

HIERARCHIES OF AMBIGUOUS BELIEFS*

David S. Ahn[†]

August 2006

Abstract

We present a theory of interactive beliefs analogous to Mertens and Zamir [31] and Brandenburger and Dekel [10] that allows for hierarchies of ambiguity. Each agent is allowed a compact set of beliefs at each level, rather than just a single belief as in the standard model. We propose appropriate definitions of coherency and common knowledge for our types. Common knowledge of coherency closes the model, in the sense that each type homeomorphically encodes a compact set of beliefs over the others' types. This space universally embeds every implicit type space of ambiguous beliefs in a beliefs-preserving manner. An extension to ambiguous conditional probability systems [4] is presented. The standard universal type space and the universal space of compact continuous possibility structures are epistemically identified as subsets.

JEL classification: C72; D81

Keywords: ambiguity, Knightian uncertainty, Bayesian games, universal type space

1 Introduction

The idea of a player's type introduced by Harsanyi [19] provides a useful and compact representation of the interactive belief structures that arise in a game, encoding a player's beliefs on some "primitive" parameter of uncertainty, her belief about the others' beliefs, their beliefs about her belief about their beliefs, and so on. Mertens and Zamir [31], hereafter MZ, constructed a universal type space encoding all internally consistent streams of beliefs, ensuring that Bayesian games with Harsanyi types lose no analytic generality.¹

*NOTICE: this is the author's version of a work that was accepted for publication in *Journal of Economic Theory*. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in *Journal of Economic Theory* 136(1): 286-301, September 2007, doi:10.1016/j.jet.2006.08.004.

[†]Department of Economics, University of California, 549 Evans Hall #3880, Berkeley, CA 94720; dahn@econ.berkeley.edu. The author thanks the associate editor and an anonymous referee, Peter Hammond, Bob Wilson, and seminar participants at the Harvard–M.I.T. theory seminar, the 2005 Workshop on Risk, Uncertainty, and Decision, and the 2005 World Congress of the Econometric Society for helpful comments. This paper is a substantially revised version of the third chapter's of the author's dissertation.

¹An earlier discussion of the problem can be found in [2] and [8].

There remains a fundamental caveat. This notion of type implicitly assumes probabilistic sophistication, in the sense that each player has precise beliefs. In reality, the decision maker's beliefs can be ambiguous, as pointed out by Ellsberg [13], and she may consider multiple beliefs to be plausible [7, 17]. Even given a precise assessment of the natural uncertainty, she may be ambiguous of her opponents' beliefs, or whether they hold precise beliefs. In games, agents may have multiple levels of multiple beliefs. A growing literature studies interactive situations with sets of beliefs [3, 20–28], for which the standard construction is inadequate.

We construct a model of interactive beliefs where each player is allowed a compact set of multiple priors. In turn, she is allowed multiple beliefs about the possibly multiple priors of the other player, and so on. If agents share common knowledge of the internal consistency of their orders of ambiguous beliefs, then an agent's type completely specifies her set of joint beliefs on the primitive state and the other's type. This space is universal in the sense that it can embed any other type space with this property in a manner that preserves the implicit hierarchies of belief. Two significant subspaces are the standard universal type space and the universal space of compact continuous possibility models [30].

The preceding criticism of the standard construction is hardly new. In fact, Epstein and Wang [15] address these concerns with hierarchies of preferences over acts. This approach has been recently extended by Di Tillio [12] for finite games. Instead of working with preferences, we explicitly model ambiguity with sets of beliefs. The comparison is clearer after formally introducing our model, hence postponed until Section 4.

2 Model

We build our model of interactive ambiguity, extending the economical construction of Brandenburger and Dekel [10], henceforth BD. Our main line of proof, establishing conditions for the Kolmogorov Extension Theorem, parallels their development and many mathematical steps are appropriately adapted. The technical contribution is mild; such adaptations are now endemic to the literature on universal spaces.

