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COMBINATORIAL VOTING

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We study elections that simultaneously decide multiple issues, where voters have independent private values over bundles of issues. The innovation is in considering nonseparable preferences, where issues may be complements or substitutes. Voters face a political exposure problem: the optimal vote for a particular issue will depend on the resolution of the other issues. Moreover, the probabilities that the other issues will pass should be conditioned on being pivotal. We prove that equilibrium exists when distributions over values have full support or when issues are complements. We then study large elections with two issues. There exists a nonempty open set of distributions where the probability of either issue passing fails to converge to either 1 or 0 for all limit equilibria. Thus, the outcomes of large elections are not generically predictable with independent private values, despite the fact that there is no aggregate uncertainty regarding fundamentals. While the Condorcet winner is not necessarily the outcome of a multi-issue election, we provide sufficient conditions that guarantee the implementation of the Condorcet winner.

KEYWORDS: Combinatorial voting, multi-issue elections, strategic voting.

1. INTRODUCTION

PROPOSITIONS 1A AND 1B of the 2006 California general election both aimed to increase funding for transportation improvements.[2] Suppose a voter prefers some increased funding and supports either proposition by itself, but given the state's fiscal situation also prefers that both measures fail together than pass together. She views the propositions as substitutes. However, the ballot does not elicit her preferences over bundles of transportation measures, but only a separate up–down vote on each proposition. If she votes up on Proposition 1A while Proposition 1B passes, she contributes to the undesired passage of both measures. On the other hand, if Proposition 1B were to fail, she would like to see Proposition 1A pass to fund some transportation improvements.

How should she vote? Some subtle considerations complicate the answer to this question. What is the likelihood that she is pivotal on either proposition or both? The issue here is that there are multiple pivotal events. If she is pivotal on some proposition, what is the conditional likelihood that the other will pass or fail? The issue here is that central strategic conjectures must be appropriately conditioned on the particular pivotal event. The natural model for these questions is a game of incomplete information. The model begs other

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[2]Proposition 1A dedicated gasoline taxes for transportation improvements, at the exclusion of other uses, while Proposition 1B issued $20 billion in bonds to fund improvements. Both measures passed by large margins.

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questions. Does equilibrium exist? What does it look like? Does it exhibit special properties in large elections? Are equilibrium outcomes predictable? Are these outcomes ordinally efficient? For elections with nonseparable issues, these basic questions are still unanswered. To our knowledge, this is the first paper to follow the strategic implications of electoral complementarity or substitution to their equilibrium conclusions, and makes initial progress in addressing these concerns.

1.1. *An Example*

The following example illustrates the strategic delicacy of elections with multiple issues. There are two issues, say propositions 1 and 2. Each voter's private values for the four possible bundles $\emptyset, \{1\}, \{2\}$, and $\{1, 2\}$ can be represented as a four-dimensional type $\theta = (\theta_\emptyset, \theta_1, \theta_2, \theta_{12})$, where θ_A denotes the value for bundle A. Voters' types are independent and identically distributed with the discrete distribution

$$\theta = \begin{cases} (\delta, 0, 0, 1) & \text{with probability } 1 - 2\varepsilon, \\ (1, 0, 0, 0) & \text{with probability } \varepsilon, \\ (0, 1, 0, 0) & \text{with probability } \varepsilon, \end{cases}$$

where $\delta, \varepsilon > 0$ are arbitrarily small. With high probability $1 - 2\varepsilon$, a voter wants both issues to pass, but slightly prefers both issues to fail than to have either issue pass alone. With small probability ε, a voter is either type $(1, 0, 0, 0)$ and wants both issues to fail or type $(0, 1, 0, 0)$ and wants issue 1 to pass alone. In either case, she is indifferent between her less preferred alternatives. It is a dominant strategy for type $(1, 0, 0, 0)$ to vote down on both issues and for type $(0, 1, 0, 0)$ to support issue 1 and vote against issue 2. The question is how type $(\delta, 0, 0, 1)$ should vote.

A natural conjecture is that type $(\delta, 0, 0, 1)$ should vote up on both issues in any large election. Then the conjectured equilibrium strategy s^* as a function of types is

$$s^*(\delta, 0, 0, 1) = \{1, 2\},$$
$$s^*(1, 0, 0, 0) = \emptyset,$$
$$s^*(0, 1, 0, 0) = \{1\},$$

where $s^*(\theta)$ refers to the issues that type θ supports. When voters play this strategy, both issues will have majority support in large elections, which is efficient. The suggested strategy might appear to be incentive compatible, since $(\delta, 0, 0, 1)$ should vote up for either issue when she is confident that the other issue will pass.

However, the proposed strategy is *not* an equilibrium in large elections because the conditional probability that the residual issue passes is starkly different from the unconditional probability. Consider a voter deciding whether to support issue 1. She correctly reasons that her support only matters when she is pivotal for issue 1. When the other votes on issue 1 are split, she is in the unlikely state of the world where half of the other voters are of type $(1, 0, 0, 0)$, since this is the only type who vote against issue 1. Moreover, in large elections, there will be some voters of type $(0, 1, 0, 0)$. Then voters of type $(\delta, 0, 0, 1)$ comprise a strict minority. Since these are the only types who support issue 2, this voter should conclude that issue 2 will surely fail whenever she is pivotal for issue 1 in a large election. Therefore, type $(\delta, 0, 0, 1)$ should vote down on issue 1 because she prefers the bundle $\emptyset$ yielding utility δ to the bundle $\{1\}$ yielding utility 0. In fact, the only equilibrium in weakly undominated strategies is for type $(\delta, 0, 0, 1)$ to vote down on both issues, inducing the ex ante inefficient social outcome of the empty bundle in large elections.[3]

Finally observe that had δ been equal to 0, then type $(\delta, 0, 0, 1) = (0, 0, 0, 1)$ would have had a dominant strategy to vote up on both issues. In this case, the suggested strategy where $s^*(\delta, 0, 0, 1) = \{1, 2\}$ would be an equilibrium and the efficient bundle would be implemented in large elections. So a small amount of nonseparability, that is, a slightly positive $\delta > 0$, is enough to remove efficiency and change the outcome of the election.

1.2. *A Political Exposure Problem*

The basic complication for elections with nonseparable issues is the wedge between the unconditional probability that an issue will pass and the conditional probability when a voter is pivotal on another issue. This resonates with existing analyses of strategic voting on a single issue with interdependent values; for example, see Austen-Smith and Banks (1996) or Feddersen and Pesendorfer (1997). In these models, being pivotal provides additional information regarding other voters' signals about an unknown state of the world. The intuition for the "swing voter's curse" is analogous to the importance of strategic conditioning in common value auctions for a single item, where it leads to the winner's curse and strategic underbidding. In both single-object auctions and single-issue elections with common values, strategic conditioning complicates information aggregation and efficiency. This is because the expected value of the object or the proposal is different when the player conditions on being the winner of the auction or the pivotal voter of the election.

The intuition for multi-issue elections also has a relationship with auction theory, but with a different branch. Here, the wedge is related to the exposure problem in combinatorial auctions for multiple items, which exists even with private values. Suppose two items are sold in separate auctions. Consider a bidder with complementary valuations who desires only the bundle of both items.

[3]A related example on voting over binary agendas is Ordeshook and Palfrey (1988). There, being pivotal in the first round of a tournament changes the expected winner in later rounds. This reasoning can lead to inefficient sequential equilibria in their model.

She must bid in both auctions to have any chance of obtaining this package, but she should recognize that doing so exposes her to the risk of losing the second auction while winning the first, forcing her to pay for an undesired single item bundle. Moreover, the unconditional probability of winning the second auction is not appropriate in computing her exposure, but rather the conditional probability of winning the second auction assuming that she wins the first auction. Likewise, a voter in an election who desires a bundle of two issues to pass, but does not want either issue to pass alone, faces an exposure problem. In deciding her vote for issue 1, she should consider whether issue 2 will pass, but also condition this probability on the assumption that she is pivotal on issue 1.

This exposure problem disappears when values are separable across issues, in which case each issue can be treated like a separate election. However, with nonseparable preferences, the following intuitions from single-issue elections break down. First, with one issue, voting sincerely for the preferred outcome (pass or fail) is a weakly dominant strategy for every voter. In contrast, with nonseparable preferences, voting sincerely is never an equilibrium. Instead, a voter's equilibrium strategy must correctly condition the other voters' ballots on the assumption that she is pivotal for some issue. Second, with a single issue, there is a generic class of distributions over values for which the outcome is predictable in large elections. We assume independent private values, so the composition of preferences is known for large electorates. Nevertheless, with multiples issues and nonseparable preferences, there exists a nontrivial set of type distributions that generate unpredictable election outcomes. This aggregate *endogenous* uncertainty exists despite the fact that there is no aggregate *primitive* uncertainty in large elections. Third, the Condorcet winner is always implemented in single-issue elections. With multiple issues, the Condorcet winning bundle can fail to be the outcome of large elections. Instead, additional assumptions will suffice for implementation of the Condorcet winner.

1.3. *Outline*

The paper proceeds as follows. Immediately following is a review of related literature. Section 2 introduces the Bayesian game of voting over multiple issues. Section 3 shows the existence of equilibrium using two arguments. One is topological and converts the infinite-dimensional fixed point problem over strategies to a finite-dimensional problem over probabilities regarding which issues a voter is pivotal for and which issues will pass irrespective of her vote. This conversion yields later dividends in characterizing equilibrium. The second argument assumes complementarity between issues and shows the existence of a monotone equilibrium, where types with a stronger preference for passing more issues also vote for more issues. This proof relies on recent general monotone existence results due to Reny (2011).

Section 4 presents a nonempty open set of densities that exhibit aggregate uncertainty regarding the outcome of the election. Even though there is no uncertainty regarding the primitives of the model, unpredictability of the outcomes is required to maintain incentives in equilibrium. This establishes that predictability of outcomes is not a generic feature of large elections with multiple issues.

Section 5 uses the limit results to study the relationship between combinatorial voting and the majority preference relation. While our example demonstrates that the Condorcet winner is not generally the outcome of the election, we provide sufficient conditions for implementation of the Condorcet winner. Finally, we provide results that suggest that ordinal separability of the majority preference relation is conducive to implementation of the Condorcet winner and hence to predictability.

Section 6 concludes and reviews open questions. Proofs are collected in the Appendix.

1.4. *Related Literature*

Several papers in political science recognize the potential problems introduced by nonseparable preferences over multiple issues. Brams, Kilgour, and Zwicker (1998) pointed out that the final set of approved issues may not match any single submitted ballot, which they call the "paradox of multiple elections."[4] Lacy and Niou (2000) constructed an example with three strategic voters and complete information where the final outcome is not the Condorcet winner. An empirical literature in political science documents nonseparable preferences. A public opinion survey conducted by Lacy (2001) documents nonseparable preferences by a large portion of respondents across pairs of policy proposals. Our model enriches this literature in two directions. First, while existing work focuses on sincere voting, we consider fully rational voters who vote strategically.[5] Second, we introduce uncertainty regarding others' preferences. Incomplete information is crucial with nonseparable preferences, since the desired resolution of a particular issue depends on the uncertainty regarding the resolution of other issues.

A natural application of our model is to simultaneous two-candidate elections for multiple political offices. Split tickets, such as those supporting a Republican president but a Democratic legislator, are increasingly common and constitute about a quarter of all ballots in recent presidential elections (Fiorina (2003, Figure 2-1)). Fiorina (2003) introduced the idea that voters have an inherent preference for divided government as a means to moderate policy choices. In our model, such voters would treat offices as substitutes. These preferences are studied in political science, where an active area of empirical

[4]The paradox was extended by Özkal-Sanver and Sanver (2006) and reinterpreted by Saari and Sieberg (2001).

[5]The exceptions are the mentioned example by Lacy and Niou (2000) and a single-person model of sequential survey responses by Lacy (2001).

investigation is the preponderance of "cognitive Madisonians" who intentionally split tickets to check government power. The literature has not reached a consensus on their importance, but at least some studies conclude that the number is "small but large enough to affect election outcomes" (Mebane (2000, p. 37)) or "over 20% of the electorate and maybe largely responsible for observed patterns of division at the aggregate level" (Lewis-Beck and Nadeau (2004, p. 97)).

Some papers provide formal models of ticket splitting. Alesina and Rosenthal (1996) presented a spatial model where voters split their ticket to moderate policy location. Chari, Jones, and Marimon (1997) presented a fiscal model where voters split tickets to increase local spending and restrain national taxation. These motivations provide foundations for nonseparable preferences, but also restrict the implied preferences and thus the implied predictions; Alesina and Rosenthal (1996) predicted that all split tickets in a particular election support the same candidates, while Chari, Jones, and Marimon (1997) predicted that all split tickets support a conservative president and a liberal legislator. While we are agnostic about the source of nonseparability, we allow arbitrary preferences over the composition of government, for example, some voters may have a desire for unified government, and predict the full spectrum of split tickets. Finally, the existing models move directly to large elections with a continuum of voters. While we examine limits as they tend large, the finite electorates in this paper are essential to maintain the political exposure problem in equilibrium.

Plurality rule over bundles is a potential alternative to combinatorial voting, especially considering the large theoretical literature on plurality rule. While more specific comparisons are made as results are presented in the paper, we now highlight some differences between large elections under combinatorial rule and plurality rule. We focus on plurality rule as an alternative aggregation scheme because it shares the same space of ballots or messages as combinatorial voting.[6] As originally observed by Palfrey (1989), limit equilibria of plurality rule typically satisfy Duverger's law and involve two active candidates.[7] These equilibria have qualitatively different features than the limit equilibria of this model. First, predictability of the outcome is a generic feature of any Duvergerian equilibrium under plurality rule. There are multiple Duvergerian equilibria involving different pairs of candidates, but for any fixed equilibrium, the outcome is determinate for a generic set of type distributions. In contrast,

[6]In our setting, the set of candidates is the power set 2^X of bundles. Combinatorial rule is not a scoring rule. In particular, combinatorial voting invokes the structure of the power set in an essential way, while this structure is irrelevant to a scoring rule. Moreover, general scoring rules require larger message spaces than combinatorial rule. For example, in this environment approval voting requires that the space of ballots be the power set of the power set or 2^{2^X}. General treatments of scoring rules can be found in Myerson and Weber (1993) and Myerson (2002).

[7]Fey (1997) showed that only the Duvergerian limit equilibria are stable in a variety of senses.

combinatorial voting can yield unpredictability for an open set of type distributions. Second, plurality rule always has at least one limit equilibrium which selects a Condorcet winner when the winner exists. As the example shows, combinatorial rule can fail to have any limit equilibria which implement the Condorcet winner. Finally, in our view the strategic considerations under plurality rule are relatively simpler than under combinatorial rule. Once an equilibrium is fixed, each voter should support whichever of the two active candidates she prefers. In our model, if the voter assumes that she is pivotal for some issue, she must then condition her conjecture regarding the residual issues on that pivot event.

One feature common to both plurality rule and combinatorial rule is that the distribution of types conditional on being pivotal diverges from the ex ante distribution of types. Other multi-candidate models with independent private values and incomplete information also share this feature. The most closely related in terms of the strategic intuitions are models that have a dynamic element to the aggregation. In such models, the wedge between the unconditional probability of an event and its probability conditional on being pivotal can also lead to inefficiencies. The earliest example of which we are aware is the treatment of strategic voting on dynamic agendas by Ordeshook and Palfrey (1988); there the winner between alternatives a and b in the first round faces alternative c in the second round. With incomplete information, being pivotal for a against b in the first round can reverse the expected resolution of a vote between a and c. In particular, as in our initial example, this wedge can prevent a Condorcet winner from being the final outcome of the tournament. While a wedge between pivotal and unconditional probabilities appears in existing work, to our knowledge this paper is the first to observe the wedge between pivotal and unconditional probabilities in the context of the exposure problem and nonseparabilities in multi-issue elections.

2. MODEL

There is a finite and odd set of I voters. The voters decide a finite set of binary issues X, whose power set is denoted $\mathcal{X}$. Each voter i submits a ballot $A_i \in \mathcal{X}$, with $x \in A_i$ meaning that i votes "up" on issue x and with $y \notin A_i$ that she votes "down" on y. An issue can be a policy referendum which will pass or fail, or an elected office decided between two political parties where one is labeled "up." The social outcome $F(A_1, \dots, A_I) \in \mathcal{X}$ is decided by what we call combinatorial rule:

$$F(A_1, \dots, A_I) = \{x \in X : \#\{i \in I : x \in A_i\} > I/2\}.$$

Each voter knows her own private values over outcomes, but is uncertain about the others' values. Her type is drawn from the (normalized) type space $\Theta_i = [0, 1]^{\#\mathcal{X}}$, with typical element θ_i. Then $\theta_i(A)$ denotes type θ_i's utility for all the issues in A passing and all those in its complement failing: so θ_i's utility for the profile of ballots $(A_1, \dots, A_I)$ is $\theta_i(F(A_1, \dots, A_I))$. Let $\Theta = \prod_i \Theta_i$ denote the space of all type profiles and let $\Theta_{-i} = \prod_{j \neq i} \Theta_j$. Voters are identical ex ante, with their types realized independently from the distribution $\mu \in \Delta\Theta$. We assume that μ admits a density. When it engenders no confusion, we also let μ denote the product distribution over the profile of types.

