not $y \succsim x$, so $x \notin \{z : y \succsim z\}$. Then $\{z : y \succsim z\}$ and $\{x\}$ are disjoint, and both subsets of $\{z : x \succsim z\}$. Then

$$
\begin{aligned}
\{z : y \succsim z\} \cup \{x\} &\subseteq \{z : x \succsim z\} \\
|\{z : y \succsim z\} \cup \{x\}| &\leq |\{z : x \succsim z\}| \\
|\{z : y \succsim z\}| + |\{x\}| &\leq |\{z : x \succsim z\}| \\
u(y) + 1 &\leq u(x) \\
u(y) &< u(x).
\end{aligned}
$$

This argument does not follow directly in the countable case, because the cardinalities of the lower contour sets can be infinite. Therefore, we need a normalization to make sure we have a convergent sum. Suppose X is countable, then enumerate $X = \{z_1, z_2, \ldots\}$.[7] Define

$$u(x) = \sum_{\{n : x \succsim z_n\}} 2^{-n}.$$

[7] This enumeration is unrelated to the preference relation. For example, it is not generally the case that $z_{n+1} \succsim z_n$.

Then $0 < u(x) \leq \sum_{i=1}^{\infty} 2^{-n} < \infty$, so u is real-valued. Suppose $x \succsim y$. By transitivity, $\{n : y \succsim z_n\} \subseteq \{n : x \succsim z_n\}$, so $u(x) \geq u(y)$. Suppose $x \succ y$. There exists some $z_k = x$. Since $x \succ y$, $k \notin \{n : y \succsim z_n\}$. Also $k \in \{n : x \succsim z_n\}$, since $x \succsim x = z_k$ by completeness. By transitivity, $\{n : y \succsim z_n\} \subseteq \{n : x \succsim z_n\}$. Finally, $u(y) = \sum_{\{n:y\succsim z_n\}} 2^{-n} < \sum_{\{n:y\succsim z_n\}} 2^{-n} + 2^{-k} \leq \sum_{\{n:x\succsim z_n\}} 2^{-n} = u(x)$. □

Problem 3.9. Prove that if $\succsim$ is a reflexive (but not necessarily complete) and transitive binary relation on a finite set X, then there exists a function $u : X \to \mathbb{R}$ such that $x \succsim y$ implies $u(x) \geq u(y)$. (Observe that u is NOT a utility representation, because it does not generally satisfy $u(x) \geq u(y)$ implies $x \succsim y$.)

Example 3.10 (Lexicographic preference)**.** Let $X = \mathbb{R}^2$ and define $\succsim$ by

$$(x_1, x_2) \succsim (y_1, y_2) \iff \begin{cases} x_1 > y_1 \\ \text{or} \\ (x_1 = y_1 \text{ and } x_2 \geq y_2), \end{cases}.$$

the so-called "lexicographic preference." The lexicographic preference is complete and transitive, yet admits no utility representation.

We prove there exists no representation by contradiction. Suppose there exists a representation u. For any $x_1 \in \mathbb{R}$, $(x_1, 1) \succ (x_1, 0)$. Since u is a utility representation, $u(x_1, 1) > u(x_1, 0)$. By the denseness of the rational numbers, there then exists a rational number $r(x_1) \in \mathbb{Q}$ such that

$$u(x_1, 1) > r(x_1) > u(x_1, 0).\text{[8]} \tag{A}$$

Define the function $r : \mathbb{R} \to \mathbb{Q}$ by selecting $r(x_1)$ to satisfy $u(x_1, 1) > r(x_1) > u(x_1, 0)$.

We now show that r is a 1–1 function. Suppose $x_1 \neq x_1'$; we can further assume $x_1 > x_1'$. Then:

$$\begin{aligned} r(x_1) &> u(x_1, 0) && \text{by (A)} \\ &> u(x_1', 1) && \text{since } (x_1, 0) \succ (x_1', 1) \\ &> r(x_1') && \text{by (A).} \end{aligned}$$

Thus r is an a 1–1 function from the real numbers to the rational numbers, which contradicts the fact the real line is uncountable.