We first introduce some notation. For any metric space X , let ΔX denote its Borel probability measures endowed with the topology of weak convergence, metrized by the Prohorov distance ρ . If X is compact Polish, then ΔX is compact Polish. If Y is also metric, for any measurable $f : X \rightarrow Y$, let $\mathcal{L}_f : \Delta X \rightarrow \Delta Y$ denote the law or image measure on Y induced by f , defined by $[\mathcal{L}_f(\mu)](E) = \mu(f^{-1}(E))$ for any $\mu \in \Delta X$ and any Borel set $E \subseteq Y$. If $\mu \in \Delta(X \times Y)$, its marginal measure on X is defined as $\text{marg}_X \mu = \mathcal{L}_{\text{Proj}_X}(\mu)$, where Proj_X denotes projection to X .

Lemma 1. *Suppose X, Y, Z are compact metric spaces and $f : X \rightarrow Y$, $g : Y \rightarrow Z$ are measurable. Then:*

1. $\mathcal{L}_{g \circ f} = \mathcal{L}_g \circ \mathcal{L}_f$
2. *If f is continuous, then \mathcal{L}_f is continuous;*

3. If f is injective, then \mathcal{L}_f is injective;
4. If f is surjective, then \mathcal{L}_f is surjective.

Let $\mathcal{K}(X)$ denote the family of nonempty compact subsets of X , endowed with the Hausdorff distance metric $d_h : \mathcal{K}(X) \rightarrow \mathbb{R}$:

$$d_h(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right\},$$

where d is the metric on X . If X is compact Polish, then $\mathcal{K}(X)$ is compact Polish. For any continuous $f : X \rightarrow Y$, define its extension $f^{\mathcal{K}} : \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ by $f^{\mathcal{K}}(A) = f(A)$. In particular, $\text{marg}_X^{\mathcal{K}}(A) = \{\text{marg}_X \mu : \mu \in A\}$ denotes the set of marginal measures on X induced by the product measure in A .

Lemma 2. *Suppose X, Y, Z are compact metric spaces and $f : X \rightarrow Y, g : Y \rightarrow Z$ are continuous. Then:*

1. $(g \circ f)^{\mathcal{K}} = g^{\mathcal{K}} \circ f^{\mathcal{K}}$;
2. $f^{\mathcal{K}}$ is continuous;
3. If f is injective, then $f^{\mathcal{K}}$ is injective;
4. If f is surjective, then $f^{\mathcal{K}}$ is surjective.

We now introduce the interactive model. Two agents, i and j , share a fundamental space of uncertainty S , which is compact metric, hence compact Polish. The set of first order ambiguous beliefs is the family of sets of probabilities on S , or elements of $\mathcal{K}(\Delta S)$. This expands on the standard beliefs ΔS , which are naturally embedded as singletons in $\mathcal{K}(\Delta S)$. Here, each agent is allowed multiple beliefs on S , with the proviso that their limit points are also included.

Moreover, i is uncertain of j 's beliefs. Her second order ambiguous beliefs are then compact sets of joint probabilities on S and first order ambiguous beliefs of j . Formally continuing this process, let

$$\begin{aligned} X_0 &= S \\ X_1 &= X_0 \times \mathcal{K}(\Delta X_0) \\ &\vdots \\ X_{n+1} &= X_n \times \mathcal{K}(\Delta X_n) \\ &\vdots \end{aligned}$$

An ambiguous type is then a hierarchy of ambiguous beliefs: $t = (A_1, A_2, \dots) \in \prod_{n=0}^{\infty} \mathcal{K}(\Delta X_n)$. The space of all possible types is denoted T_0 and carries the product topology. A type $\bar{t} = (\bar{A}_1, \bar{A}_2, \dots)$ is unambiguous if \bar{A}_n is a singleton for every n . The space of all unambiguous types is

denoted \bar{T}_0 and obviously embeds $\prod_{n=0}^{\infty} \Delta X_n$. As a general mnemonic, for any set of types $T \subseteq T_0$, the overlined $\bar{T} = T \cap \bar{T}_0$ will denote its unambiguous types. Also, for any function f from a subset T of types, the overlined $\bar{f} = f|_{\bar{T}}$ will denote its restriction to unambiguous types.