A (pure) strategy s_i for each voter i is a measurable function $s_i : \Theta_i \to \mathcal{X}$ assigning a ballot to each of her types. The space of strategies for each voter is S_i. The space of strategy profiles is $S = \prod_i S_i$, and let $S_{-i} = \prod_{j \neq i} S_j$. Voter i's ex ante expected utility for the joint strategy profile $s(\theta) = (s_1(\theta_1), \dots, s_I(\theta_I))$ is $\mathrm{EU}_i(s) = \int_\Theta \theta_i(F(s(\theta)))\, d\mu$.

DEFINITION 1: A strategy profile s^* is a *voting equilibrium* if it is a symmetric Bayesian Nash equilibrium in weakly undominated strategies.

A voter's values might exhibit certain structural characteristics. For example, she might view the issues as complements, as substitutes, or as having no interaction. These are captured by the following definitions.

DEFINITION 2: θ_i is *supermodular* if, for all $A, B \in \mathcal{X}$,

$$\theta_i(A \cup B) + \theta_i(A \cap B) \geq \theta_i(A) + \theta_i(B).$$

θ_i is *submodular* if, for all $A, B \in \mathcal{X}$,

$$\theta_i(A \cup B) + \theta_i(A \cap B) \leq \theta_i(A) + \theta_i(B).$$

θ_i is *additively separable* if it is both supermodular and submodular.

As mentioned in the Introduction, sincere voting is not an equilibrium when μ has full support. The result has a simple intuition. Optimal voting is determined cardinally by utility differences across bundles, while sincere voting is determined ordinally by the best bundle.

PROPOSITION 1: *If each μ admits a density with full support, then sincere voting, where*

$$s(\theta) = \arg\max_A \theta(A),$$

is not a voting equilibrium.

3. EXISTENCE OF EQUILIBRIUM

We begin by proving the existence of voting equilibria. We present topological and lattice-theoretic arguments for existence.

3.1. *General Existence of Equilibrium*

PROPOSITION 2: *Suppose μ_i admits a density function with full support. There exists a voting equilibrium s^*.*[8]

The proof lifts the infinite-dimensional problem of finding a fixed point in the space of strategy profiles to a finite-dimensional space of probabilities. Specifically, the strategically relevant information for voter i is summarized as the set of issues C for which voter i is pivotal and the set of issues D which will pass irrespective of voter i's ballot. The outcome of submitting the ballot A is that those issues which she supports and on which she is pivotal will pass, along with those issues which will pass no matter how she votes: $[A \cap C] \cup D$. The relevant uncertainty can therefore be summarized as a probability over the ordered disjoint pairs of subsets of X, which we write as $\mathcal{D} = \{(C, D) \in \mathcal{X} \times \mathcal{X} : C \cap D = \emptyset\}$. Each strategy $s \in S$ induces a probability $\pi(s) \in \Delta\mathcal{D}$ over $\mathcal{D}$, where $\Delta\mathcal{D}$ denotes the space of probabilities on $\mathcal{D}$. The function $\pi : S \to \Delta\mathcal{D}$ is continuous.[9]

In turn, each belief $P \in \Delta\mathcal{D}$ over these ordered pairs induces an optimal ballot $[\sigma(P)](\theta) \in \mathcal{X}$ for type θ, which is the ballot A maximizing the expected utility $\sum_{\mathcal{D}} \theta([A \cap C] \cup D) \cdot P(C, D)$. Observe that this expression is a linear function with coefficients $P(C, D)$ on θ. Then the types for which A is an optimal ballot are those where $\sum_{\mathcal{D}} \theta(A \cap C \cup D) \cdot P(C, D) \geq \sum_{\mathcal{D}} \theta([A' \cap C] \cup D) \cdot P(C, D)$, which defines a finite intersection of half-spaces. Small changes in P induce small geometric changes in these half-spaces. The density assumption implies that these small geometric changes also have small measure, proving that $\sigma : \Delta\mathcal{D} \to S$ is continuous.

Before applying a fixed point theorem, we need to restrict attention to undominated strategies. Consider a weakly undominated strategy s. The induced probability $\pi(s)$ that a voter will be pivotal for the issues in C while the issues in D pass is at least as large as the probability that half the other voters submit $C \cup D$ while the other half submits D. By the full support assumption, there is a strictly positive probability any voter submits C, $C \cup D$, and D in any weakly undominated strategy. Since types are independent, we conclude that $[\pi(s)](C, D)$ is strictly positive because there is some chance that half the voters submit D while the other half submit $C \cup D$. Then the probabilities induced by weakly undominated strategies S^U live in a compact subsimplex Δ^U in the interior of the entire simplex $\Delta\mathcal{D}$: $\pi(S^U) \subseteq \Delta^U \subseteq \text{int}(\Delta\mathcal{D})$. Since all strategically relevant events (C, D) have strictly positive probability in Δ^U, the induced best

[8]The full support assumption can be replaced with a weaker assumption that guarantees every ballot is submitted with positive probability. For example, if for every bundle A there is a positive measure of types whose unique weakly undominated strategy is to submit the ballot A, then equilibrium would exist. Alternatively, assuming a set of naive voters who submit the ballot A in all circumstances would also guarantee an equilibrium among the sophisticated voters.

[9]The topology on S is defined by the distance $d(s, s') = \mu(\{\theta : s(\theta) \neq s'(\theta)\})$.

replies must be weakly undominated. The restriction $\pi \circ \sigma : [\Delta^U] \to S^U \to [\Delta^U]$ defines a continuous function from a compact subset of a finite-dimensional space to itself. By Brower's theorem, there exists a fixed point P^* with strictly positive probabilities on all pairs. Then $\sigma(P^*)$ is a Bayesian Nash equilibrium in weakly undominated strategies.

The key insight, moving from the infinite-dimensional space of strategies to a finite-dimensional space of probabilities, is adapted from Oliveros (2009). The broad approach of reducing the fixed point problem to a finite simplex is reminiscent of the distributional approach of Radner and Rosenthal (1982) and Milgrom and Weber (1985). However, these earlier results are not immediately applicable given the restriction to weakly undominated strategies. As an important benefit, expressing equilibrium as a fixed point of probabilities over pivot and passing events enables sharper characterizations of equilibria in large elections.

3.2. *Existence of Monotone Equilibrium*

With complementary issues, equilibrium can be sharpened to be monotone in the increasing differences order: those types who have a stronger preference for more issues passing also support more issues. Monotonicity of ballots with respect to types is useful for empirical identification. Monotonicity justifies the inference that those who are observed to vote for more issues have a preference for larger bundles. For example, suppose X is a number of political offices and voting up corresponds to voting for the Republican candidate while voting down corresponds to voting for the Democratic candidate. If all voters prefer to have politicians of the same party in government, then we can infer that those who vote for more Republicans are more right-leaning than those who vote for fewer Republicans. However, if some voters are concerned with balancing party representation, that is, if issues are substitutes, then this inference is no longer justified, as it confounds ideological centrism with a desire for party balance.

Consider the partial order $\geq$ on types defined as $\theta' \geq \theta$ if the inequality $\theta'(A) - \theta'(B) \geq \theta(A) - \theta(B)$ holds for all $A \supseteq B$. This order captures the notion that a larger type θ' has a uniformly stronger preference for more issues to pass, as the difference in her utility between a larger bundle A and a smaller bundle B always dominates that difference for a smaller type θ. If up is coded as a Republican candidate for that office, the difference in utility between a more Republican (A) and a less Republican (B) legislature is greater for a right-leaning type θ' then it is for a left-leaning type θ. The following proposition demonstrates that assuming issues are complementary, that is, that a more unified legislature is more desirable, suffices for the desired inference that more ideologically conservative voters will support more Republican candidates.

PROPOSITION 3: *Suppose μ admits a density whose support is the set of all supermodular type profiles. Define the increasing differences order $\geq$ on Θ by $\theta' \geq \theta$ if*

$$\theta'(A) - \theta'(B) \geq \theta(A) - \theta(B) \quad \forall A \supseteq B.$$

Then there exists a monotone voting equilibrium s^*, where $s^*(\theta') \supseteq s^*(\theta)$ whenever $\theta' \geq \theta$.

Note that the election is not a supermodular game. Sufficiently large strategies by all voters guarantee that no voter is ever pivotal on any issue and eliminate the difference in interim utility between any of two strategies. Moreover, the restriction to weakly undominated strategies is important, since the trivial equilibrium where all types submit the same ballot is monotone. To handle these considerations, the proof relies on recent monotone existence results by Reny (2011) that improve earlier theorems by Athey (2001) and McAdams (2003) by allowing for general orders on types, such as the increasing differences order, and for restrictions on strategies, such as the exclusion of weakly dominated strategies.

4. UNPREDICTABILITY

We specialize to the case with two issues for the remainder of the paper.

This section studies the predictability of outcomes in combinatorial elections. Since the distribution μ over types is fixed, there is no aggregate uncertainty in large elections about the proportion of types in the population. Nevertheless, we prove that the outcomes of large elections remain uncertain for a nontrivial set of type distributions. This unpredictability is not an artifact of primitive statistical uncertainty, but rather is necessary to maintain incentives in equilibrium.[10]

The unpredictability of outcomes under combinatorial rule is qualitatively distinct from the indeterminacy of equilibrium under plurality rule. Under plurality rule, there are multiple limit equilibria where different pairs of candidates are active. But fixing any equilibrium selection, the outcome is generically certain. In contrast, the unpredictability of outcomes under combinatorial rule is not due to multiplicity of equilibria. Rather, for all equilibria the probability of an issue passing is uniformly bounded away from 0 or 1.

Unpredictability also contrasts nonseparable preferences from separable preferences in our model. When there is only a single issue or when preferences are separable, predictable outcomes are relatively generic. Excepting

[10]In a two-period version of their model with aggregate uncertainty regarding the distribution of preference, Alesina and Rosenthal (1996) predicted uncertain presidential winners for a nontrivial range of parameters. However, this assumes primitive uncertainty on the distribution of preferences. The unpredictability disappears in their basic model where the distribution of preferences is common knowledge.

knife-edge distributions where voters are equally likely to prefer passing and failing, the outcome of each issue is certain in large elections. Unpredictability in election outcomes is therefore difficult to reconcile with a model of costless voting with private separable values. With nonseparable preferences, the predictability of large elections depends on the type distribution μ.

Given a sequence of strategies, the following definition explains whether an issue becomes certain to pass or fail at the limit, that is, whether the outcome of that issue is predictable in large elections.

DEFINITION 3: Consider a sequence of strategies $s_I \to s$. Issue 1 is *unconditionally certain to pass* (*fail*) if

$$\mathbf{P}\left(\#\{i : 1 \in s_I^*(\theta_i)\} > (<) \frac{I}{2}\right) \to 1.$$

If every issue $x = 1, 2$ is either certain to pass or to fail, we write that the set

$$A = \{x \in X : x \text{ is certain to pass}\}$$

is a *limit outcome* of the election. If an issue $x = 1, 2$ is neither certain to pass nor to fail, we write that issue x is *unpredictable*.

We focus on the following set of densities.

EXAMPLE 1: Pick some small $\varepsilon > 0$.[11] Consider the class of densities $\mathcal{C}$ which satisfy the following restrictions:

$$\frac{1-\varepsilon}{4} < \mu(\theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) < \frac{1}{4},$$
$$\frac{1-\varepsilon}{4} < \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) < \frac{1}{4},$$
$$\frac{1-\varepsilon}{4} < \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) < \frac{1}{4},$$
$$\frac{1-\varepsilon}{4} < \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) < \frac{1}{4}.$$

This class is open and nonempty.[12]

For all sequences of equilibria for all distributions in $\mathcal{C}$, the probability that either issue will pass is uniformly bounded away from 0 or 1. In other words,

[11] In fact, ε can be as large as $\frac{1}{16}$.
[12] It is open in both the sup and weak convergence topologies.

even given the exact distribution $\mu \in \mathcal{C}$ and an arbitrarily large number of voters that are independently drawn from that distribution, an observer would not be able to predict the outcome of either issue.

PROPOSITION 4: *Consider any density in $\mathcal{C}$. For every sequence of equilibria, both issues are unpredictable.*

We now sketch the proof of Proposition 4. Many of the statistical details that are potentially useful for future applications are included in Appendix A.4.

As the introductory example demonstrated, the conditional probability is the strategically relevant one and can sharply diverge from the unconditional probability. A main objective in analyzing combinatorial elections is to connect the unconditional probabilities, which are of empirical relevance to the analyst, to the conditional probabilities, which are of strategic relevance to the voters. An important statistic for equilibrium analysis is the conditional probability that issue 1 passes when a voter is pivotal for issue 2. Consider the following definition.

DEFINITION 4: Fix a sequence of strategies $s_I \to s$. Issue 1 is *conditionally certain to pass* (*fail*) if:

$$\mathbf{P}\left(\#\{j \neq i : 1 \in s_I(\theta_j)\} > (<) \, \frac{I-1}{2} \,\middle|\, \#\{j \neq i : 2 \in s_I(\theta_j)\} = \frac{I-1}{2}\right) \to 1. \tag{1}$$

The bundle $A \subseteq \{1, 2\}$ is *conditionally certain* if x is conditionally certain to pass at $x' \neq x$ for every $x \in A$ and x is conditionally certain to fail at x' for every $x \notin A$. If each issue is conditionally certain to pass or to fail, then we simply write that the limit equilibrium exhibits *conditional certainty*. If it does not exhibit conditional certainty, we write that it exhibits *conditional uncertainty.*

When the conditional probabilities converge to 1 or 0, the analysis of limit equilibria simplifies because optimal ballots are determined from a few ordinal inequalities. In particular, the equilibrium strategy can be decided issue by issue. For example, if issue 2 is conditionally certain to pass, then a voter should support issue 1 if and only if she prefers $\{1, 2\}$ to $\{2\}$. This is because the probability of being jointly pivotal for both issues vanishes much more quickly than the probability of being pivotal for either issue alone, by virtue of the full support assumption that ensures each ballot is played with strictly positive probability. However, the conditional probability of an issue passing is defined with respect to endogenous equilibrium strategies. So the first task in the proof is to provide primitive conditions that are necessary and sufficient for conditional certainty.

As a starting point, suppose that the first voter is pivotal for issue 1. In this case, the probability that another voter supports issue 1 is exactly: $\mathbf{P}(1 \in s^*(\theta)) = \frac{1}{2}$. So the conditional probability that another voter supports issue 2 is

$$\begin{aligned}
&\mathbf{P}(1 \in s^*(\theta)) \times \mathbf{P}(2 \in s^*(\theta)|1 \in s^*(\theta)) \\
&\quad + \mathbf{P}(1 \notin s^*(\theta)) \times \mathbf{P}(2 \in s^*(\theta)|1 \notin s^*(\theta)) \\
&= \frac{1}{2} \times \mathbf{P}(2 \in s^*(\theta)|1 \in s^*(\theta)) + \frac{1}{2} \times \mathbf{P}(2 \in s^*(\theta)|1 \notin s^*(\theta)).
\end{aligned}$$

If this quantity is strictly greater than $\frac{1}{2}$, then a simple application of the triangular law of large numbers would imply that issue 2 will pass whenever a voter is pivotal for issue 1, that is, that issue 2 is conditionally certain to pass. However, a straightforward application is erroneous, because the other votes, while unconditionally independent, are correlated when a voter is pivotal. The other votes on issue 1 are drawn without replacement from an urn that contains equal numbers of yes and no votes. If supporting issue 1 and issue 2 are correlated in the equilibrium strategy, this then induces correlation across the votes on issue 2.

To handle the statistical dependence, we divide the set of voters into two subsamples: the first consists of those who supported issue 1; the second consists of those who voted against issue 1. Within each subsample, the ballots are independent, because the dependence was introduced through the split count on issue 1. Since we assume every voter in the first subsample supported issue 1, this dependence is removed. We can then apply the law of large numbers to the first subsample, which is a triangular sequence of binary variables with success probability approaching $\mathbf{P}(2 \in s^*(\theta)|1 \in s^*(\theta))$. In particular, the number of votes for issue 2, after dividing for the number $\frac{I-1}{2}$ of voters in the subsample, will approach the success probability. Similarly, the normalized vote count for issue 2 in the second subsample of voters that voted against issue 1 approaches $\mathbf{P}(2 \in s^*(\theta)|1 \notin s^*(\theta))$. Then the fraction of support for issue 2 in the entire sample is the average of these two numbers, namely the quantity highlighted above. Thus, through a more involved argument, the desired implication is indeed true.

Given the characterization of best response through ordinal conditions, some algebra delivers the characterization in Lemma 1, relating conditional certainty to the primitive distribution μ over types. In particular, the preference for $\{1, 2\}$ in relation to its neighboring bundles $\{1\}$ and $\{2\}$ determines whether both issues are conditionally certain to pass.