Exercise 3.11. Draw the upper contour set of $(1, 1)$ under the lexicographic preference in the Cartesian plane.

Exercise 3.12. Prove that the lexicographic preference $\succsim$ is antisymmetric, i.e. if $x \succsim y$ and $y \succsim x$, then $x = y$.

[8]Recall that the rational numbers $\mathbb{Q}$ are dense in the real line: for any open interval (a, b), there exists a rational number $r \in \mathbb{Q}$ such that $r \in (a, b)$.

Problem 3.13. Construct a utility representation of the lexicographic preference restricted to $\mathbb{Q} \times \mathbb{R}$, the product of the rational and real numbers.

Problem 3.14 (P. Fishburn)**.** Suppose $X = [-1, 1]$ and define $\succsim$ on X by

$$x \succsim y \iff \begin{cases} |x| > |y| \\ \text{or} \\ (|x| = |y| \text{ and } x \geq y), \end{cases},$$

Prove that $\succsim$ cannot be represented by any utility function on X.

Definition 3.15. A binary relation $\succsim$ on the metric space X is *continuous* if, for all $x \in X$, the upper and lower contour sets, $\{y \in X : y \succsim x\}$ and $\{y \in X : x \succsim y\}$, are closed.

Proposition 3.16. *A binary relation $\succsim$ is continuous if and only if:*

1. *If $x_n \succsim y$ for all n and $x_n \to x$, then $x \succsim y$; and*
2. *If $x \succsim y_n$ for all n and $y_n \to y$, then $x \succsim y$.*

Proof. This follows immediately from the fact C is closed if and only if C contains all its limit points. □

Exercise 3.17. Suppose $X = \mathbb{R}^2$ and

$$(x_1, x_2) \succsim (y_1, y_2) \iff x_1 \geq y_1 \text{ and } x_2 \geq y_2.$$

Verify that $\succsim$ is continuous.

Exercise 3.18. Verify that the lexicographic preference relation on $\mathbb{R}^2$ is not continuous.

Theorem 3.19 (Debreu)**.** *Suppose $X \subseteq \mathbb{R}^n$. The binary relation $\succsim$ on X is complete, transitive, and continuous if and only if there exists a continuous utility representation $u : X \to \mathbb{R}$.*

Proof. The proof of necessity is a homework problem. A fully general proof of sufficiency is beyond our scope. Instead, we will make the additional assumption that $X = \mathbb{R}^n$ and $\succsim$ is *strictly monotone*, i.e. if $x_i \geq y_i$ for all dimensions i and $x \neq y$, then $x \succ y$. We prove that if $\succsim$ is a continuous and strictly monotone preference relation on $\mathbb{R}^n$, then there exists a continuous utility representation of $\succsim$. Let $e = (1, 1, \ldots, 1)$. Verify that, if $\succsim$ is strictly monotone, then

$$\alpha \geq \beta \iff \alpha e \succsim \beta e. \tag{B}$$

Step 1: There exists a unique $\alpha^*(x)$ such that $\alpha^* e \sim x$. Let

$$\alpha^* = \inf \underbrace{\{\beta \in \mathbb{R} : \beta e \succsim x\}}_{B \subset \mathbb{R}}.$$

We want to show that $\alpha^* e \sim x$. For future reference, let $B = \{\beta \in \mathbb{R} : \beta e \succsim x\} \subset \mathbb{R}$. We know $(\max_i x_i)e \geq x$. Then strict monotonicity implies $(\max_i x_i)e \succsim x$, so B is nonempty. Moreover, B is bounded from below by $\alpha = (\min_i x_i) - 1$ by monotonicity. So, α^* is well-defined.