Recall that $\text{marg}_{X_{n-1}} \mu_{n+1} = \mathcal{L}_{\text{Proj}_{X_{n-1}}}(\mu_{n+1})$ refers to the marginal measure on X_{n-1} induced by $\mu_{n+1} \in \Delta X_n = \Delta(X_{n-1} \times \mathcal{K}(\Delta X_{n-1}))$. Then, carrying our earlier notation, $\text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}$ refers to the compact set of marginal measures on X_{n-1} induced by elements of A_{n+1} .

Definition 1. A type $t = (A_1, A_2, \dots) \in T_0$ is *coherent* if for all $n \geq 1$,

$$A_n = \text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}.$$

Definition 1 may be more transparent as its two set containments. First $\{\text{marg}_{X_{n-1}} \mu_n : \mu_{n+1} \in A_{n+1}\} \subseteq A_n$. This means i 's ambiguous beliefs are supported by her lower level beliefs. She cannot suddenly add an unrelated possible belief in the dimension X_n of her joint beliefs on $X_{n+1} = X_n \times \mathcal{K}(\Delta X_n)$. We can think of this set containment as a form of forward consistency.

However, this direction does not imply that *all* lower level beliefs are necessary to support higher level beliefs. For example, the set of all probabilities $A_n = \Delta X_{n-1}$ can support any specification of A_{n+1} . The other set containment, $A_n \subseteq \{\text{marg}_{X_{n-1}} \mu_{n+1} : \mu_{n+1} \in A_{n+1}\}$, eliminates superfluous lower level beliefs and requires that each element of A_n be in the service of some higher level belief. This direction can therefore be considered a form of backward consistency. The combination of directions generalizes [10, Definition 1], and is identical when A_n is restricted to the singletons.

Let T_1 denote the space of all coherent types. The space of unambiguous coherent types is $\bar{T}_1 = T_1 \cap \bar{T}_0 = \{(A_1, A_2, \dots) \in T_1 : |A_n| = 1, \forall n\}$. Both spaces are compact.

Lemma 3. T_1 and \bar{T}_1 are compact.

Proof. Let $T_{1,n} = \{(A_1, A_2, \dots) \in T_0 : A_n = \text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}\}$. We prove each of these sets $T_{1,n}$ is closed. Take a convergent sequence $\{t^k\}_{k=1}^{\infty} \subseteq T_{1,n}$ with $t^k \rightarrow t$ as $k \rightarrow \infty$; we need to show $t \in T_{1,n}$. Identify each $t^k = (A_1^k, A_2^k, \dots)$ and $t = (A_1, A_2, \dots)$. By the definition of the product topology, $A_{n+1}^k \rightarrow A_{n+1}$ as $k \rightarrow \infty$. Projection is continuous and $\text{marg}_{X_{n-1}} = \mathcal{L}_{\text{Proj}_{X_{n-1}}}$, so Lemmata 1 and 2 imply the function $\text{marg}_{X_{n-1}}^{\mathcal{K}} : \mathcal{K}(\Delta X_n) \rightarrow \mathcal{K}(\Delta X_{n-1})$ is continuous. Then the images of A_{n+1}^k approach the image of A_{n+1} : $\text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}^k \rightarrow \text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}$ as $k \rightarrow \infty$. Since each $t^k \in T_{1,n}$, by construction $\text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}^k = A_n^k$ for all k . Hence $A_n^k \rightarrow A_n$. Collecting equalities,

$$A_n = \lim_{k \rightarrow \infty} A_n^k = \lim_{k \rightarrow \infty} \left[\text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}^k \right] = \text{marg}_{X_{n-1}}^{\mathcal{K}} A_{n+1}.$$

Therefore $t \in T_{1,n}$. Since $T_1 = \bigcap_{n=1}^{\infty} T_{1,n}$ is an intersection of closed sets, the set of coherent types T_1 is itself a closed subset of T_0 .