LEMMA 1: *If*

$$(\dagger) \qquad \frac{\mu[\theta_{12} \geq \max\{\theta_1, \theta_2\}]}{\mu[\theta_{12} \leq \min\{\theta_1, \theta_2\}]} > \max\left\{\frac{\mu[\theta_1 \geq \theta_{12} \geq \theta_2]}{\mu[\theta_2 \geq \theta_{12} \geq \theta_1]}, \frac{\mu[\theta_2 \geq \theta_{12} \geq \theta_1]}{\mu[\theta_1 \geq \theta_{12} \geq \theta_1]}\right\},$$

then there exists a sequence of equilibria $s_I^* \to s^*$ *such that* $\{1, 2\}$ *is conditionally certain. Moreover, if there exists a sequence of equilibria* $s_I^* \to s^*$ *such that* $\{1, 2\}$ *is conditionally certain, then* (†) *holds weakly.*[13]

The necessary weak inequality is violated by the construction of $\mathcal{C}$: the fraction $\frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}$ is large because the denominator $\mu(\theta_1 \geq \theta_{12} \geq \theta_2)$ is less than ε. This means that both issues cannot be conditionally certain to pass. Considering the symmetry of the considered distributions, no outcome can be conditionally certain, so at least one issue must be conditionally uncertain. We then argue that conditional uncertainty on one issue implies conditional uncertainty on the other. We do this by contradiction. As one case, assume issue 1 is conditionally uncertain while issue 2 is conditionally certain to pass. We can parameterize the equilibrium decision to support issue 2 by some probability $\alpha \in (0, 1)$. In particular, in large elections a type θ will support issue 2 if and only if

$$\alpha\theta_{12} + (1 - \alpha)\theta_2 \geq \alpha\theta_1 + (1 - \alpha)\theta_\emptyset.$$

This identifies the sets of types who support and oppose issue 2 in large elections, and the conditional certainty of issue 2 to pass imposes an inequality on the size of these sets that is parameterized by α. The construction of the set $\mathcal{C}$ provides additional inequalities; for example, the probability that $\theta_{12} \geq \theta_2$ is at most $\frac{1}{4} + \varepsilon$. These inequalities turn out to be mutually inconsistent, regardless of the selection of α. The other cases can be argued similarly to also lead to contradictions. Therefore, conditional uncertainty on one issue must be accompanied by conditional uncertainty on the other.

This proves that there must be *conditional* uncertainty on either issue when a voter is pivotal on the other. But the relevant empirical uncertainty regards the *unconditional* uncertainty of the vote count. The final step in the proof links the two uncertainties: in particular, conditional uncertainty on both issues is equivalent to unconditional uncertainty on both issues. To begin proving this equivalence, we reapply the conditioning argument and split the voters into two subsamples to work around the conditional dependence of the ballots and allow application of the central limit theorem to the conditional distribution of the vote count on issue 1. Thus, the conditional vote count on issue 1 can be approximated by a normal cumulative distribution function. For there to be conditional uncertainty on issue 1, the conditional probability that a random voter supports issue 1 when there is a split on issue 2 must converge to $\frac{1}{2}$ at a rate faster than $\sqrt{I - 1}$:

$$\sqrt{I-1}\left|\frac{1}{2}\mu(1 \in s_I^*(\theta)|2 \in s_I^*(\theta)) + \frac{1}{2}\mu(1 \in s_I^*(\theta)|2 \notin s_I^*(\theta)) - \frac{1}{2}\right| < \infty.$$

[13]Consideration of the bundle $\{1, 2\}$ is without loss of generality, since any bundle can be expressed as $\{1, 2\}$ through appropriately reinterpreting up and down.

Otherwise the distribution function will collapse too quickly and will be degenerately 0 or 1 at $\frac{1}{2}$. A similar rate of convergence must hold for the conditional probability of supporting issue 2 given a split on issue 1.

To move from the conditional probabilities to the unconditional probabilities, observe that the unconditional probability of a voter supporting issue 1 can be written as a convex combination of the two conditional probabilities given her vote, either up or down, on issue 2. This can also be viewed as a linear equation. A similar linear equation can be written for the unconditional probability of a voter supporting issue 2 as a function of conditional probabilities. Jointly, the pair defines a system of two linear equations with two unknowns, namely the unconditional probabilities. The coefficients of the system are given by the conditional probabilities. The resulting solutions for the unconditional probabilities imply that root convergence to $\frac{1}{2}$ for both conditional probabilities is equivalent to root convergence to $\frac{1}{2}$ for both unconditional probabilities. In fact, conditional uncertainty on both issues is the only case where this conversion is possible, because this guarantees that the coefficients of the linear system are finite.

In general, characterizing unconditional uncertainty and limit outcomes faces two obstacles. First, a full characterization of the relevant limits would involve not only deciding whether there is convergence of equilibrium probabilities, but also controlling the *rate* of this convergence. Moreover, while is is relatively easier to control the *conditional* uncertainty of an issue, we are ultimately interested in the *unconditional* uncertainty. There is a tight connection between these concepts when both issues are uncertain, but this connection is lost in all other cases. For example, we cannot rule out the possibility that both issues are unconditionally certain while a single issue is conditionally uncertain.

In the example, the majority preference ranking exhibits two key features. First, a Condorcet cycle exists. Second, the majority preference is not ordinally separable. A majority prefers issue 2 to pass if issue 1 were to pass ($\mu(\theta_{12} \geq \theta_1) > \frac{1}{2}$), but also prefer issue 2 to fail if issue 1 were to fail ($\mu(\theta_\emptyset \geq \theta_2) > \frac{1}{2}$). We explore the extent to which transitivity and separability of the majority preference ensure predictability in Section 5.

5. CONDORCET ORDERS AND COMBINATORIAL RULE

In this section, we examine the Condorcet consistency of majority rule with two issues. We make two remarks at the outset. Since we assume independent private values, any Vickrey–Clarke–Groves mechanism will implement the utilitarian outcome in dominant strategies. However, the communication demands and the implied transfers of such mechanisms are often impracticable. Considering mechanisms without transfers (such as in combinatorial or plurality rule), implementing the Condorcet winner, when it exists, is a standard objective.

Second, a distinguishing feature of combinatorial rule is its dependence on the structure of the power set of bundles. This manifests itself in the earlier characterizations, which all appeal exclusively to the relationship between a bundle and its neighboring bundles (those that differ on exactly one issue). Consequently, the more useful concept is not whether $\{1,2\}$ is the Condorcet winner, but whether $\{1,2\}$ is preferred by a majority to both $\{1\}$ and $\{2\}$. For example, when Lemma 1 characterizes whether $\{1,2\}$ is conditionally certain to pass, the value of the empty set $\theta_\emptyset$ is unmentioned. Again, this is because the probability of being pivotal for both issues simultaneously vanishes at a much faster rate than the probability of being pivotal for either issue alone.

We now define the Condorcet order, or majority rule preference. The maximal and minimal bundles of this order are the Condorcet winner and loser. In addition, in combinatorial voting, local comparisons with neighboring bundles are particularly important. So we also define a local Condorcet winner as a bundle which is preferred by a majority to its neighbors; an analogous notion defines a local Condorcet loser.

DEFINITION 5: The *Condorcet order* $\succ_C$ on $\mathcal{X}$ is defined by $A \succ_C B$ if $\mu(\theta_A \geq \theta_B) > \frac{1}{2}$.[14] The bundle $A \in \mathcal{X}$ is a *Condorcet winner* $A \succ_C B$ for all $B \neq A$. It is a *local Condorcet winner* if $A \succ_C B$ whenever $B \neq A, X \setminus A$. The bundle $A \in \mathcal{X}$ is a *Condorcet loser* if $B \succ_C A$ for all $B \neq A$. It is a *local Condorcet loser* if $B \succ_C A$ whenever $B \neq A, X \setminus A$.

A minimal criterion for ordinal efficiency is that when a Condorcet loser exists, it is not the outcome of the election. When a Condorcet loser exists, it cannot be a limit outcome of plurality rule for any sequence of equilibria. For combinatorial voting, the Condorcet loser is generically not a limit outcome.

PROPOSITION 5: *If A is a local Condorcet loser and $\succ_C$ is complete, then A is not a limit outcome.*

Proposition 5 proves that a Condorcet loser cannot be the determinate outcome of the election. However, it remains open whether a Condorcet loser can have strictly positive probability of being enacted; Proposition 5 only proves that this probability is strictly less than 1.

The Condorcet criterion for ordinal efficiency is that an electoral mechanism implements the Condorcet winner whenever such a winner exists. Under plurality rule, there exists some limit equilibrium of plurality rule that selects the Condorcet winner. However, plurality rule also yields other equilibria which fail to pass the Condorcet winner, for example, when the Condorcet winner is

[14]The inequality $\theta_A \geq \theta_B$ is weak to maintain notational consistency with the rest of the paper. Given the density assumption on μ, it is equivalent to the strict version, that is, $A \succ_C B$ if and only if $\mu(\theta_A > \theta_B) > \frac{1}{2}$.

not one of the two active candidates. This logic for inefficiency similarly riddles combinatorial rule, where some limit equilibria can miscoordinate on the wrong outcome. But more troublingly, combinatorial voting can also fail to yield *any* limit equilibrium that selects the Condorcet winner. So combinatorial rule generates other factors beyond miscoordination which can preclude ordinal efficiency.

We now provide conditions that suffice for implementation of the Condorcet winner, and hence for predictability. These results suggest that a quasiseparable and transitive majority ranking might imply predictability, but the general conjecture remains open. The difficulties in proving the general result are similar to those in completely characterizing limit outcomes in this model: the need to control rates of convergence and the need to pass these rates from conditional to unconditional probabilities.

The first result provides sufficient inequalities on the type distribution to implement the Condorcet winner.

PROPOSITION 6: *If* $\{1, 2\}$ *is a local Condorcet winner and*

$$\mu(\theta_{12} \geq \theta_1) > \frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2) + \mu(\theta_2 \geq \theta_{12} \geq \theta_1)},$$

$$\mu(\theta_{12} \geq \theta_2) > \frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2) + \mu(\theta_2 \geq \theta_{12} \geq \theta_1)},$$

then $\{1, 2\}$ *is a limit outcome*.

To interpret the inequalities, consider the quantity on either side. On the left hand side is the statistical electoral advantage that $\{1, 2\}$ enjoys against its neighbor $\{1\}$ or $\{2\}$. The right hand side is a ratio which measures, conditional on $\{1, 2\}$ being between its neighbors, the likelihood that the most preferred bundle among them is the opposite neighbor. This ratio is close to 1 if, for example, the likelihood that $\theta_2 \geq \theta_{12} \geq \theta_1$ is much larger than $\theta_2 \geq \theta_{12} \geq \theta_1$. This reflects a local asymmetry across the bundle $\{1, 2\}$ and its neighbors. So two factors will make the sufficient inequalities more likely to carry:

(i) Local electoral advantage, that is, a large proportion of the population favors the bundle $\{1, 2\}$ to either of its neighbors (this increases the quantities on the left hand sides).

(ii) Local symmetry, that is, the distribution of rankings treats the two neighbors as nearly identical (this decreases the quantities on the right hand sides).

Proposition 6 is actually a special case of the following more generally useful result.

LEMMA 2: *Suppose* $\{1, 2\}$ *is a local Condorcet winner*. *If issue* 1 (*or issue* 2) *is conditionally certain to pass*, *then* $\{1, 2\}$ *is a limit outcome*.

Contrapositively, Lemma 2 implies that for the Condorcet winner to fail, it must be the case that *both* issues conditionally disagree with the winner. Proposition 6 follows from this implication because the inequalities suffice for both issues to be conditionally certain to pass. This is stronger than required: having either issue conditionally certain to pass would ensure Condorcet consistency.

Recall that Example 1, which is an open set of distributions with unpredictable election outcomes, had two features. The first is the lack of a Condorcet winner. The second is that the Condorcet order was not separable; a majority preferred issue 2 to pass if issue 1 were to pass, but also preferred issue 2 if issue 1 were to fail. We now examine whether excluding these two pathologies, cyclicity and nonseparability of the Condorcet order, generates predictability.[15] We first introduce an ordinal definition of separability, since the cardinal notion of additive separability is not sensible for the majority ranking.

DEFINITION 6: A binary relation $\succ$ on $\mathcal{X}$ is *quasiseparable* if

$$A \succ A \cap B \quad \Longleftrightarrow \quad A \cup B \succ B.$$[16]

Quasiseparability means that the preference for issue 1 to pass is independent of the resolution of issue 2 and vice versa. When $\succ_C$ is a quasiseparable weak order, conditional certainty of any bundle, even a bundle that is not the Condorcet winner, is sufficient for the Condorcet winner to be the outcome of the election.

PROPOSITION 7: *Suppose $\succ_C$ is quasiseparable and there is conditional certainty on both issues. If $\{1, 2\}$ is the Condorcet winner, then both issues are conditionally certain to pass and $\{1, 2\}$ is a limit outcome.*[17]

While conditional certainty is not a statement on primitives, the hypothesis of Proposition 7 can be converted to primitive statements through the characterization in Lemma 1.

There are two observations in proving Proposition 7. This first observation is that if the Condorcet ranking is quasiseparable and $\{1, 2\}$ is the Condorcet winner, then its complement $\emptyset$ must be the Condorcet loser. The second observation is that a Condorcet loser cannot be conditionally certain. Then since

[15]An open question is whether the existence of a Condorcet winner, by itself, implies predictability. Note that this is different than the question of Condorcet consistency; in the introductory example, the Condorcet winner $\{1, 2\}$ fails but the outcome of $\emptyset$ is still predictable.

[16]This is equivalent to the relation being quasi-submodular and quasi-supermodular in the set containment order.

[17]We cannot conclude that the Condorcet winner is the unique limit outcome because there could be other limit equilibria without conditional certainty.

both issues cannot be conditionally certain to fail, one issue must be conditionally certain to pass. However, in the proof of Proposition 7, this was shown to be enough to insure that $\{1,2\}$ is conditionally certain whenever it is a Condorcet winner.

What is particularly appealing about the equilibria implied in Proposition 7 is that they require little strategic sophistication by the voters. This is because the conditional and unconditional outcomes of the issues are equivalent. So if a voter submits her optimal ballot assuming that the residual issue will pass or fail according to sincere poll data, then she will be submitting her equilibrium ballot.

One condition that pairs with quasiseparability to ensure Condorcet consistency is common knowledge of supermodularity. So if all voters agree that the issues are complements, then a quasiseparable $\succ_C$ suffices to make the Condorcet winner a limit outcome of the election. In this case, we can also conclude that the Condorcet winner is the unique outcome of the election, across all limit equilibria.

PROPOSITION 8: *Suppose $\succ_C$ is quasiseparable and the support of μ is the set of supermodular (or submodular) types. If A is a Condorcet winner, then A is the unique limit outcome.*

Consider the supermodular case. If either $\{1\}$ or $\{2\}$ is the Condorcet winner, supermodularity by itself, without quasiseparability, ensures Condorcet consistency. If $\{1,2\}$ is the Condorcet winner, then quasiseparability of the majority ranking implies that its complement $\emptyset$ is the Condorcet loser. So $\{1\} \succ_C \emptyset$, that is, $\mu(\theta_1 \geq \theta_\emptyset) > \frac{1}{2}$. Under supermodularity, we prove that this inequality implies that issue 1 is (unconditionally) certain to pass. A symmetric argument proves that issue 2 is also certain to pass. Thus $\{1,2\}$ is the limit outcome of the election. An analogous argument applies when $\emptyset$ is the winner.

One attractive consequence of supermodularity is the uniqueness of the Condorcet winner as the outcome of the election. When the majority ranking is quasiseparable and issues are complementary, combinatorial voting avoids the coordination problems that are endemic to plurality rule.

6. CONCLUSION

This paper introduced and analyzed a model of elections with nonseparable preferences over multiple issues. We provided topological and lattice-theoretic proofs for the existence of equilibrium. Predictable outcomes are not a generic feature of large elections. While the Condorcet winner is not generally the outcome of the multi-issue elections, we provided sufficient conditions for its implementation. We conclude by posing some open questions.

Multi-issue elections induce a political exposure problem that is analogous to the exposure problem in multiunit auctions. The political exposure problem was at the core of our strategic analysis. We assumed that voters understand this exposure and that they correctly condition the exposure on being pivotal for some issue. Alternative assumptions regarding the strategic sophistication of voters will generate different predictions.

An unsettled question is whether the existence of a Condorcet winner implies predictability. We were not able to construct a distribution with both a Condorcet winner and unpredictability; neither were we able to prove that the existence of a Condorcet winner guarantees predictability. Relatedly, does quasiseparability of the majority ranking imply Condorcet consistency? We reported several results that suggest a positive answer, but were unable to provide a general proof.

Our results regarding predictability and ordinal efficiency were restricted to the setting with two issues. While two issues were enough to construct the negative counterexamples, it remains open whether our positive results have analogs with three or more issues. We suspect that there is an extension of our methods—comparing bundles that are connected in the set-containment lattice—to more general settings.

In conjunction with existing results on plurality rule, our findings provide an initial comparison of combinatorial voting and plurality rule. Plurality rule always has an equilibrium which selects a Condorcet winner, but the multiplicity of equilibria introduces a coordination problem. Like plurality rule, combinatorial voting can also yield multiple equilibria. More distinctively, however, combinatorial voting can fail to have any equilibria which implement the Condorcet winner. On the other hand, in some cases the Condorcet winner is the unique limit outcome across equilibria of combinatorial rule, eliminating the coordination problem altogether. Our general understanding of the comparison is still incomplete.

The worst-case distributions for combinatorial voting can be very inefficient. Why is the institution so pervasive despite its theoretical inefficiency? We took the set of issues on the ballot as exogenous. In reality, the set of referendums or initiatives is a consequence of strategic decisions by political agents. For example, a new substitute measure can be introduced to siphon votes away from an existing measure or two complementary policies can be bundled as a single referendum. The ubiquity of combinatorial voting might be due to the considered introduction or bundling of issues. If agents anticipate the electoral consequences of their decisions, our model provides a first step in the analysis of strategic ballot design.