Since α^* is the infimum of B, there exists a sequence $\beta_n \in B$ such that $\beta_n \to \alpha^*$ (in $\mathbb{R}$). Then $\beta_n e \to \alpha^* e$ (in $\mathbb{R}^n$) and $\beta_n e \succsim x$ because $\beta_n \in B$. Applying continuity, $\alpha^* e \succsim x$.

Since α^* is a lower bound of B, if $\alpha \in B$, then $\alpha \geq \alpha^*$. Stated contrapositively:

$$\alpha < \alpha^* \Longrightarrow \alpha e \prec x. \tag{C}$$

Let $\alpha_n = \alpha^* - \frac{1}{n}e$. By (C), $\alpha_n e \prec x$, so $\alpha_n e \precsim x$. Also, $\alpha_n e \to \alpha^* e$. By continuity, $\alpha^* e \precsim x$. Since $\alpha^* e \succsim x$ and $\alpha^* e \precsim x$, we have $\alpha^* e \sim x$, proving existence.

To prove uniqueness, suppose $\alpha e \sim x$. By transitivity, $\alpha e \sim \alpha^* e$. Then, by (B), $\alpha = \alpha^*$.

Step 2: Define $u(x) = \alpha^*(x)$, where $\alpha^*(x)$ is the unique α^* such that $\alpha^* e \sim x$. Then $u(x)$ represents $\succsim$. Suppose $\alpha^*(x) \geq \alpha^*(y)$. Then $\alpha^*(x)e \succsim \alpha^*(y)e$, by (B). By construction of α^*, $x \sim \alpha^*(x)e \succsim \alpha^*(y)e \sim y$. By transitivity, $x \succsim y$.

The argument for $\alpha^*(x) > \alpha^*(y)$ implies $x \succ y$ is identical.

Step 3: The defined $u(x)$ is continuous. Recall that to prove $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, it suffices to show that $f^{-1}((a,b))$ is open for all $a, b \in \mathbb{R}$. We have that $u(ae) = \alpha^*(ae) = a$ for any $a \in \mathbb{R}$, since obviously $ae \sim ae$. Next,

$$u^{-1}((a,b)) = u^{-1}((a,\infty) \cap (-\infty,b)) = u^{-1}((a,\infty)) \cap u^{-1}((-\infty,b)).$$

But $u(ae) = a$, so

$$u^{-1}((a,\infty)) = u^{-1}((u(ae),\infty)) = \{x \in \mathbb{R}^n : x \succ ae\},$$

which is open because the strict upper contour set of ae is open whenever $\succsim$ is complete and continuous. An entirely symmetric argument proves that $u^{-1}((-\infty,b))$ is the strict lower contour set of be, hence also open. Then $u^{-1}((a,b))$ is the intersection of two open sets, therefore open.[9] □

Debreu's Theorem only asserts that *one* of the utility representations for $\succsim$ must be continuous, not that *all* of the utility representations must be continuous. Therefore, continuity is a *cardinal* feature of the utility function, not an *ordinal* feature, since it is not robust to strictly increasing transformations.

Exercise 3.20. Construct a preference relation on $\mathbb{R}$ that is not continuous, but admits a utility representation. Construct a discontinuous utility representation of a continuous preference relation on $\mathbb{R}$.

Proposition 3.21. *If $\succsim$ is a continuous preference relation and $A \subseteq \mathbb{R}^n$ is nonempty and compact, then $C_{\succsim}(A)$ is nonempty and compact.*

[9] Maximilian Kasy came up with this nice proof for Step 3.

Proof. Suppose $\succsim$ is continuous. By Debreu's Theorem, there exists some continuous representation u. Then

$$C_{\succsim}(A) = \arg\max_{x \in A} u(x).^{10}$$

Nonemptiness follows from continuity of u and the Extreme Value Theorem. Compactness follows from the fact $u^{-1}(y)$ is bounded, since it is a subset of the bounded set A, and closed, since the inverse image of a closed set under a continuous function is closed. □

[10] Verify this.