$\mathcal{K}(\Delta S)$ is compact Polish, so each $\mathcal{K}(\Delta X_n)$ is inductively compact Polish. Because $T_0 = \prod_{n=0}^{\infty} \mathcal{K}(\Delta X_n)$ is a product of compact sets, T_0 is compact by the Tychonoff Product Theorem. Thus T_1 is a closed subset of a compact space, hence compact.

Let $\bar{T}_{0,n} = \{(A_1, A_2, \dots) \in T_0 : |A_n| = 1\}$. In Hausdorff distance, any convergent sequence of singletons approaches a singleton limit, so $\bar{T}_{0,n}$ is closed. Then $\bar{T}_0 = \bigcap_{n=1}^{\infty} \bar{T}_{0,n}$ is also closed. Thus $\bar{T}_1 = T_1 \cap \bar{T}_0$ is a closed subset of T_0 , hence compact. \square

3 The universal type space

Coherence removes one level of uncertainty from the model: each coherent type for i identifies i 's set of beliefs on j 's type.

Proposition 4. *There exists a homeomorphism $f : T_1 \rightarrow \mathcal{K}(\Delta(S \times T_0))$ that is canonical in the following sense: for all $n \geq 0$,*

$$\text{marg}_{X_n}^{\mathcal{K}} \circ f = \text{Proj}_{\mathcal{K}(\Delta X_n)}.$$

Proof. S is compact Polish, so $\mathcal{K}(\Delta S)$ is compact Polish. By induction, each $\mathcal{K}(\Delta X_n)$ is hereditarily compact and Polish. Recall that \bar{T}_1 is the subset of coherent unambiguous beliefs in $\prod_{n=0}^{\infty} \mathcal{K}(\Delta X_n)$, and also naturally identified as a subset of $\prod_{n=0}^{\infty} \Delta X_n$. Define $Z_0 = X_0$ and $Z_n = \mathcal{K}(\Delta X_{n-1})$. Since each Z_n is Polish, a version of the Kolmogorov Extension Theorem [10, Lemma 1] produces a homeomorphism \bar{f} from

$$D = \left\{ (\delta_1, \delta_2, \dots) : \begin{array}{l} \delta_n \in \Delta(Z_0 \times \dots \times Z_{n-1}), \forall n \geq 1, \text{ and} \\ \text{marg}_{Z_0 \times \dots \times Z_{n-2}} \delta_n = \delta_{n-1}, \forall n \geq 2 \end{array} \right\}$$

to $\Delta(\prod_{n=0}^{\infty} Z_n)$. But, recalling the construction, $Z_0 \times \dots \times Z_n = X_n$ and $\prod_{n=0}^{\infty} Z_n = S \times T_0$, while $\bar{T}_1 = D$. So $\bar{f} : \bar{T}_1 \rightarrow \Delta(S \times T_0)$. Applying Lemma 2 to $\bar{f}^{\mathcal{K}} : \mathcal{K}(\bar{T}_1) \rightarrow \mathcal{K}(\Delta(S \times T_0))$ and its inverse proves that $\bar{f}^{\mathcal{K}}$ is a homeomorphism.

For any nonempty compact set of unambiguous coherent types $K \subseteq \bar{T}_1$, define

$$G(K) = (\text{Proj}_{\Delta X_0}^{\mathcal{K}}(K), \text{Proj}_{\Delta X_1}^{\mathcal{K}}(K), \dots).$$

In words, $G(K)$ associates a set of unambiguous beliefs with its induced sets of beliefs at each level. We have

$$\begin{aligned} G_n(K) &= \text{Proj}_{\Delta X_{n-1}}^{\mathcal{K}}(K) \\ &= \{\text{Proj}_{\Delta X_{n-1}}(\bar{t}) : \bar{t} \in K\} \\ &= \{\text{marg}_{X_{n-1}} \text{Proj}_{\Delta X_n}(\bar{t}) : \bar{t} \in K\} \\ &= \text{marg}_{X_{n-1}}^{\mathcal{K}} G_{n+1}(K), \end{aligned}$$

where the third equality follows from the coherence of each $\bar{t} \in K \subseteq \bar{T}_1$. Therefore the hierarchy $G(K)$ is coherent, so $G(\mathcal{K}(\bar{T}_1)) \subseteq T_1$. Thus $G : \mathcal{K}(\bar{T}_1) \rightarrow T_1$ takes a compact set of coherent unambiguous hierarchies and maps to a single coherent ambiguous hierarchy.