APPENDIX

A.1. *Proof of Proposition 2*

Note that, contrary to the order of presentation in the main text, we prove Proposition 1 immediately after the proof of Proposition 2.

We first verify that every undominated strategy assigns an open set of types to each ballot. Additive separability is too strong for this purpose because the set of additively separable types is Lebesgue null. This motivates the following weaker notion of separability:

DEFINITION 7: The type θ is *quasiseparable* if

$$\theta(A) \geq [>]\theta(A \cap B) \quad \Longleftrightarrow \quad \theta(A \cup B) \geq [>]\theta(B).$$

When θ is quasiseparable, the voter's preference for whether any issue x is voted up or down is invariant to which of the other issues in $A \setminus \{x\}$ pass or fail. The following observation was also made by Lacy and Niou (2000, Result 4).

LEMMA 3: *Suppose θ is quasiseparable and $A^*(\theta) = \arg\max_{A \in \mathcal{X}} \theta(A)$ is unique. Then $s(\theta) = A^*(\theta)$ whenever s is weakly undominated.*

PROOF: Suppose θ is quasiseparable and $A^* = A^*(\theta)$ is unique. We first prove that if $x \in A^*$, then s is weakly dominated whenever $x \notin s(\theta)$. Since $\theta(A^*) > \theta(A^* \setminus \{x\})$, we have $\theta(\{x\}) > \theta(\emptyset)$. Consider any strategy s where $x \notin s(\theta)$. Compare this to the strategy $s'(\theta) = s(\theta) \cup \{x\}$ for θ and equal to s for all other types. Now, for any fixed ballot profile A_{-i} for the other voters, either i is pivotal for issue x or is not. If not, then the same set of issues passes under both strategies, so there is no loss of utility to θ. If she is pivotal on x, then the set of issues $F(s(\theta), A) \cup \{x\}$ passes, which leaves her strictly better off by quasiseparability. So s is weakly dominated.

Similarly, if $x \notin A^*$, then s is weakly dominated whenever $x \in s(\theta)$. Therefore, $A^* = s(\theta)$ for every weakly undominated strategy s. *Q.E.D.*

We now begin the proof of existence. We endow each voter's strategy space S_i with the topology induced by the distance $d(s_i, s_i') = \mu_i(\{\theta_i : s_i(\theta_i) \neq s_i'(\theta_i)\})$ and endow the space of strategy profiles S with the product topology.[18] For a fixed strategy profile s, let the function $G^{s_{-i}} = (G_0^{s_{-i}}, G_+^{s_{-i}}) : \Theta_{-i} \to \mathcal{X} \times \mathcal{X}$ be defined by

$$G_0^{s_{-i}}(\theta_{-i}) = \left\{x \in X : \#\{j \neq i : x \in s_j(\theta_j)\} = \frac{I-1}{2}\right\},$$

that is, the set of issues where voter i is pivotal, and

$$G_+^{s_{-i}}(\theta_{-i}) = \left\{x \in X : \#\{j \neq i : x \in s_j(\theta_j)\} > \frac{I-1}{2}\right\},$$

[18]To be precise, this is defined over equivalence classes of strategies whose differences are μ_i null.

that is, the set of issues which pass irrespective of voter i's ballot. Then, for type θ_i, her utility for a fixed ballot profile $(A_1, \dots, A_I)$ is $\theta_i([A_i \cap G_0^{A_{-i}}] \cup G_+^{A_{-i}})$, that is, the union of two sets: first, the set of issues where she is pivotal and she votes up; second, the set of issues which are passed irrespective of her ballot. Let $\mathcal{D} \subseteq \mathcal{X} \times \mathcal{X}$ denote the space of ordered disjoint pairs of sets of issues, $\mathcal{D} = \{(C, D) \in \mathcal{X} \times \mathcal{X} : C \cap D = \emptyset\}$. For a type θ_i, her expected utility for a strategy profile s is

$$\sum_{(C,D)\in\mathcal{D}} \theta_i([s_i \cap C] \cup D) \times \mu_{-i}([G^{s_{-i}}]^{-1}(C, D)).$$

An important observation is that the type's expected utility for a ballot depends only on her belief about for which issues she will be pivotal and which issues will pass irrespective of her ballot. Let $\Delta\mathcal{D}$ denote the probability distributions over $\mathcal{D}$. For $P, P' \in \Delta\mathcal{D}$, define the sup metric $\|P - P'\| = \max_{(C,D)\in\mathcal{D}} |P(C, D) - P'(C, D)|$.

Define the probability $\pi_i(s) \in \Delta\mathcal{D}$ by

$$[\pi_i(s)](C, D) = \mu_{-i}([G^{s_{-i}}]^{-1}(C, D)). \tag{2}$$

In words, $[\pi_i(s)](C, D)$ is the probability, from voter i's perspective, that she will be pivotal on the issues in C and that the issues in D will pass no matter how she votes, given that the strategy s is being played by the other voters. Fix $(C, D) \in \mathcal{D}$. If s_j is weakly undominated, by Lemma 3 there exists some quasiseparable type θ_j for whom $s_j(\theta_j) = D$. Moreover, these conditions are satisfied in an open neighborhood U^D of θ_i. By the full support assumption, there is some strictly positive probability $\mu_j(U^D) > 0$ of a type for j with $s_j(U^D) = D$ and, similarly, $\mu_j^{C\cup D}$ of a set of types $U^{C\cup D}$ for which $s_j(U^{C\cup D}) = C \cup D$. Enumerate $I \setminus \{i\} = \{j_1, \dots, j_{I-1}\}$. By independence of μ across voters, for any weakly undominated strategy profile, the joint probability that D is submitted for the first $\frac{I-1}{2}$ other voters and $C \cup D$ is submitted by the second $\frac{I-1}{2}$ other voters is at least

$$L_i(C, D) = \prod_{k=1}^{(I-1)/2} \mu_{j_k}^D \times \prod_{k=(I+1)/2}^{I-1} \mu_{j_k}^{C\cup D} > 0.$$

Thus $[\pi_i(s)](C, D) \geq L_i(C, D)$ for all $(C, D) \in \mathcal{D}$, whenever s is weakly undominated. Let

$$L = \min\{L_i(C, D) : i \in I, (C, D) \in \mathcal{D}\}$$

and define the compact convex subset of probabilities

$$\Delta^U = \Big\{P \in \Delta\mathcal{D} : \min_{(C,D)\in\mathcal{D}} P(C, D) \geq L\Big\}.$$

Letting S^U denote the space of weakly undominated strategy profiles, we can consider the function $\pi_i : S^U \to \Delta^U$. By independence of μ,

$$\begin{aligned}[\pi_i(s)](C,D) &= \mu_{-i}([G^{s_{-i}}]^{-1}(C,D)) \\ &= \sum_{\{A_{-i}\in\mathcal{X}^{I-1} : G_i^{A_{-i}}=(C,D)\}} \mu_{-i}(\{\theta_{-i} : s_{-i}(\theta_{-i}) = A_{-i}\}) \\ &= \sum_{\{A_{-i}\in\mathcal{X}^{I-1} : G_i^{A_{-i}}=(C,D)\}} \left[\prod_{j\neq i} \mu_j\big(\{\theta_j : s_j(\theta_j) = [A_{-i}]_j\}\big)\right].\end{aligned}$$

The last expression is a sum of products of probabilities $\mu_j(\{\theta_j : s_j(\theta_j) = [A_{-i}]_j\})$ which, considered as functions dependent on S^U, are immediately continuous in the defined topology on S^U. Hence π_i is continuous. Then the function $\pi : S^U \to [\Delta^U]^I$ defined by $\pi(s) = (\pi_1(s), \dots, \pi_I(s))$ is continuous.

Fix a belief $P_i \in \Delta^U$. Then the set of types for voter i for which it is optimal to submit the ballot A_i is defined by

$$A_i(P_i) = \bigcap_{A'_i\in\mathcal{X}} \left\{\theta_i : \sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i(C,D) \geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P_i(C,D)\right\}.$$

Fix an enumeration $\mathcal{X} = \{A^1, \dots, A^{|\mathcal{X}|}\}$ and define the function $\sigma_i : \Delta\mathcal{D} \to S$ as follows. Let A^0 denote the set of types which are not quasiseparable or do not have a unique $\arg\max_{A\in\mathcal{X}} \theta_i(A)$. For all $\theta_i \in A^k(P_i) \setminus [A^0 \cup \cdots \cup A^{k-1}]$, let $[\sigma_i(P_i)](\theta_i) = A^k$.[19] Since P_i is in the interior of $\Delta\mathcal{D}$, $\sigma_i(P_i)$ is not weakly dominated: $\sigma_i(P_i) \in S_i^U$. Observe that the set of types θ_i which play A_i is a convex polytope (with open and closed faces).

We now prove that $\sigma_i : \Delta^U \to S_i^U$ is continuous. Since $P_i \in \Delta^U$ is strictly bounded away from zero, the set of types which have multiple optimal ballots given belief P_i is of strictly lower dimension than Θ_i, hence is μ_i null since μ_i admits a density. Then $\mu_i(A_i(P_i) \setminus [\sigma(P_i)]^{-1}(A_i)) = 0$, so it suffices to show that $\mu_i(A_i(P_i))$ is continuous in P_i. Fix $\varepsilon > 0$. The set $A_i(P_i)$ is nonempty, since there exists a nonempty neighborhood of quasiseparable types which submit A_i in any undominated strategy. By outer regularity of μ_i, the probability of the closed set $A_i(P_i)$ is arbitrarily well approximated by the probabilities of its neighborhoods (cf. Parthasarathy (1967, Theorem 1.2)), that is,

[19] This construction is to avoid ambiguous assignments on the μ_i-null set of types with multiple optimal ballots given P_i. Alternatively, one can consider the space S modulo differences of μ measure zero, in which case the ambiguous assignment is irrelevant.

there exists some δ neighborhood of $A_i(P_i)$, denoted $U_\delta[A_i(P_i)]$, such that $\mu_i(U_\delta[A_i(P_i)]) < \mu_i(A_i(P_i)) + \varepsilon$. Moreover, there exists a sufficiently small $\gamma > 0$ such that if, for all $A_i' \in \mathcal{X}$,

$$\begin{aligned}\sum_{\mathcal{D}} &\theta_i([A_i \cap C] \cup D) \times P_i(C, D) \\ &\geq \sum_{\mathcal{D}} \theta_i([A_i' \cap C] \cup D) \times P_i(C, D) - \gamma,\end{aligned}$$

then $\theta_i \in U_\delta[A_i(P_i)]$, because both sides of the inequality are continuous in θ_i. Suppose $\|P_i - P_i'\| < \gamma/2$. The difference in expected utility for any action across the two probabilities is bounded by $\gamma/2$, since values were normalized to live in the unit interval. Then fixing $\theta_i \in A_i(P_i')$, that is, a type θ_i for whom A_i is optimal given conjecture P_i', we have, for all $A_i' \in \mathcal{X}$,

$$\begin{aligned}\sum_{\mathcal{D}} &\theta_i([A_i \cap C] \cup D) \times P_i(C, D) \\ &\geq \sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i'(C, D) - \gamma/2 \\ &\geq \sum_{\mathcal{D}} \theta_i([A_i' \cap C] \cup D) \times P_i'(C, D) - \gamma/2 \\ &\geq \sum_{\mathcal{D}} \theta_i([A_i' \cap C] \cup D) \times P_i(C, D) - \gamma.\end{aligned}$$

So $A_i(P_i')$ is contained in $U_\delta[A_i(P_i)]$. Then $\mu_i(A_i(P_i') \setminus A_i(P_i)) \leq \mu_i(U_\delta[A_i(P_i)] \setminus A_i(P_i)) < \varepsilon$. Similarly, there exists a sufficiently small distance $\gamma' > 0$ such that if $|P_i - P_i'| < \gamma'$, then $\mu_i(A_i(P_i) \setminus A_i(P_i')) < \varepsilon$. But

$$\begin{aligned}\mu_i&\big(\{\theta_i : [\sigma_i(P_i)](\theta_i) \neq [\sigma_i(P_i')](\theta_i)\}\big) \\ &\leq \sum_{A_i \in \mathcal{X}} \mu_i(A_i(P_i) \Delta A_i(P_i')) \\ &= \sum_{A_i \in \mathcal{X}} \big(\mu_i(A_i(P_i') \setminus A_i(P_i)) + \mu_i(A_i(P_i) \setminus A_i(P_i'))\big) \\ &< 2|\mathcal{X}|\varepsilon\end{aligned}$$

whenever $\|P_i - P_i'\| < \min\{\gamma/2, \gamma'\}$. So the function $\sigma : [\Delta^U]^I \to S^U$ defined by $\sigma(P_1, \ldots, P_I) = (\sigma_1(P_1), \ldots, \sigma_I(P_I))$ is continuous.

Then the composition $\pi \circ \sigma : [\Delta^U]^I \to S^U \to [\Delta^U]^I$ is continuous and hence yields a fixed point $(P_1^*, \ldots, P_I^*)$ by Brower's theorem. Then $s^* = \sigma(P_1^*, \ldots, P_I^*)$ is, by construction, a best response to itself, hence the desired equilibrium in weakly undominated strategies.

A.2. *Proof of Proposition 1*

We maintain the notation from the proof of Proposition 2. Let s^0 denote the sincere voting profile, where $s_i^0(\theta_i) \in \arg\max_A \theta_i(A)$. By full support assumption, for all A and i, there is a strictly positive measure of types θ_i, where the sincere ballot is A. Hence the induced probability $P_i^0 = \pi_i(s^0)$ is in the interior of $\Delta\mathcal{D}$. Therefore, $P_i^0(C, D) > 0$. Consider θ_i with $\theta_i(\emptyset) = 1$ and $\theta_i(\{1\}) = 0$, and $\theta_i(A) = 1 - \delta$ whenever $A \neq \emptyset, \{1\}$. When $2 \in C$ and $D = \{1\}$,

$$\theta_i\big([\{2\} \cap C] \cup \{1\}\big) - \theta_i([\emptyset \cap C] \cup \{1\}) = \theta_i(\{1, 2\}) - \theta_i(\{1\}) = 1 - \delta.$$

For all other (C, D), the difference $\theta_i([\{2\} \cap C] \cup D) - \theta_i([\emptyset \cap C] \cup D)$ is either 0 or $-\delta$. Since P_i^0 has full support, we have

$$\sum_{\mathcal{D}} \theta_i\big([\{2\} \cap C] \cup D\big) \times P_i^0(C, D) > \sum_{\mathcal{D}} \theta_i([\emptyset \cap C] \cup D) \times P_i^0(C, D)$$

for sufficiently small $\delta > 0$. So submitting the ballot $\{2\}$ is a strictly better reply than the sincere ballot $\emptyset$ for this θ_i. This property is maintained in a neighborhood of θ_i, so by the full support assumption, sincere voting is not a best reply for a strictly positive measure of types.

A.3. *Proof of Proposition 3*

We use recent results due to Reny (2011). Namely, we verify the assumptions of Theorems 4.1 and 4.2, which we summarize in the following statement.[20]

THEOREM 1—Reny (2011): *Suppose that the following statements hold*:

G.1. *Θ is a complete separable metric space endowed with a measurable partial order*.

G.2. *μ assigns probability zero to any Borel subset of T_i having no strictly ordered points*.

G.3. *A is a compact locally complete metric semilattice*.

G.4. *$u(\cdot, \theta) : A \to \mathbb{R}$ is continuous for every $\theta \in \Theta$*.

Let C be a join-closed, piecewise-closed, and pointwise-limit-closed subset of pure strategies containing at least one monotone pure strategy, such that the intersection of C with i's set of monotone best replies is nonempty whenever every other player j employs a monotone pure strategy in C. Then there exists a symmetric monotone (*pure strategy*) *equilibrium in which each player i's pure strategy is in C*.

We first show that the election is weakly quasisupermodular and obeys single crossing in $\geq$, which will be useful later.

[20]We consider appropriately symmetrized statements of these results.

LEMMA 4: *The voting game is weakly quasisupermodular in actions*, *that is*,

$$\int_{\Theta_{-1}} \theta(A, s(\theta_{-1}))\, d\mu_{-1} \geq \int_{\Theta_{-1}} \theta(A \cap B, s(\theta_{-1}))\, d\mu_{-1}$$
$$\implies \int_{\Theta_{-1}} \theta(A \cup B, s(\theta_{-1}))\, d\mu_{-1} \geq \int_{\Theta_{-1}} \theta(A, s_{-1}(\theta_{-1}))\, d\mu_{-1}.$$

PROOF: We show that supermodularity *in outcomes* of the ex post utilities implies weak quasisupermodularity *in actions* of the interim utilities. So suppose the hypothesis inequality holds. Carrying the notation from the proof of Proposition 2, this can be rewritten as

$$\sum_{C,D \in \mathcal{D}} \theta([A \cap C] \cup D) \times [\pi(s)](C, D)$$
$$\geq \sum_{C,D \in \mathcal{D}} \theta([A \cap B \cap C] \cup D) \times [\pi(s)](C, D).$$

Applying supermodularity of θ to the sets $[A \cap C] \cup D$ and $[B \cap C] \cup D$, then

$$\sum_{C,D \in \mathcal{D}} \big[\theta([A \cap C] \cup D) - \theta([A \cap B \cap C] \cup D)\big] \times [\pi(s)](C, D) \geq 0$$

implies

$$\sum_{C,D \in \mathcal{D}} \big[\theta\big([(A \cup B) \cap C] \cup D\big) - \theta([A \cap C] \cup D)\big] \times [\pi(s)](C, D) \geq 0,$$

which can be rewritten as the desired conclusion. *Q.E.D.*

LEMMA 5: *The voting game satisfies weak single crossing in* $\geq$, *that is*, *if* $\theta' \geq \theta$ *and* $A' \supseteq A$, *then*

$$\int_{\Theta_{-1}} \theta\big(F(A', s(\theta_{-1}))\big)\, d\mu_{-1} \geq \int_{\Theta_{-1}} \theta\big(F(A, s(\theta_{-1}))\big)\, d\mu_{-i}$$
$$\implies \int_{\Theta_{-1}} \theta'\big(F(A', s(\theta_{-1}))\big)\, d\mu_{-1} \geq \int_{\Theta_{-1}} \theta'\big(F(A, s(\theta_{-1}))\big)\, d\mu_{-1}$$

for any profile s_{-i} *of monotone strategies by the other voters*.

PROOF: Suppose $\theta' \geq \theta$ and fix a monotone symmetric strategy profile s. Suppose $A' \supseteq A$. Then $F(A', s(\theta_{-1})) \supseteq F(A, s(\theta_{-i}))$ for any $\theta_{-i} \in \Theta_{-i}$. By construction of the partial order $\geq$,

$$\theta'\big(F(A', s(\theta_{-1}))\big) - \theta'\big(F(A, s(\theta_{-1}))\big) \\ \geq \theta\big(F(A', s(\theta_{-1}))\big) - \theta\big(F(A, s(\theta_{-1}))\big).$$

This inequality is preserved by integration:

$$\int_{\Theta_{-1}} \theta'\big(F(A', s(\theta_{-1}))\big)\, d\mu_{-1} - \int_{\Theta_{-1}} \theta'\big(F(A, s(\theta_{-1}))\big)\, d\mu_{-1} \\ \geq \int_{\Theta_{-1}} \theta\big(F(A', s(\theta_{-1}))\big)\, d\mu_{-1} - \int_{\Theta_{-1}} \theta\big(F(A, s(\theta_{-1}))\big)\, d\mu_{-1}.$$

Then if

$$\int_{\Theta_{-1}} \theta\big(F(A', s(\theta_{-1}))\big)\, d\mu_{-1} \geq \int_{\Theta_{-1}} \theta\big(F(A, s(\theta_{-1}))\big)\, d\mu_{-1},$$

the inequality implies

$$\int_{\Theta_{-1}} \theta'\big(F(A', s(\theta_{-1}))\big)\, d\mu_{-1} \geq \int_{\Theta_{-1}} \theta'\big(F(A, s(\theta_{-1}))\big)\, d\mu_{-1}.$$

Q.E.D.

We can now check the assumptions in Reny's theorem. The technical conditions G.1–G.4 are straightforward to verify. We restrict attention to a space of strategies which will induce weakly undominated best responses. Let R be the subset of strategies such that (μ almost surely (μ-a.s.)) if θ is quasiseparable and $A^*(\theta) = \arg\max_{A\in\mathcal{X}} \theta(A)$ is unique, then $s(\theta) = A^*(\theta)$. This space is join-closed, pointwise-limit-closed, and piecewise-closed because it is the intersection of two measurable order inequalities (cf. Reny (2011, Remark 4)). To see this, let

$$f(\theta) = \begin{cases} A^*(\theta), & \text{if } \theta \text{ is quasiseparable and} \\ & A^*(\theta) = \arg\max_{A\in\mathcal{X}} \theta(A) \text{ is unique,} \\ \emptyset, & \text{otherwise,} \end{cases}$$

and, similarly,

$$g(\theta) = \begin{cases} A^*(\theta), & \text{if } \theta \text{ is quasiseparable and} \\ & A^*(\theta) = \arg\max_{A\in\mathcal{X}} \theta(A) \text{ is unique} \\ X, & \text{otherwise.} \end{cases}$$

In addition, $R = \{s \in S : f(\theta) \subseteq s(\theta) \subseteq g(\theta), \mu\text{-a.s.}\}$.

We next show that there exists a monotone strategy in R. Define the strategy

$$s(\theta) = \bigcup_{\theta' \leq \theta} f(\theta').$$

By construction, s is monotone. Now suppose $\theta' \geq \theta$ are quasiseparable with respective unique maximizers $A^*(\theta')$ and $A^*(\theta')$. By repeated application of quasiseparability, we have

$$\theta(A^*(\theta) \cup A^*(\theta')) - \theta(A^*(\theta')) \geq 0.$$

Considering the definition of $\geq$,

$$\theta'(A^*(\theta') \cup A^*(\theta)) - \theta'(A^*(\theta')) \geq 0.$$

Since $A^*(\theta')$ is the unique maximizer of θ', this forces $A^*(\theta') \cup A^*(\theta) = A^*(\theta')$, that is, $A^*(\theta') \supseteq A^*(\theta)$. So if θ' is separable and has a unique maximizer $A^*(\theta')$, then $f(\theta') \supseteq f(\theta)$ for all $\theta \leq \theta'$. Hence $s(\theta') = A^*(\theta)$. So $s \in R$.

Finally, we prove that any monotone strategy in R will induce a monotone best reply in R. Since R is a superset of the weakly undominated strategies, clearly for any symmetric strategy profile s of the other voters, there is some element of R which is a best response. Moreover, any best reply to a strategy profile from R must be weakly undominated. This is because the quasiseparable types with unique maximizer A constitute a relatively open subset of the supermodular types, so every ballot has positive probability for each voter by the full support assumption. Standard lattice arguments show that weak quasisupermodularity and weak single crossing imply that the pointwise join of each type's best replies in weakly undominated strategies constitutes a monotone best reply itself; for example, see the proof of Corollary 4.3 in Reny (2011). Since R is join-closed, this monotone best reply lives in R. So there exists an equilibrium in strategies in R and by construction, this equilibrium must be in weakly undominated strategies.

A.4. *Asymptotic Results*

LEMMA 6: *Consider a sequence of equilibrium strategies $s_I^* \to s^*$. If issue 2 is conditionally certain to pass, then*

$$1 \in s^*(\theta) \iff \theta_{12} \geq \theta_2.$$ [21]

PROOF: Recall from the proof of Proposition 2 that for a fixed electorate size I, the expression (2) for $[\pi(s)](C, D)$ reflects the probability induced by strategy profile s that voter i will be pivotal on the issues in C and the issues in D will pass irrespective of her vote. For ease of notation, let $P_I^* = \pi(s_I^*)$ in the election with I voters.

[21] While this lemma is written for the two-issue case, a similar result holds for the general case: Let A be the issues on $X \setminus x$ that are conditionally certain to fail. Then $x \in s^*(\theta)$ if and only if $\theta_{A\cup\{x\}} \geq \theta_A$. The authors or an earlier working version of this paper can be consulted for details.

Consider a pair of ballots A and A' with $1 \in A$ and $1 \notin A'$. The incentive condition for A being a better reply than A' (for type θ) to s_I^* is

$$\sum_{C,D\in\mathcal{D}} P_I^*(C,D)\cdot\theta([A\cap C]\cup D) \geq \sum_{C,D\in\mathcal{D}} P_I^*(C,D)\cdot\theta([A'\cap C]\cup D). \tag{3}$$

However, if $1 \notin C$, we have $C\cap\{1\}\cup D = C\cap A'\cup D$, the only relevant components of the sums on both sides of this inequality are

$$\sum_{C,D\in\mathcal{D}:1\in C} P_I^*(C,D)\cdot\theta([A\cap C]\cup D) \geq \sum_{C,D\in\mathcal{D}:1\in C} P_I^*(C,D)\cdot\theta([A'\cap C]\cup D).$$

Dividing both sides by $\sum_{(C,D):1\in C} P_I^*(C,D) > 0$, we can replace the unconditional probabilities P_I^* with the conditional probabilities $P_I^*(C,D|1\in C)$:

$$\sum_{C,D\in\mathcal{D}:1\in C} P_I^*(C,D|1\in C)\cdot\theta([C\cap A]\cup D) \geq \sum_{C,D\in\mathcal{D}:1\in C} P_I^*(C,D|1\in C)\cdot\theta([C\cap A']\cup D). \tag{4}$$

We can rewrite the left hand side as

$$\begin{aligned}&\sum_{C,D\in\mathcal{D}:1\in C} P_I^*(C,D|1\in C)\cdot\theta([C\cap A]\cup D)\\ &\quad= P_I^*(\{1\},\{2\}|1\in C)\cdot\theta_{12}\\ &\qquad+\sum_{C,D:1\in C,D\neq\{2\}} P_I^*(C,D|1\in C)\cdot\theta([C\cap A]\cup D).\end{aligned}$$

Similarly rewriting the right hand side, the incentive inequality (4) can be rewritten as

$$\begin{aligned}&P_I^*(\{1\},\{2\}|1\in C)\cdot\theta_{12}+\sum_{C,D:1\in C,D\neq\{2\}} P_I^*(C,D|1\in C)\cdot\theta([C\cap A]\cup D)\\ &\quad\geq P_I^*(\{1\},\{2\}|1\in C)\cdot\theta_2\\ &\qquad+\sum_{C,D:1\in C,D\neq\{2\}} P_I^*(C,D|1\in C)\cdot\theta([C\cap A']\cup D).\end{aligned}$$

This is rearranged as

$$\theta_{12}\geq\theta_2+\Delta_I, \tag{5}$$

where

$$\Delta_I = \frac{\sum_{C,D:1\in C,D\neq\{2\}} P_I^*(C,D|1\in C)[\theta_i([C\cap A']\cup D)-\theta_i([C\cap A]\cup D)]}{P_I^*(\{1\},\{2\}|1\in C)}.$$

If $|\theta_{12}-\theta_2| > \Delta_I$ for all $x \in X$, then A is a better reply than A' if and only if $\theta_{12} \geq \theta_2$. But since the only assumption is that $1 \in A$ and $1 \notin A'$, this is also equivalent to 1 being included in the best reply. Observe that the set of types for which $|\theta_{12}-\theta_2| > \Delta_I$ for all $x \in X$ is of full Lebesgue measure at the limit, since $\Delta_I \to 0$. Invoking the density assumption, this set approaches probability 1 at the limit. *Q.E.D.*

LEMMA 7: *Let $x' \neq x$. There exists $\alpha_x \in [0,1]$ such that*

$$x \in s^*(\theta) \quad \Longleftrightarrow \quad \alpha_x\theta_{12} + (1-\alpha_x)\theta_x \geq \alpha_x\theta_{x'} + (1-\alpha_x)\theta_\emptyset.$$

PROOF: Without loss of generality, consider the case $x = 2$. Let

$$\alpha_I = \mathbf{P}\left(\#\{j\neq i : 1\in s_I^*(\theta_k)\} > \frac{I-1}{2} \middle| \#\{j\neq i : 2\in s_I^*(\theta_k)\} = \frac{I-1}{2}\right)$$

and

$$\beta_I = \mathbf{P}\left(\#\{j\neq i : 1\in s_I^*(\theta_k)\} < \frac{I-1}{2} \middle| \#\{j\neq i : 2\in s_I^*(\theta_k)\} = \frac{I-1}{2}\right).$$

By the full support assumption, the conditional probability of being pivotal on issue 1 when pivotal on issue 2 vanishes, so $\alpha_I + \beta_I \to 1$.

Fix a voter i with type θ and consider a ballot $A \subseteq \{1\}$ which does not include 2. The incentive condition for $\{2\} \cup A$ being a better reply than A to the strategy s_I^* is

$$\alpha_I\theta_{12} + \beta_I\theta_2 + (1-\alpha_I-\beta_I)\theta_{2\cup A} \geq \alpha_I\theta_1 + \beta_I\theta_\emptyset + (1-\alpha_I-\beta_I)\theta_A.$$

Passing to a subsequence if necessary, there exists an α such that $\alpha_I \to \alpha$. The incentive inequality can be rewritten as

$$\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset + \Delta_I,$$

where

$$\begin{aligned}\Delta_I = {} & [\alpha_I - \alpha](\theta_1-\theta_{12}) + [\beta_I - (1-\alpha)](\theta_\emptyset - \theta_2) \\ & + [1-\alpha_I-\beta_I](\theta_A - \theta_{2\cup A}).\end{aligned}$$

However, $\Delta_I \to 0$. From here, we can replicate the arguments that conclude the proof of Lemma 6 to conclude that, at the limit, the set of types which support issue 2 is characterized by the inequality

$$\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset. \qquad \textit{Q.E.D.}$$

LEMMA 8: *Let* $s_I^* \to s^*$. *If*

$$(\star) \qquad \mu(y \in s(\theta) | x \in s(\theta)) > (<)\, \mu(y \notin s(\theta) | x \notin s(\theta)),$$

then y *is conditionally certain to pass* (*fail*) *at* x. *Moreover*, *if* y *is conditionally certain to pass* (*fail*) *at* x, *then* ($\star$) *holds weakly*.

PROOF: Without loss of generality, suppose voter I is pivotal on issue 1 and consider whether issue 2 is conditionally certain to pass or fail. Let X_k^{Ii} ($I = 1, 3, \ldots$; $i = 1, \ldots, I-1$; $k = 1, 2$) denote the triangular array of indicator functions on the events $\{\theta_i : k \in s_I^*(\theta_i)\}$. While X_2^{Ii} are unconditionally rowwise independent, this independence is broken once we condition on voter I being pivotal on issue 1. This precludes a straightforward application of the law of large numbers to the array and the proof requires more delicacy.

The basic logic is to split the sample of $I-1$ other voters into two subsamples: those $\frac{I-1}{2}$ who voted for issue 1 and those $\frac{I-1}{2}$ who did not. Within each subsample, the votes on issue k are conditionally identical and independent because the votes on issue 1 are fixed. However, by exchangeability, the particular identity of voters in each subsample is irrelevant, so we can proceed without loss of generality by assuming the first half of other voters constitute the first subsample, while the remainder constitute the second.

Formally, consider the arrays of rowwise independent binary random variables

$$Y^{Ii} = \begin{cases} 1 & \text{with probability } \mu(2 \in s_I(\theta_i) | 1 \notin s_I(\theta_I)), \\ 0 & \text{with probability } \mu(2 \notin s_I(\theta_i) | 1 \notin s_I(\theta_I)) \end{cases}$$

and

$$Z^{Ii} = \begin{cases} 1 & \text{with probability } \mu(2 \in s_I(\theta_i) | 1 \in s_I(\theta_I)), \\ 0 & \text{with probability } \mu(2 \notin s_I(\theta_i) | 1 \in s_I(\theta_I)). \end{cases}$$

We first prove that the distribution of $\sum_{i=1}^{I-1} X_2^{Ii}$ conditional on $\sum_{i=1}^{I-1} X_1^{Ii} = \frac{I-1}{2}$ is identical to the distribution of the sum

$$\sum_{i=1}^{(I-1)/2} Y^{Ii} + \sum_{i=1}^{(I-1)/2} Z^{Ii}.$$

Suppressing the I superscript for the size of the electorate and fixing any integer n yields

$$\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i=1}^{I-1} X_1^i = \frac{I-1}{2}\right) \\
&\quad = \sum_{A\subset I-1:\#A=(I-1)/2}\left[\mathbf{P}\left(\sum_{i\in A} X_2^i = 0 \,\middle|\, \sum_{i=1}^{I-1} X_1^i = \frac{I-1}{2}\right)\right. \\
&\qquad \left.\times\, \mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i\in A} X_1^i = 0, \sum_{j\notin A} X_1^j = \frac{I-1}{2}\right)\right].
\end{aligned}$$

By exchangeability across voters, the particular identities of the voters in the set A who voted down on issue 1 is irrelevant. In other words, we can assume without loss of generality that the first $\frac{I-1}{2}$ other voters included 1 in their ballots and the last $\frac{I-1}{2}$ other voters excluded 1 from their ballots. The prior expression is, therefore, equal to

$$\begin{aligned}
&= \sum_{A\subset I-1:\#A=(I-1)/2}\left[\mathbf{P}\left(\sum_{i=1}^{(I-1)/2} X_2^i = 0 \,\middle|\, \sum_{i=1}^{I-1} X_1^i = \frac{I-1}{2}\right)\right. \\
&\qquad \left.\times\, \mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i=1}^{(I-1)/2} X_1^i = 0, \sum_{j=(I+1)/2}^{I-1} X_1^j = \frac{I-1}{2}\right)\right] \\
&= \mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i=1}^{(I-1)/2} X_1^i = 0, \sum_{j=(I+1)/2}^{I-1} X_1^j = \frac{I-1}{2}\right) \\
&= \sum_{m=0}^{(I-1)/2}\left[\mathbf{P}\left(\sum_{i=1}^{(I-1)/2} X_2^i = m \,\middle|\, \sum_{i=1}^{(I-1)/2} X_1^i = 0\right)\right. \\
&\qquad \left.\times\, \mathbf{P}\left(\sum_{j=(I+1)/2}^{I-1} X_2^j = n-m \,\middle|\, \sum_{j=(I+1)/2}^{I-1} X_1^j = \frac{I-1}{2}\right)\right] \\
&= \sum_{m=0}^{(I-1)/2}\left[\mathbf{P}\left(\sum_{i=1}^{(I-1)/2} Y^{Ii} = m\right)\mathbf{P}\left(\sum_{i=1}^{(I-1)/2} Z^{Ii} = n-m\right)\right] \\
&= \mathbf{P}\left(\sum_{i=1}^{(I-1)/2} Y^{Ii} + \sum_{i=1}^{(I-1)/2} Z^{Ii} = n\right).
\end{aligned}$$

We now show that the normalized sum $\frac{\sum_{i=1}^{I-1} X_2^{Ii}}{I-1}$ conditional on $\sum_{i=1}^{I-1} X_1^{Ii} = \frac{I-1}{2}$ converges in probability to $\frac{1}{2}\mu(2 \in s_I^*(\theta)|1 \in s_I^*(\theta)) + \frac{1}{2}\mu(2 \in s_I^*(\theta)|1 \notin s_I^*(\theta))$. We can apply the strong law of large numbers for triangular arrays to Y^{Ii} and Z^{Ii} to conclude that

$$\left(\frac{I-1}{2}\right)^{-1} \sum_{i=1}^{(I-1)/2} Y^{Ii} \to \mu(2 \in s_I^*(\theta_i)|1 \notin s_I^*(\theta_i))$$

and

$$\left(\frac{I-1}{2}\right)^{-1} \sum_{i=1}^{(I-1)/2} Z^{Ii} \to \mu(2 \in s_I^*(\theta_i)|1 \in s_I^*(\theta_i))$$

almost surely, hence in distribution. By the continuous mapping theorem, the sum

$$\frac{1}{2}\left(\frac{I-1}{2}\right)^{-1} \sum_{i=1}^{(I-1)/2} Y^{Ii} + \frac{1}{2}\left(\frac{I-1}{2}\right)^{-1} \sum_{i=1}^{(I-1)/2} Z^{Ii} \tag{6}$$

converges in distribution to the constant

$$\frac{1}{2}\mu(2 \in s_I^*(\theta)|1 \in s_I^*(\theta)) + \frac{1}{2}\mu(2 \in s_I^*(\theta)|1 \notin s_I^*(\theta)).$$

As show in the previous paragraph, since the conditional distribution of $\frac{\sum_{i=1}^{I-1} X_2^{Ii}}{I-1}$ shares the distribution of (6), it also converges in distribution to the same constant. As convergence in distribution to a constant implies convergence in probability, this delivers the desired conclusion.