We now prove G is injective. Suppose $K, K' \in \mathcal{K}(\bar{T}_1)$ and $G(K) = G(K')$. Fix an unambiguous type $\bar{t} = (\mu_1, \mu_2, \dots) \in K$. We show $(\mu_1, \mu_2, \dots) \in K'$. Let $K'_n = \{(\mu'_1, \mu'_2, \dots) \in K' : \mu'_n = \mu_n\}$,

$$\begin{array}{ccc}
T_1 & \xrightarrow{f} & \mathcal{K}(\Delta(S \times T_0)) \\
\swarrow F & & \nearrow \bar{f}^{\mathcal{K}} \\
& & \mathcal{K}(\bar{T}_1) \\
& \nwarrow G &
\end{array}$$

Figure 1: Commutative diagram of proof of Proposition 4

so K'_n consists of the unambiguous types in K' whose beliefs on X_n agree with those of \bar{t} . Since $K'_n = K' \cap G_n^{-1}(\{\bar{t}\})$, each K'_n is closed. We will prove $\{K'_n\}_{n=1}^\infty$ has the finite intersection property. By coherence, if $\mu'_n = \mu_n$, then iteratively $\mu'_m = \mu_m$ for all $m \leq n$, since these lower order beliefs are determined by marginalization. So, for any finite n , $K'_n \subseteq \bigcap_{m \leq n} K'_m$. Thus, to demonstrate the finite intersection property, it suffices to show that each K'_n is nonempty. Since $G(K) = G(K')$, we have that $G_n(K) = G_n(K')$. Since μ_n is obviously an element of $G_n(K)$, it is also an element of $G_n(K')$. So, there must exist some $(\mu'_1, \mu'_2, \dots) \in K'$ such that $\mu'_n = \mu_n$, i.e. $K'_n = \{(\mu'_1, \mu'_2, \dots) \in K' : \mu'_n = \mu_n\}$ is nonempty. As K' is compact and $\{K'_n\}$ is a family of closed subsets with the finite intersection property, the intersection $\bigcap_{n=1}^\infty K'_n$ is nonempty. Thus there exists $(\mu'_1, \mu'_2, \dots) \in K'$ such that $\mu'_n = \mu_n$ for all levels n . Then $(\mu_1, \mu_2, \dots) = (\mu'_1, \mu'_2, \dots) \in K'$. This proves $K \subseteq K'$. Mutatis mutandis, $K' \subseteq K$. So $K = K'$. Hence G is injective.

Let

$$F(A_1, A_2, \dots) = \{\bar{t} \in \bar{T}_1 : \text{Proj}_{\Delta X_{n-1}}(\bar{t}) \in A_n, \forall n\}.$$

For any $t \in T_1$, $G(F(t)) = t$, so G is a surjection onto T_1 . Lemma 2 implies each component G_n is continuous, hence G is continuous in the product topology. Since \bar{T}_1 is compact by Lemma 3, so is $\mathcal{K}(\bar{T}_1)$. Therefore $G : \mathcal{K}(\bar{T}_1) \rightarrow T_1$ is a continuous bijection from a compact space to a metric, hence Hausdorff, space. So G and its inverse $F : T_1 \rightarrow \mathcal{K}(\bar{T}_1)$ are homeomorphisms. Finally, $f = \bar{f}^{\mathcal{K}} \circ F$ is a composition of homeomorphisms, hence the required function in the theorem, as shown in the commutative diagram of Figure 1. \square

The homeomorphism F from T_1 to $\mathcal{K}(\bar{T}_1)$ demonstrates the mathematical bite of coherency for our ambiguous hierarchies: a coherent set of beliefs is completely characterized by its coherent unambiguous selections. This bridges the homeomorphism produced by the Kolmogorov Extension Theorem on the space of coherent unambiguous beliefs to the space of all coherent beliefs.