To conclude the proof, suppose $\mu(2 \in s(\theta)|1 \in s(\theta)) > \mu(2 \notin s(\theta)|1 \notin s(\theta))$; the argument for the opposite strict inequality is symmetric. Then

$$\begin{aligned}
&\mu(2 \in s(\theta)|1 \in s(\theta)) > \mu(2 \notin s(\theta)|1 \notin s(\theta)),\\
&\mu(2 \in s(\theta)|1 \in s(\theta)) + 1 - \mu(2 \notin s(\theta)|1 \notin s(\theta)) > 1,\\
&\mu(2 \in s(\theta)|1 \in s(\theta)) + \mu(2 \in s(\theta)|1 \notin s(\theta)) > 1,\\
&\frac{1}{2}\mu(2 \in s(\theta)|1 \in s(\theta)) + \frac{1}{2}\mu(2 \in s(\theta)|1 \notin s(\theta)) > \frac{1}{2}.
\end{aligned}$$

Let $E = \frac{1}{2}\mu(2 \in s(\theta)|1 \in s(\theta)) + \frac{1}{2}\mu(2 \in s(\theta)|1 \notin s(\theta))$ and pick a strictly positive $\delta < E - \frac{1}{2}$. By the previous paragraph, the probability that the normalized vote count on issue 2, conditional on voter I being pivotal on 1, is greater than $E - \delta > \frac{1}{2}$ approaches 1. Thus, the conditional probability that 2 passes converges to 1.

The necessity of the weak inequality follows from the contraposition of the sufficiency claim, that is, if y is conditionally certain to pass, then it is not conditionally certain to fail. *Q.E.D.*

We now restate Lemma 1 from the text.

LEMMA 1: *If*

$$(\dagger) \qquad \frac{\mu[\theta_{12} \geq \max\{\theta_1, \theta_2\}]}{\mu[\theta_{12} \leq \min\{\theta_1, \theta_2\}]} > \max\left\{\frac{\mu[\theta_1 \geq \theta_{12} \geq \theta_2]}{\mu[\theta_2 \geq \theta_{12} \geq \theta_1]}, \frac{\mu[\theta_2 \geq \theta_{12} \geq \theta_1]}{\mu[\theta_1 \geq \theta_{12} \geq \theta_1]}\right\},$$

then there exists a sequence of equilibria $s_I^* \to s^*$ *such that* $\{1, 2\}$ *is conditionally certain. Moreover, if there exists a sequence of equilibria* $s_I^* \to s^*$ *such that* $\{1, 2\}$ *is conditionally certain, then* ($\dagger$) *holds weakly.*

PROOF: We first prove the sufficiency of the strict inequality. So let

$$\Theta_{12} = \{\theta : \theta_{12} \geq \max\{\theta_1, \theta_2\}\},$$
$$\Theta_1 = \{\theta : \theta_1 \geq \theta_{12} \geq \theta_2\},$$
$$\Theta_2 = \{\theta : \theta_2 \geq \theta_{12} \geq \theta_1\},$$
$$\Theta_\emptyset = \{\theta : \theta_{12} \leq \min\{\theta_1, \theta_2\}\}.$$

These four sets of types cover Θ. Since μ has full support and admits a density, these sets have strictly positive probability but null pairwise intersections. By assumption,

$$\frac{\mu(\Theta_{12})}{\mu(\Theta_\emptyset)} > \max\left\{\frac{\mu(\Theta_1)}{\mu(\Theta_2)}, \frac{\mu(\Theta_2)}{\mu(\Theta_1)}\right\}.$$

Let

$$\mathcal{P}_n = \left\{P \in \Delta^U : P(x, x'|x \in C) \geq 1 - \frac{1}{n}, \forall x = 1, 2\right\}.$$

Recall that $P(C, D)$ is the probability that an anonymous voter is pivotal on the issues in C and that the issues in D will pass irrespective of her ballot. Let $A \subset \{x'\}$ and consider $A' = \{x\} \cup A$. The incentive condition for A' being a better reply than A given the belief $P \in \mathcal{P}_n$ over pivotal and passing events is

$$\theta_{12} \geq \theta_x + \Delta_n(\theta),$$

where

$$\Delta_n(\theta) = \frac{P(x, \emptyset|x \in C)[\theta_\emptyset - \theta_x] + P(\{1, 2\}, \emptyset|x \in C)[\theta_A - \theta_{A'}]}{P(x, x'|x \in C)}.$$

Observe that $\Delta_n(\theta) \to 0$ as $n \to \infty$. Let $\Theta_n = \{\theta : |\theta_{12} - \theta_x| > \Delta_n\}$ and notice that $\mu(\Theta_n) \to 1$. As $n \to \infty$, the proportion of types in Θ_{12} which include 1 and 2 in their optimal ballots covers the entire subset Θ_{12}, while the proportion of types in $\Theta \setminus \Theta_{12}$ which include x becomes null. Similarly arguing for Θ_1, Θ_2, and Θ_0, we have

$$\sigma_I(P_n) \to s,$$

where $s_A(\theta_A) = A$ for all $\theta_A \in \Theta_A$, for any sequence of selections $P_n \in \mathcal{P}_n$. Then

$$\frac{\mu([\sigma_I(P)](\theta) = \{1,2\})}{\mu([\sigma_I(P)](\theta) = \emptyset)} - \max\left\{\frac{\mu([\sigma_I(P)](\theta) = \{1\})}{\mu([\sigma_I(P)](\theta) = \{2\})}, \frac{\mu([\sigma_I(P)](\theta) = \{2\})}{\mu([\sigma_I(P)](\theta) = \{1\})}\right\},$$

which is arbitrarily close to

$$\frac{\mu(\Theta_{12})}{\mu(\Theta_\emptyset)} - \max\left\{\frac{\mu(\Theta_1)}{\mu(\Theta_2)}, \frac{\mu(\Theta_2)}{\mu(\Theta_1)}\right\} > 0$$

for any $P \in \mathcal{P}_n$ as $n \to \infty$. So there exists some n_0 such that if $n > n_0$, then

$$\frac{\mu([\sigma_I(P)](\theta) = \{1,2\})}{\mu([\sigma_I(P)](\theta) = \emptyset)} > \max\left\{\frac{\mu([\sigma_I(P)](\theta) = \{1\})}{\mu([\sigma_I(P)](\theta) = \{2\})}, \frac{\mu([\sigma_I(P)](\theta) = \{2\})}{\mu([\sigma_I(P)](\theta) = \{1\})}\right\}$$

for all $P \in \mathcal{P}_n$.

So let $n > n_0$ and consider the sequence of strategies $s_I = \sigma_I(P)$ for any $P \in \mathcal{P}_n$. Fix I and let $\mu_A = \mu([\sigma_I(P)](\theta) = A)$. Then

$$\frac{\mu_{12}}{\mu_\emptyset} > \frac{\mu_1}{\mu_2},$$

$$\mu_{12}\mu_2 > \mu_\emptyset\mu_1,$$

$$\mu_{12}(\mu_2 + \mu_\emptyset) > \mu_\emptyset(\mu_1 + \mu_{12}),$$

$$\frac{\mu_{12}}{\mu_1 + \mu_{12}} > \frac{\mu_\emptyset}{\mu_2 + \mu_\emptyset},$$

$$\mu\big(2 \in [\sigma_I(P)](\theta) | 1 \in [\sigma_I(P)](\theta)\big) > \mu\big(2 \notin [\sigma_I(P)](\theta) | 1 \notin [\sigma_I(P)](\theta)\big).$$

A symmetric condition identifies when issue 2 is conditionally certain to pass. By Lemma 6 (see also the remark immediately following the proof), for I sufficiently large, the probability $\pi_I(\sigma_I)$ satisfies

$$\big[\pi_I(\sigma_I(P))\big](x, x'|x \in C) \geq 1 - \frac{1}{n}$$

for $x = 1, 2$. This means for sufficiently large $I > I_0$, the image $[\pi \circ \sigma_I](\mathcal{P}_n) \subseteq \mathcal{P}_n$. It therefore admits a fixed point $P_I^* \in \mathcal{P}_n$ which defines an equilibrium s_I^*, for all sufficiently large $I > I_0(n)$.

Finally, define $n(I) = \max\{n : I > I_0(n)\} \vee I$. Observe that as $I \to \infty$, we have $n(I) \to \infty$. For each I, select a fixed point $P_I^* \in [\pi \circ \sigma](\mathcal{P}_{n(I)})$. The induced equilibrium strategy s_I^* satisfies

$$\mu\big(\sigma_I^*(\theta)|1 \in [\sigma_I(P)](\theta)\big) > \mu\big(2 \notin [\sigma_I(P)](\theta)|1 \notin [\sigma_I(P)](\theta)\big).$$

By Lemma 6, the set $\{1, 2\}$ is conditionally certain. Also, recalling the construction, $s_I^* \to s^*$, where $s^*(\Theta_A) = A$.

We now prove the necessity of the weak inequality. Suppose each issue is conditionally certain to pass. In particular, 2 is conditionally certain to pass at 1. By Lemma 8,

$$\mu[2 \in s^*(\theta_i)|1 \in s^*(\theta_i)] \geq \mu[2 \notin s^*(\theta_i)|1 \notin s^*(\theta_i)].$$

For notational convenience, let $\mu_A^* = \mu(\{\theta_i : s^*(\theta_i) = A\})$. Since 2 is conditionally certain to pass at 1, we have

$$\begin{aligned}
&\frac{\mu_{12}^*}{\mu_{12}^* + \mu_1^*} \geq \frac{\mu_\emptyset^*}{\mu_\emptyset^* + \mu_2^*}, \\
&\mu_{12}^*(\mu_\emptyset^* + \mu_2^*) \geq \mu_\emptyset^*(\mu_{12}^* + \mu_1^*), \\
&\mu_{12}^*\mu_2^* \geq \mu_\emptyset^*\mu_1^*, \\
&\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_1^*}{\mu_2^*}.
\end{aligned}$$

Symmetrically, since 1 is conditionally certain to pass at 2,

$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_2^*}{\mu_1^*}.$$

The prior two inequalities imply

$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \max\left\{\frac{\mu_1^*}{\mu_2^*}, \frac{\mu_2^*}{\mu_1^*}\right\}. \tag{7}$$

By Lemma 6, we have

$$x \in s^*(\theta_i) \quad \Longleftrightarrow \quad \theta_i(\{1,2\}) \geq \theta(\{x'\}).$$

Thus

$$s^*(\theta) = \begin{cases} \{1,2\}, & \text{if } \theta_{12} \geq \max\{\theta_1, \theta_2\}, \\ \{1\}, & \text{if } \theta_1 \geq \theta_{12} \geq \theta_2, \\ \{2\}, & \text{if } \theta_2 \geq \theta_{12} \geq \theta_1, \\ \emptyset, & \text{if } \theta_{12} \leq \min\{\theta_1, \theta_2\}. \end{cases}$$

Substituting these cases into condition (7) delivers the result. *Q.E.D.*

LEMMA 9: *The following conditions are equivalent*:
(i) *Inequality* (†) *in Lemma* 1.
(ii) $\mu(\theta_{12} \geq \theta_1 | \theta_2 \geq \theta_{12}) > \mu(\theta_1 \geq \theta_{12} | \theta_{12} \geq \theta_2)$ *and* $\mu(\theta_{12} \geq \theta_2 | \theta_1 \geq \theta_{12}) > \mu(\theta_2 \geq \theta_{12} | \theta_{12} \geq \theta_1)$.
(iii) $\mu(\theta_{12} \geq \theta_1) > \mu(\theta_2 \geq \theta_{12} \geq \theta_1)/(\mu(\theta_1 \geq \theta_{12} \geq \theta_2) + \mu(\theta_1 \geq \theta_{12} \geq \theta_2))$ *and* $\mu(\theta_{12} \geq \theta_2) > \mu(\theta_1 \geq \theta_{12} \geq \theta_2)/(\mu(\theta_1 \geq \theta_{12} \geq \theta_2) + \mu(\theta_1 \geq \theta_{12} \geq \theta_2))$.
Moreover, *the weak versions of these conditions are equivalent*.

PROOF: We first prove that (i) and (ii) are equivalent. Observe that the equality

$$\mu(\theta_{12} \geq \max\{\theta_1, \theta_2\}) = \mu(\theta_{12} \geq \theta_2 \geq \theta_1) + \mu(\theta_{12} \geq \theta_1 \geq \theta_2)$$

holds. This can be rewritten as

$$\begin{aligned} &\mu(\theta_{12} \geq \max\{\theta_1, \theta_2\}) + \mu(\theta_2 \geq \theta_{12} \geq \theta_1) \\ &\quad = \mu(\theta_2 \geq \theta_{12} \geq \theta_1) + \mu(\theta_{12} \geq \theta_2 \geq \theta_1) + \mu(\theta_{12} \geq \theta_1 \geq \theta_2). \end{aligned}$$

This is equivalent to

$$\mu(\theta_{12} \geq \max\{\theta_1, \theta_2\}) = \mu(\theta_{12} \geq \theta_1) - \mu(\theta_2 \geq \theta_{12} \geq \theta_1).$$

Reasoning analogously, we obtain the four equalities

$$\begin{aligned} \mu(\theta_{12} \geq \max\{\theta_1, \theta_2\}) &= \mu(\theta_{12} \geq \theta_1) - \mu(\theta_2 \geq \theta_{12} \geq \theta_1) \\ &= \mu(\theta_{12} \geq \theta_2) - \mu(\theta_1 \geq \theta_{12} \geq \theta_2), \\ \mu(\theta_{12} \leq \min\{\theta_1, \theta_2\}) &= \mu(\theta_{12} \leq \theta_1) - \mu(\theta_2 \leq \theta_{12} \leq \theta_1) \\ &= \mu(\theta_{12} \leq \theta_2) - \mu(\theta_1 \leq \theta_{12} \leq \theta_2). \end{aligned}$$

Then condition (i) can be rewritten as the two inequalities

$$\frac{\mu(\theta_{12} \geq \theta_2) - \mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_{12} \leq \theta_2) - \mu(\theta_1 \leq \theta_{12} \leq \theta_2)} > \frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}, \tag{8}$$

$$\frac{\mu(\theta_{12} \geq \theta_1) - \mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_{12} \leq \theta_1) - \mu(\theta_2 \leq \theta_{12} \leq \theta_1)} > \frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}. \tag{9}$$

Inequality (8) can be expressed as any of the equivalent inequalities

$$\frac{\mu(\theta_{12} \geq \theta_2)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)} > \frac{\mu(\theta_{12} \leq \theta_2)}{\mu(\theta_1 \leq \theta_{12} \leq \theta_2)},$$

$$\frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)} > \frac{1 - \mu(\theta_{12} \geq \theta_2)}{\mu(\theta_{12} \geq \theta_2)},$$

$$\frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_2 \geq \theta_{12})} > \frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_{12} \geq \theta_2)},$$

$$\mu(\theta_{12} \geq \theta_1 | \theta_2 \geq \theta_{12}) > \mu(\theta_1 \geq \theta_{12} | \theta_{12} \geq \theta_2).$$

This is the first inequality in condition (ii). Similarly, inequality (9) can be rewritten as the second inequality in condition (ii).

We now prove that (ii) and (iii) are equivalent. First observe that the first inequality in (ii) is equivalent to the second inequality in (iii) through the steps

$$\frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_2 \geq \theta_{12})} > \frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_{12} \geq \theta_2)},$$

$$\frac{\mu(\theta_{12} \geq \theta_2)}{\mu(\theta_2 \geq \theta_{12})} > \frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_2 \geq \theta_{12} \geq \theta_2)},$$

$$\frac{\mu(\theta_2 \geq \theta_{12})}{\mu(\theta_{12} \geq \theta_2)} < \frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)},$$

$$\frac{1 - \mu(\theta_{12} \geq \theta_2)}{\mu(\theta_{12} \geq \theta_2)} < \frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)},$$

$$\frac{1}{\mu(\theta_{12} \geq \theta_2)} - 1 < \frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)},$$

$$\frac{1}{\mu(\theta_{12} \geq \theta_2)} < \frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_2) + \mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)},$$

$$\mu(\theta_{12} \geq \theta_2) > \frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_2 \geq \theta_{12} \geq \theta_2) + \mu(\theta_1 \geq \theta_{12} \geq \theta_2)}.$$

Similarly, the second inequality in condition (ii) is equivalent to the first inequality in condition (iii). *Q.E.D.*

A.5. *Proof of Proposition 4*

Suppose $\varepsilon < \frac{1}{16}$. Let μ be any density in the class described in Example 1. We first prove that there must exist at least a single issue which exhibits conditional uncertainty.

LEMMA 10: *For any density in $\mathcal{C}$, there is no sequence of equilibria that exhibits conditional certainty.*

PROOF: To see that $\{1, 2\}$ cannot be conditionally certain, observe that

$$\frac{\mu(\theta_{12} \geq \max\{\theta_1, \theta_2\})}{\mu(\theta_{12} \leq \min\{\theta_1, \theta_2\})} \leq \frac{\frac{1}{4} + \varepsilon}{\frac{1}{4} - \varepsilon}.$$

For small ε, this ratio approximates 1. On the other hand,

$$\frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)} \geq \frac{\frac{1}{2} - \varepsilon}{\varepsilon}.$$

For small ε, this ratio becomes arbitrarily large. This precludes the required inequality of the necessity direction of Lemma 1 for $A = \{1, 2\}$. An entirely similar argument proves that the inequality also fails for $A = \{1\}, \{2\}, \emptyset$. By Lemma 1, there cannot be an equilibrium with conditional certainty. *Q.E.D.*

By Lemma 10, we can assume that there is some issue with conditional uncertainty. We now prove that this implies that the other issue must also be conditionally uncertain.

LEMMA 11: *For every density in $\mathcal{C}$, all convergent sequences of equilibria exhibit conditional uncertainty on both issues.*

PROOF: The proof shows that assuming one issue is conditionally certain while the other is conditionally uncertain leads to a contradiction. So either both issues are conditionally certain or both issues are conditionally uncertain. By Lemma 10, it must be the latter case. We now demonstrate that if issue 1 is conditionally uncertain, then issue 2 cannot be conditionally certain to pass. The other cases can be argued symmetrically.

So suppose issue 2 is conditionally certain to pass. Recall that μ_A^* denotes the probability that an anonymous voter submits the ballot A when playing the limit strategy s^*. Since 2 is conditionally certain to pass, then $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_1^*}{\mu_2^*}$. Since 1 is conditionally uncertain, it is not conditionally certain to fail. We therefore conclude $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_2^*}{\mu_1^*}$. So we have the inequality

$$\text{(10)} \qquad \frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \max\left\{\frac{\mu_1^*}{\mu_2^*}, \frac{\mu_2^*}{\mu_1^*}\right\}.$$

In view of Lemmas 6 and 7, there exists some $\alpha \in (0,1)$ such that the limit strategy in terms of type is described by

$$s^*(\theta) = \begin{cases} \{1,2\}, & \text{if } \theta_{12} \geq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset, \\ \{1\}, & \text{if } \theta_{12} \geq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset, \\ \{2\}, & \text{if } \theta_{12} \leq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset, \\ \emptyset, & \text{if } \theta_{12} \leq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset. \end{cases}$$

Let

$$\phi_{12}(\alpha) = \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2),$$
$$\phi_\emptyset(\alpha) = \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1).$$

Observe that since μ has full support and admits a density, ϕ_{12} and $\phi_\emptyset$ are increasing and continuous functions with $\phi_{12}(0) = \phi_\emptyset(0) = 0$ and $\phi_{12}(1) = \phi_\emptyset(1) = 1$.

Now we can rewrite the limit probability of voting for both issues as

$$\begin{aligned} \mu_{12}^* &= \mu(\theta_{12} \geq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_{12} \geq \theta_2) \\ &\leq \mu(\theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) \\ &\quad \times \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) \\ &\quad + \varepsilon \\ &= \frac{1}{4}\phi_{12}(\alpha) + \varepsilon. \end{aligned}$$

Likewise, the limit probability of voting down on both issues can be rewritten as

$$\begin{aligned} \mu_\emptyset^* &= \mu(\theta_{12} \leq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_2 \geq \theta_{12}) \\ &\geq \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) \\ &\quad \times \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) \\ &\quad + \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\ &\quad \times \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\ &\quad + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \\ &\quad \times \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \end{aligned}$$

$$
\begin{aligned}
&= \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) \\
&\quad + \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1)(1 - \phi_\emptyset(\alpha)) \\
&\geq \frac{1}{4} + \frac{1}{4}(1 - \phi_\emptyset(\alpha)) - \varepsilon,
\end{aligned}
$$

so

$$
\frac{\mu_{12}^*}{\mu_\emptyset^*} \leq \frac{\frac{1}{4}\phi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \phi_\emptyset(\alpha)) - \varepsilon}. \tag{11}
$$

Inequality (10) provides that $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_2^*}{\mu_1^*}$, so (11) implies

$$
\frac{\mu_2^*}{\mu_1^*} \leq \frac{\frac{1}{4}\phi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \phi_\emptyset(\alpha)) - \varepsilon}. \tag{12}
$$

Inequality (10) provides that $\frac{\mu_{12}^*}{\mu_\emptyset^*}$ is larger than a fraction and its reciprocal, so we have $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq 1$. Therefore, (11) also implies

$$
\begin{aligned}
&\frac{1}{4}\phi_{12}(\alpha) + \varepsilon \geq \frac{1}{4} + \frac{1}{4}(1 - \phi_\emptyset(\alpha)) - \varepsilon, \\
&\phi_{12}(\alpha) \geq 2 - \phi_0(\alpha) - 8\varepsilon, \\
&\phi_{12}(\alpha) > 1 - 8\varepsilon.
\end{aligned} \tag{13}
$$

Arguing symmetrically,

$$
\begin{aligned}
&\frac{1}{4}\phi_{12}(\alpha) + \varepsilon \geq \frac{1}{4} + \frac{1}{4}(1 - \phi_\emptyset(\alpha)) - \varepsilon, \\
&\phi_{12}(\alpha) \geq 2 - \phi_0(\alpha) - 8\varepsilon, \\
&\phi_0 \geq 2 - \phi_1 2(\alpha) - 8\varepsilon, \\
&\phi_0(\alpha) > 1 - 8\varepsilon.
\end{aligned} \tag{14}
$$

We can rewrite the limit probability of voting only for issue 2 as

$$
\begin{aligned}
\mu_2^* &= \mu(\theta_{12} \leq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_{12} \geq \theta_2) \\
&\geq \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) \\
&\quad \times \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12})
\end{aligned}
$$

$$
\begin{aligned}
&+ \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\
&\times \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\
&+ \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \\
&\times \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset | \theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \\
= {}& \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1)\phi_\emptyset(\alpha) + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \\
\geq {}& \frac{1}{4}\phi_0(\alpha) + \frac{1}{4} - \varepsilon.
\end{aligned}
$$

Similarly, rewriting the probability of voting only for issue 1 yields

$$
\mu_1^* \leq \frac{1}{4}(1 - \phi_{12}(\alpha)) + \varepsilon.
$$

So

$$
\frac{\mu_2^*}{\mu_1^*} \geq \frac{\frac{1}{4}\phi_0(\alpha) + \frac{1}{4} - \varepsilon}{\frac{1}{4}(1 - \phi_{12}(\alpha)) + \varepsilon}. \tag{15}
$$

Combining (12) and (15),

$$
\frac{\frac{1}{4}\phi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \phi_\emptyset(\alpha)) - \varepsilon} \geq \frac{\frac{1}{4}\phi_0(\alpha) + \frac{1}{4} - \varepsilon}{\frac{1}{4}(1 - \phi_{12}(\alpha)) + \varepsilon}.
$$

This can be rewritten as

$$
(\phi_{12}(\alpha) + 4\varepsilon)(1 - \phi_{12}(\alpha) + 4\varepsilon) \geq (\phi_0(\alpha) + 1 - 4\varepsilon)(2 - \phi_0(\alpha) - 4\varepsilon). \tag{16}
$$

At the same time, recalling earlier inequalities,

$$
\begin{aligned}
\phi_{12}(\alpha) + 4\varepsilon &< 1 + 4\varepsilon \\
&< 2 - 12\varepsilon && \text{(since } \varepsilon < \tfrac{1}{16}\text{)} \\
&< \phi_0(\alpha) + 1 - 4\varepsilon && \text{(by (14))}
\end{aligned}
$$

and

$$
\begin{aligned}
1 - \phi_{12}(\alpha) + 4\varepsilon &< 1 - (1 - 8\varepsilon) + 4\varepsilon && \text{(by (13))} \\
&= 12\varepsilon \\
&< 1 - 4\varepsilon && \text{(since } \varepsilon < \tfrac{1}{16}\text{)} \\
&< 2 - \phi_0(\alpha) - 4\varepsilon.
\end{aligned}
$$

But the prior two series of inequalities contradict (16), since they imply the left hand side of (16) is the product of strictly smaller positive quantities than those in the product on the right hand side of (16). *Q.E.D.*

We now prove that if there is conditional uncertainty on both issues, then there is unconditional uncertainty on both issues. Given Lemma 11, this will suffice to show that there is unconditional uncertainty on both issues for every density in $\mathcal{C}$.

For notational ease, we now define

$$\mu^I(x|x') = \mu(x \in s_I(\theta_i)|x' \in s_I(\theta_i))$$

and

$$\mu^I(x|\neg x') = \mu(x \in s_I(\theta_i)|x' \notin s_I(\theta_i)).$$

Let

$$\mu^I(x) = \mu(x \in s_I(\theta_i)).$$

LEMMA 12: *Issue x is conditionally uncertain if and only if*

$$\lim_{I\to\infty}\left|\sqrt{(I-1)}(\mu^I(x|x') + \mu^I(x|\neg x') - 1)\right| < \infty.$$

PROOF: Take $x = 2$; the case $x = 1$ is identical. Recall the two arrays defined in the proof of Lemma 8, rowwise independent binary random variables Y^{Ii} and Z^{Ii} whose success probabilities are $\mu(2 \in s_I(\theta_i)|1 \in s_I(\theta_I))$ and $\mu(2 \in s_I(\theta_i)|1 \notin s_I(\theta_I))$. In that proof, we demonstrated that the conditional distribution of the vote count on issue 2 is equal to the distribution of $\sum_{i=1}^{(I-1)/2} Y^{Ii} + \sum_{i=1}^{(I-1)/2} Z^{Ii}$. Let $W^{Ii} = Y^{Ii} + Z^{Ii}$. As Y^{Ii} and Z^{Ii} are mutually independent, the array W^{Ii} defines a rowwise independent array of random variables. We can write that

$$\mathbf{P}\left(\#\{j \neq i : 2 \in s_I^*(\theta_j)\} > \frac{I-1}{2}\middle|\#\{j \neq i : 1 \in s_I^*(\theta_j)\} = \frac{I-1}{2}\right)$$
$$= \mathbf{P}\left(\sum_{i=1}^{(I-1)/2} W^{Ii} > \frac{I-1}{2}\right).$$

Recalling the definition of the binary random variables $Y^{Ii}(\theta)$ and $Z^{Ii}(\theta)$, we have that

$$\mathbf{E}(W^{Ii}) = \mu^I(2|1) + \mu^I(2|\neg 1)$$

and

$$\text{Var}(W^{Ii}) = \mu^I(2|1)[1-\mu^I(2|1)] + \mu^I(2|\neg 1)[1-\mu^I(2|\neg 1)].$$

Applying the central limit theorem for triangular arrays,

$$\mathbf{P}\left(\frac{\sum_{i=1}^{(I-1)/2} W^{Ii} - \left(\frac{I-1}{2}\right)[\mu^I(2|1)+\mu^I(2|\neg 1)]}{\sqrt{\left(\frac{I-1}{2}\right)\left(\mu^I(2|1)[1-\mu^I(2|1)] + \mu^I(2|\neg 1)[1-\mu^I(2|\neg 1)]\right)}} < y\right) \longrightarrow \Phi(y), \tag{17}$$

where Φ denotes the standard normal cumulative distribution function.

The conditional probability that issue i fails is $\mathbf{P}(\sum_{i=1}^{(I-1)/2} W^{Ii} < \frac{I-1}{2})$. By manipulation of (17), this converges to

$$\Phi\left(\frac{1}{2}\cdot\frac{\sqrt{(I-1)}(1-(\mu^I(2|1)+\mu^I(2|\neg 1)))}{\sqrt{\mu^I(2|1)[1-\mu^I(2|1)]+\mu^I(2|\neg 1)[1-\mu^I(2|\neg 1)]}}\right).$$

Therefore, $\lim_{I\to\infty} |\sqrt{(I-1)}(\mu^I(2|1) + \mu^I(2|\neg 1) - 1)| < \infty$ is necessary and sufficient for issue 2 to be conditionally uncertain. *Q.E.D.*

LEMMA 13: *Issue x is unconditionally uncertain if and only if* $\lim_{I\to\infty} |\sqrt{I}(\mu_k^I - \frac{1}{2})| < \infty$.

PROOF: Define the binary random variable

$$V^{Ii} = \begin{cases} 1 & \text{with probability } \mu^I(x), \\ 0 & \text{with probability } 1-\mu^I(x), \end{cases}$$

with mean $\mu^I(x)$ and variance $\mu^I(x)(1-\mu^I(x))$. The probability that issue k will pass (fail) is

$$\mathbf{P}\left(\sum_{i=1}^{I} V_k^{Ii} > (<)\, \frac{I}{2}\right).$$

Arguing as in the proof of Lemma 12, we have that the asymptotic (unconditional) probability that issue x will pass is equal to

$$\Phi\left(\frac{\sqrt{I}\left(\frac{1}{2}-\mu^I(x)\right)}{\sqrt{\mu^I(x)(1-\mu^I(x))}}\right).$$

Therefore,

$$\lim_{I\to\infty}\left|\sqrt{I}\left(\frac{1}{2}-\mu^I(x)\right)\right|<\infty$$

is necessary and sufficient for unconditional uncertainty. *Q.E.D.*

LEMMA 14: *There is unconditional uncertainty for both issues if and only if there is conditional uncertainty for both issues*.

PROOF: Let

$$x^I=\mu^I(1),$$
$$y^I=\mu^I(2),$$

and

$$a^I=\mu^I(1|2),$$
$$b^I=\mu^I(1|\neg 2),$$
$$c^I=\mu^I(2|1),$$
$$d^I=\mu^I(2|\neg 1).$$

We have a system of two equations with two unknowns, x^I and y^I:

$$x^I=a^Iy^I+b^I(1-y^I), \tag{18}$$

$$y^I=c^Ix^I+d^I(1-x^I). \tag{19}$$

The corresponding solutions for x and y are

$$x^I=\frac{(a^I-b^I)d^I+b^I}{1-(a^I-b^I)(c^I-d^I)}, \tag{20}$$

$$y^I=\frac{(c^I-d^I)b^I+d^I}{1-(c^I-d^I)(a^I-b^I)}. \tag{21}$$

We first prove that if there is conditional uncertainty on both issues, then there must be unconditional uncertainty on both. Subtracting $\frac{1}{2}$ from both sides in equations (20) and (21) yields, after some manipulation,

$$x^I-\frac{1}{2}=\frac{1}{2}\frac{(b^I-a^I)(1-(c^I+d^I))+(1-(a^I+b^I))}{1-(a^I-b^I)(c^I-d^I)}, \tag{22}$$

$$y^I-\frac{1}{2}=\frac{1}{2}\frac{(d^I-c^I)(1-(a^I+b^I))+(1-(c^I+d^I))}{1-(c^I-d^I)(a^I-b^I)}. \tag{23}$$

By Lemma 12, conditional uncertainty on both issues means

$$\lim_{I\to\infty} \sqrt{I}|1-(a^I+b^I)| < \infty$$

and

$$\lim_{I\to\infty} \sqrt{I}|1-(c^I+d^I)| < \infty.$$

Since $1-(a^I-b^I)(c^I-d^I)$ is uniformly bounded away from 0 and $|(a^I-b^I)|$ is bounded by 1, this suffices to show that $\lim_{I\to\infty}\sqrt{I}|x^I-\frac{1}{2}|$ and $\lim_{I\to\infty}\sqrt{I}|x^I-\frac{1}{2}|$ given the expressions in (22) and (23) are both finite. By Lemma 13, this implies unconditional uncertainty on both issues.