We now inductively impose common knowledge of coherency in our model. We first introduce a notion of knowledge for multiple beliefs. If E is a closed subset of $S \times T_0$, slightly abuse notation and let ΔE denote $\{\mu \in \Delta(S \times T_0) : \mu(E) = 1\}$, which defines a closed face. Then type $t \in T_1$ knows event $E \subseteq S \times T_0$ if *all* her canonically associated beliefs put probability one on E : $f(t) \subseteq \Delta E$.² Let $\mathbf{K}_0(E) = E$ and $\mathbf{K}_m(E) = \{t \in T_1 : f(t) \subseteq \Delta(S \times \mathbf{K}_{m-1}(E))\}$ denote m -th order knowledge. Common knowledge of E is defined as $\mathbf{CK}(E) = \bigcap_{m=0}^\infty \mathbf{K}_m(E)$.

²Brandenburger and Dekel [9, 10] provide a detailed analysis of why this almost sure notion of knowledge is appropriate for the infinite space $S \times T_0$.

Lemma 5. *If E is a closed subset of $S \times T_0$, then $\mathbf{K}_m(E)$ is a closed subset of $S \times T_0$.³*

Proof. We proceed by induction. Base step: $K_0(E) = E$ is closed by assumption. Inductive step: Suppose $\mathbf{K}_m(E)$ is closed. Let $D = S \times \mathbf{K}_m(E)$. Since D is closed, ΔD is closed [1, Corollary 14.6]. It remains to show that $f^{-1}(\mathcal{K}(\Delta D))$ is closed. Since f is continuous, it suffices to prove that $\mathcal{K}(\Delta D) = \{K \in \mathcal{K}(\Delta(S \times T_0)) : K \subseteq \Delta D\}$ is a closed subfamily of $\mathcal{K}(\Delta(S \times T_0))$. Suppose $K_i \in \mathcal{K}(\Delta D)$ and $K_i \rightarrow K$. Let $x \in K$. By the definition of the Hausdorff metric, there exists a sequence $x_i \rightarrow x$ with $x_i \in K_i$ for all i . But, $x_i \in K_i \subseteq \Delta D$ and ΔD is closed, so $x = \lim x_i \in \Delta D$. So $K \subseteq \Delta D$, i.e. $K \in \mathcal{K}(\Delta D)$. Thus $\mathcal{K}(\Delta D) = \mathcal{K}(\Delta(S \times \mathbf{K}_m(E))) = \mathbf{K}_{m+1}(E)$ is closed. \square

Let $T_m = \mathbf{K}_{m-1}(S \times T_1)$, the types with m -level knowledge of coherence. For example, T_3 contains each type that: is coherent; knows that her opponent is coherent; and knows that her opponent knows that she is coherent. Let T_∞ denote the set of types with common knowledge of coherence: $T_\infty = \mathbf{CK}(S \times T_1) = \bigcap_{m=1}^{\infty} T_m$. This restriction closes the model.

Proposition 6. *There exists a canonical homeomorphism $g : T_\infty \rightarrow \mathcal{K}(\Delta(S \times T_\infty))$.*

Proof. T_1 is compact by Lemma 3, so each T_m is closed by Lemma 5. So $\Delta(S \times T_m)$ is well-defined. Let

$$\bar{T}_m = \{\bar{t} \in \bar{T}_1 : f(\bar{t}) \in \Delta(S \times T_{m-1})\} = f^{-1}(\Delta(S \times T_{m-1})),$$

recalling that $f(\bar{t})$ is a singleton if and only if $\bar{t} \in \bar{T}_1$. In words, \bar{T}_m is the set of unambiguous types which have $m - 1$ knowledge of coherence. Let $\bar{T}_\infty = \bigcap_{m=1}^{\infty} \bar{T}_m$.