We finally show that unconditional uncertainty on both issues implies conditional uncertainty on both. Equations (18) and (19) imply

$$x^I-y^I=(a^I+b^I-1)y^I+2a^I\left(\frac{1}{2}-y^I\right),$$

$$y^I-x^I=(c^I+d^I-1)x^I+2c^I\left(\frac{1}{2}-x^I\right).$$

These can be rewritten as

$$(a^I+b^I-1)y^I=\left(x^I-\frac{1}{2}\right)+2\left(\frac{1}{2}-a^I\right)\left(\frac{1}{2}-y^I\right),$$

$$(c^I+d^I-1)x^I=\left(y^I-\frac{1}{2}\right)+2\left(\frac{1}{2}-c^I\right)\left(\frac{1}{2}-x^I\right).$$

By Lemma 13, unconditional uncertainty on both issues yields that $\lim_{I\to\infty}\sqrt{I}|\frac{1}{2}-x^I|$ and $\lim_{I\to\infty}\sqrt{I}|\frac{1}{2}-y^I|$ are both finite. Since both $|\frac{1}{2}-a^I|$ and $|\frac{1}{2}-c^I|$ are bounded by $\frac{1}{2}$, it suffices to show that

$$\lim_{I\to\infty} \sqrt{I}|a^I+b^I-1|y^I < \infty.$$

By Lemma 12, this implies issue 1 is conditionally uncertain. Similarly, issue 2 is also conditionally uncertain. *Q.E.D.*

A.6. *Proof of Proposition 5*

We begin by proving a useful implication of conditional certainty.

LEMMA 15: *If* $\{1,2\}$ *is conditionally certain*, *then*

$$\mu(\theta_{12}\geq\theta_1)\mu(\theta_{12}\geq\theta_2)\geq\mu(\theta_1\geq\theta_{12})\mu(\theta_2\geq\theta_{12}).$$

PROOF: Suppose that $\{1, 2\}$ is conditionally certain. By Lemmas 1 and 9, we know that

$$\mu(\theta_{12} \geq \theta_1 | \theta_2 \geq \theta_{12}) \geq \mu(\theta_1 \geq \theta_{12} | \theta_{12} \geq \theta_2), \tag{24}$$

$$\mu(\theta_{12} \geq \theta_2 | \theta_1 \geq \theta_{12}) \geq \mu(\theta_2 \geq \theta_{12} | \theta_{12} \geq \theta_1) \tag{25}$$

hold. Observe that

$$\mu(\theta_{12} \geq \theta_1 | \theta_2 \geq \theta_{12}) = \mu(\theta_2 \geq \theta_{12} | \theta_{12} \geq \theta_1) \times \frac{\mu(\theta_{12} \geq \theta_1)}{\mu(\theta_2 \geq \theta_{12})}.$$

We can then rewrite (24) as

$$\mu(\theta_2 \geq \theta_{12} | \theta_{12} \geq \theta_1) \times \frac{\mu(\theta_{12} \geq \theta_1)}{\mu(\theta_2 \geq \theta_{12})} \geq \mu(\theta_{12} \geq \theta_2 | \theta_1 \geq \theta_{12}) \times \frac{\mu(\theta_1 \geq \theta_{12})}{\mu(\theta_{12} \geq \theta_2)}.$$

This is equivalent to

$$\frac{\mu(\theta_1 \geq \theta_{12})\mu(\theta_2 \geq \theta_{12})}{\mu(\theta_{12} \geq \theta_1)\mu(\theta_{12} \geq \theta_2)} \leq \frac{\mu(\theta_2 \geq \theta_{12} | \theta_{12} \geq \theta_1)}{\mu(\theta_{12} \geq \theta_2 | \theta_1 \geq \theta_{12})}. \tag{26}$$

Moreover, (25) implies that

$$\frac{\mu(\theta_2 \geq \theta_{12} | \theta_{12} \geq \theta_1)}{\mu(\theta_{12} \geq \theta_2 | \theta_1 \geq \theta_{12})} \leq 1. \tag{27}$$

Together, (26) and (27) imply

$$\frac{\mu(\theta_1 \geq \theta_{12})\mu(\theta_2 \geq \theta_{12})}{\mu(\theta_{12} \geq \theta_1)\mu(\theta_{12} \geq \theta_2)} \leq 1,$$

which is the desired conclusion. *Q.E.D.*

Without loss of generality, assume $A = \emptyset$ is the local Condorcet loser. First observe that if both issues are conditionally uncertain, Lemma 14 implies that there is no unconditionally certain bundle. In particular, $\emptyset$ cannot be unconditionally certain.

We next argue that if either issue is conditionally certain to fail, then $\emptyset$ cannot be unconditionally certain. So suppose issue 1 is conditionally certain to fail. Then Lemma 6 implies $2 \in s^*(\theta)$ if $\theta_2 > \theta_\emptyset$. However, since $\emptyset$ is a local Condorcet loser, $\mu(\theta_2 > \theta_\emptyset) > \frac{1}{2}$. By the strong law of large numbers for triangular arrays, this means issue 2 is unconditionally certain to pass. A symmetric argument holds if issue 2 is conditionally certain to fail. So we can now assume without loss of generality that there is at least one issue that is conditionally certain to pass and that the other issue is either conditionally certain to pass or is conditionally uncertain. Consider the case where issue 1 is conditionally certain to pass; the argument for issue 2 is identical.

Case 1. *Issue* 2 *is conditionally certain to pass.* Then $\{1, 2\}$ is conditionally certain. Now, by way of contradiction, suppose $\emptyset$ is unconditionally certain. So $\mu(1 \in s^*(\theta)) \leq \frac{1}{2}$. By Lemma 6, $\mu(\theta_1 \geq \theta_{12}) \geq \frac{1}{2} \geq \mu(\theta_{12} \geq \theta_1)$ because issue 2 is conditionally certain to pass. Symmetrically, we can also conclude $\mu(\theta_2 \geq \theta_{12}) \geq \frac{1}{2} \geq \mu(\theta_{12} \geq \theta_2)$. Then

$$\mu(\theta_1 \geq \theta_{12})\mu(\theta_2 \geq \theta_{12}) \geq \mu(\theta_{12} \geq \theta_1)\mu(\theta_{12} \geq \theta_2).$$

At the same time, the fact that $\{1, 2\}$ is conditionally certain also implies, through Lemma 15, that

$$\mu(\theta_1 \geq \theta_{12})\mu(\theta_2 \geq \theta_{12}) \leq \mu(\theta_{12} \geq \theta_1)\mu(\theta_{12} \geq \theta_2).$$

The only way to maintain the prior two inequalities is for $\mu(\theta_1 \geq \theta_{12}) = \frac{1}{2}$ and $\mu(\theta_1 \geq \theta_{12}) = \frac{1}{2}$. But then the Condorcet ranking $\succ_C$ is incomplete, contradicting the hypothesis that $\succ_C$ is complete.

Case 2. *Issue* 2 *is conditionally uncertain.* Recall (22) from the proof of Lemma 14:

$$\begin{aligned}&\sqrt{I}\mu(1 \in s_I^*(\theta)) - \frac{1}{2}\\ &\quad = \frac{1}{2}\frac{\sqrt{I}(b^I - a^I)(1 - (c^I + d^I)) + \sqrt{I}(1 - (a^I + b^I))}{1 - (a^I - b^I)(c^I - d^I)},\end{aligned}$$

where

$$\begin{aligned}a^I &= \mu(1 \in s_I^*(\theta)|2 \in s_I^*(\theta)),\\ b^I &= \mu(1 \in s_I^*(\theta)|2 \notin s_I^*(\theta)),\\ c^I &= \mu(2 \in s_I^*(\theta)|1 \in s_I^*(\theta)),\\ d^I &= \mu(2 \in s_I^*(\theta)|1 \notin s_I^*(\theta)).\end{aligned}$$

By Lemma 12, we have $\lim \sqrt{I}(1 - (c^I + d^I)) < \infty$ since issue 2 is conditionally uncertain. Similarly, $\lim \sqrt{I}(1 - (a^I + b^I)) = \infty$. Since $1 - (a^I - b^I)(c^I - d^I)$ is uniformly bounded away from 0, this suffices to prove $\sqrt{I}\mu(1 \in s_I^*(\theta)) - \frac{1}{2} \to \infty$. Then by Lemma 13, we conclude that issue 1 is unconditionally certain to pass. This suffices to show that the empty set cannot be unconditionally certain.

A.7. *Proof of Proposition 6*

Begin by observing that, by Lemmas 1 and 9, both issues are conditionally certain to pass. Then the proposition follows directly from Lemma 2, which we recall here.

LEMMA 2: *Suppose* $\{1, 2\}$ *is a local Condorcet winner. If issue* 1 (*or issue* 2) *is conditionally certain to pass, then* $\{1, 2\}$ *is the limit outcome.*

PROOF: There are five cases to consider.

Case 1. $\{1, 2\}$ *is conditionally certain.* Since issue 2 is conditionally certain to pass, by Lemma 6, $1 \in s^*(\theta)$ whenever $\theta_{12} \geq \theta_2$. But since $\{1, 2\}$ is a local Condorcet winner, $\mu(\theta_{12} \geq \theta_2) > \frac{1}{2}$, that is, $\mu(1 \in s^*(\theta)) > \frac{1}{2}$. Then, by the strong law of large numbers for triangular arrays, issue 1 is unconditionally certain to pass. A similar argument establishes that issue 2 is also unconditionally certain to pass. Thus $\{1, 2\}$ is unconditionally certain.

Case 2. *The bundle* $\{1\}$ *is conditionally certain.* By Lemma 15, we have

$$\mu(\theta_1 \geq \theta_{12})\mu(\theta_1 \geq \theta_\emptyset) \geq \mu(\theta_{12} \geq \theta_1)\mu(\theta_\emptyset \geq \theta_1). \tag{28}$$

Since $\{1, 2\}$ is a local Condorcet winner, we know that $\mu(\theta_{12} \geq \theta_1) > \frac{1}{2}$, so

$$\mu(\theta_1 \geq \theta_{12}) < \mu(\theta_{12} \geq \theta_1). \tag{29}$$

To maintain the inequality (28), it must be that

$$\mu(\theta_1 \geq \theta_\emptyset) > \mu(\theta_\emptyset \geq \theta_1). \tag{30}$$

However, (29) and Lemma 6 imply that $\mu(1 \in s^*(\theta)) > \frac{1}{2}$. By the strong law of large numbers for triangular arrays, issue 1 is unconditionally certain to pass. Similarly, (30) and Lemma 6 imply that issue 2 is also unconditionally certain to pass.

Case 3. *The bundle* $\{2\}$ *is conditionally certain.* This case can be argued similarly to Case 2.

Case 4. *Issue* 1 *is conditionally certain to pass and issue* 2 *is conditionally uncertain.* The argument for Case 1 in Proposition 5 can be replicated verbatim to demonstrate that issue 1 is unconditionally certain to pass. A symmetric argument demonstrates that issue 2 is also unconditionally certain to pass.

Case 5. *Issue* 2 *is conditionally certain to pass and issue* 1 *is conditionally uncertain.* This case can be argued similarly to Case 4. *Q.E.D.*

A.8. *Proof of Proposition 7*

We first record a straightforward but useful implication of quasiseparability.

LEMMA 16: *Suppose* $\succ_C$ *is quasiseparable. If* $\{1, 2\}$ *is a Condorcet winner, then its complement* $\emptyset$ *is a Condorcet loser.*

PROOF: Without loss of generality, suppose $\{1,2\}$ is a Condorcet winner. Then $\{1,2\} \succ_C \{1\}$. By quasiseparability of $\succ_C$, we have $\{2\} \succ_C \emptyset$. Similarly, $\{1,2\} \succ_C \{2\}$ implies $\{1\} \succ_C \emptyset$. Moreover, since $\{1,2\}$ is a Condorcet winner, we have $\{1,2\} \succ_C \emptyset$. Therefore, $\emptyset$ is a Condorcet loser. *Q.E.D.*

To prove the proposition, without loss of generality consider the case where $\{1,2\}$ is the Condorcet winner. Since $\emptyset$ is the Condorcet loser, then it cannot be conditionally certain. To see this, observe that the necessary inequalities in part (iii) of Lemma 9 are impossible because both $\mu(\theta_\emptyset \geq \theta_1)$ and $\mu(\theta_\emptyset \geq \theta_2)$ are strictly less than $\frac{1}{2}$, while one of the ratios on the right hand sides of the inequality must be weakly greater than $\frac{1}{2}$. But since there is conditional certainty on both issues and $\emptyset$ is not conditionally certain, one of the issues must be conditionally certain to pass. Then by Lemma 2, this implies that $\{1,2\}$ is conditionally certain.

A.9. *Proof of Proposition 8*

We prove the case when types are supermodular; the argument for the submodular case then follows by relabeling up to down on the second issue. We begin with a preliminary observation.

LEMMA 17: *Suppose the support of μ is the set of supermodular types. Then for $x = 1, 2$,*

$$\mu(\theta_x \geq \theta_\emptyset) \leq \mu(x \in s^*(\theta)) \leq \mu(\theta_{x'} \leq \theta_{12}).$$

PROOF: Consider the case where $x = 1$. By Lemma 7, there exist $\alpha \in [0,1]$ such that

$$\begin{aligned}\mu(1 \in s^*(\theta)) &= \mu(\alpha\theta_{12} + (1-\alpha)\theta_1 \geq \alpha\theta_2 + (1-\alpha)\theta_\emptyset) \\ &= \mu(\alpha[\theta_{12} + \theta_\emptyset - \theta_1 - \theta_2] \geq \theta_\emptyset - \theta_1) \\ &= \mu((1-\alpha)[\theta_1 + \theta_2 - \theta_{12} - \theta_\emptyset] \geq \theta_2 - \theta_{12}).\end{aligned}$$

Supermodularity implies that

$$\theta_{12} + \theta_\emptyset - \theta_1 - \theta_2 \geq 0. \tag{31}$$

This provides the inequality

$$\begin{aligned}\mu(1 \in s^*(\theta)) &= \mu(\alpha[\theta_{12} + \theta_\emptyset - \theta_1 - \theta_2] \geq \theta_\emptyset - \theta_1) \\ &\geq \mu(0 \geq \theta_\emptyset - \theta_1) \\ &= \mu(\theta_1 \geq \theta_\emptyset).\end{aligned}$$

Inequality (31) also provides the inequality

$$\begin{aligned}\mu(1 \in s^*(\theta)) &= \mu((1-\alpha)[\theta_1 + \theta_2 - \theta_{12} - \theta_\emptyset] \geq \theta_2 - \theta_{12}) \\ &\leq \mu(\theta_{12} \geq \theta_2) \\ &= \mu(\theta_{12} \geq \theta_2).\end{aligned}$$

Q.E.D.

We now record the following implications of Lemma 17:

$$\mu(1 \in s^*(\theta)) \geq \mu(\theta_1 \geq \theta_\emptyset), \tag{32}$$

$$\mu(2 \in s^*(\theta)) \geq \mu(\theta_2 \geq \theta_\emptyset), \tag{33}$$

$$\mu(1 \in s^*(\theta)) \leq \mu(\theta_2 \leq \theta_{12}), \tag{34}$$

$$\mu(2 \in s^*(\theta)) \leq \mu(\theta_1 \leq \theta_{12}). \tag{35}$$

We now argue by cases that the Condorcet winning bundle A is a limit outcome of the election and that it is the unique limit outcome.

Case 1. $A = \{1\}$. Since $\{1\}$ is the Condorcet winner, $\mu(\theta_1 \geq \theta_\emptyset) > \frac{1}{2}$. By (32), we have that issue 1 is unconditionally certain to pass. Again since $\{1\}$ is the Condorcet winner, $\mu(\theta_1 \geq \theta_{12}) > \frac{1}{2}$. So by (35), it must be that issue 2 is unconditionally certain to fail.

Case 2. $A = \{2\}$. This argument is nearly identical to Case 1, using (34) and (33).[22]

Case 3. $A = \{1, 2\}$. Since $\{1, 2\}$ is the Condorcet winner, we have $\{1, 2\} \succ_C \{2\}$. By quasiseparability, $\{1\} \succ_C \emptyset$. Recalling the definition of the Condorcet order, we have $\mu(\theta_1 \geq \theta_\emptyset) > \frac{1}{2}$. But using (32), this implies $\mu(1 \in s^*(\theta)) > \frac{1}{2}$. Appealing to the strong law of large numbers for triangular arrays, this implies issue 1 is unconditionally certain to pass. Quasiseparability of $\succ_C$ similarly implies $\{2\} \succ_C \emptyset$, that is, $\mu(\theta_2 \geq \theta_\emptyset) > \frac{1}{2}$. Using (33), this similarly implies issue 2 is unconditionally certain to pass. Thus $\{1, 2\}$ is the only limit outcome.

Case 4. $A = \emptyset$. This argument is nearly identical to Case 3, using (34) and (35).

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[22]Note that quasi-separability is not needed for Cases 1 and 2; supermodularity on its own suffices.

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