We now show that $f(\bar{T}_\infty) = \Delta(S \times T_\infty)$. Pick $\bar{t} \in \bar{T}_\infty$. By definition, $[f(\bar{t})](S \times T_m) = 1$ for every m . Hence $[f(\bar{t})](S \times T_\infty) = 1$, since $S \times T_\infty = \bigcap_{m=1}^{\infty} S \times T_m$ and a countable intersection of probability one events has probability one. So $f(\bar{T}_\infty) \subseteq \Delta(S \times T_\infty)$.

In the other direction, take $\mu \in \Delta(S \times T_\infty)$. Since $T_\infty \subset T_{m-1}$ and $\mu(S \times T_\infty) = 1$, monotonicity of μ implies $\mu(S \times T_{m-1}) = 1$. Thus $\mu \in \Delta(S \times T_{m-1})$ for each $m-1$. Since $\bar{T}_m = f^{-1}(\Delta(S \times T_{m-1}))$, we have $\mu \in f(\bar{T}_m)$ for each m , i.e. $\mu \in \bigcap_{m=1}^{\infty} f(\bar{T}_m)$. Because f is bijective, it distributes intersections: $\bigcap_{m=1}^{\infty} f(\bar{T}_m) = f(\bigcap_{m=1}^{\infty} \bar{T}_m) = f(\bar{T}_\infty)$. Thus $\mu \in f(\bar{T}_\infty)$. So $\Delta(S \times T_\infty) \subseteq f(\bar{T}_\infty)$. Then $f(\bar{T}_\infty) = \Delta(S \times T_\infty)$.

Since $f = \bar{f}$ on \bar{T}_∞ , we have $\bar{f}(\bar{T}_\infty) = \Delta(S \times T_\infty)$. Then the restriction of $\bar{f}^{\mathcal{K}}$ from $\mathcal{K}(\bar{T}_\infty)$ maps onto $\mathcal{K}(S \times T_\infty)$, i.e. $\bar{f}^{\mathcal{K}} : \mathcal{K}(\bar{T}_\infty) \rightarrow \mathcal{K}(\Delta(S \times T_\infty))$ is a homeomorphism.

Recall the functions F and G from the proof of Proposition 4. Since $f = \bar{f}^{\mathcal{K}} \circ F$ and we proved that $\bar{f}^{\mathcal{K}}$ is a homeomorphism between $\mathcal{K}(\bar{T}_\infty)$ and $\mathcal{K}(\Delta(S \times T_\infty))$, it now suffices to demonstrate that F , restricted to \bar{T}_∞ , is a homeomorphism between \bar{T}_∞ and $\mathcal{K}(\bar{T}_\infty)$. Since this restriction is hereditarily injective and continuous, it only remains to show that $F(\bar{T}_\infty) = \mathcal{K}(\bar{T}_\infty)$.

Take $t \in \bar{T}_\infty$. Since $f(t) \subseteq \Delta(S \times T_m)$ and $f = \bar{f}^{\mathcal{K}} \circ F$, we have $F(t) \in \mathcal{K}(\bar{T}_m)$ for each m . Then $F(t) \subseteq \bar{T}_m$ for each m , so $F(t) \subseteq \bigcap_{m=1}^{\infty} \bar{T}_m = \bar{T}_\infty$. Thus $F(t) \in \mathcal{K}(\bar{T}_\infty)$. So $F(\bar{T}_\infty) \subseteq \mathcal{K}(\bar{T}_\infty)$.

Now take $K \subseteq \bar{T}_\infty$. Then $G(K) \in T_\infty$ because $f \circ G = \bar{f}^{\mathcal{K}}$ and $\bar{f}^{\mathcal{K}}(K) \subseteq \Delta(S \times T_\infty)$. But $F(G(K)) = K$, so $F(\bar{T}_\infty) \supseteq \mathcal{K}(\bar{T}_\infty)$. Thus $F(\bar{T}_\infty) = \mathcal{K}(\bar{T}_\infty)$, so $f(\bar{T}_\infty) = \mathcal{K}(\Delta(S \times T_\infty))$. Hence

³We thank a referee for suggesting this step and essentially providing the proof.

$$\begin{array}{ccc}
T_\infty & \xrightarrow{g} & \mathcal{K}(\Delta(S \times T_\infty)) \\
& \begin{array}{c} \swarrow F \\ \searrow G \end{array} & \nearrow f^\mathcal{K} \\
& & \mathcal{K}(\bar{T}_\infty)
\end{array}$$

Figure 2: Commutative diagram of proof of Proposition 6

the restriction $g = f|_{T_\infty}$ is the desired canonical homeomorphism. The relationship is shown in the commutative diagram in Figure 2. \square

In applications, a complete type space T , admitting a surjection onto $\mathcal{K}(\Delta(S \times T))$, would often be sufficient. Many such complete structures exist.⁴ Several features distinguish the particular type space we constructed. First, T_∞ was canonically selected from explicit hierarchies, which seems methodologically sensible and avoids redundant types. Second, and more important, we now show that T_∞ is universal. Formally, if T is an implicit type space continuously mapped to $\mathcal{K}(\Delta(S \times T))$, then T is uniquely and continuously embedded into T_∞ in a manner preserving the implicit higher-order beliefs.

This can be proven directly using arguments analogous to those used by Mertens and Zamir [31] or by Battigali and Siniscalchi [4]. Here, we adapt a novel indirect architecture, originally developed by Mariotti, Meier, and Piccione [30, Section 3.2], both for its structural elegance and for the clarity it later provides in comparing T_∞ to the infinite consumption problems of Gul and Pesendorfer [18]. This strategy first demonstrates that an alternative space is universal. Then, it proves that T_∞ is homeomorphic to this alternative space, hence itself universal.

We begin by constructing our analog to the $*$ -beliefs of Mariotti, Meier, and Piccione [30]. These $*$ -beliefs are distinct from the hierarchies considered so far, because $*$ -beliefs do not carry the lower level space as separate component of the higher level space. Instead, the higher level space is a product of the primitive state space and beliefs on prior level space. Formally, let

$$\begin{aligned}
X_0^* &= S \\
&\vdots \\
X_{n+1}^* &= S \times \mathcal{K}(\Delta X_n^*) \\
&\vdots
\end{aligned}$$

A $*$ -hierarchy is an sequence $(A_1^*, A_2^*, \dots) \in \prod_{n=0}^{\infty} \mathcal{K}(\Delta X_n^*) = T_0^*$. Let $P_1 : \mathcal{K}(\Delta X_1^*) \rightarrow \mathcal{K}(\Delta S)$ be defined by $P_1(A_2^*) = \text{marg}_S^{\mathcal{K}} A_2$. Consider the image measure $\mathcal{L}_{(\text{Id}_S; P_n)} : \Delta X_n^* \rightarrow \Delta X_{n-1}^*$ induced by the function $(\text{Id}_S; P_n) : S \times \mathcal{K}(\Delta X_{n+1}^*) \rightarrow S \times \mathcal{K}(\Delta X_n^*)$. Then inductively define $P_{n+1} :$

⁴For example, the Cantor set \mathcal{C} immediately provides a complete type space, because any compact metric space, such as $\mathcal{K}(\Delta(S \times \mathcal{C}))$ can be expressed as a continuous image of the Cantor set; this general and elegant argument was first used by Brandenburger, Friedenberg, and Keisler [11, Proof of Proposition 6.1, p. 29]. We thank an associate editor for pointing this out to us.

$\mathcal{K}(\Delta X_{n+1}^*) \rightarrow \mathcal{K}(\Delta X_n^*)$ by

$$P_{n+1} = \mathcal{L}_{(\text{Id}_S; P_n)}^{\mathcal{K}}.$$

Definition 2. A $*$ -hierarchy (A_1^*, A_2^*, \dots) is *coherent* if $A_n^* = P_n(A_{n+1}^*)$ for all $n \geq 1$.

Denote the space of all coherent $*$ -hierarchies as T_1^* .⁵ Suppose $f : T \rightarrow \mathcal{K}(\Delta(S \times T))$ is continuous, so T is a type space. We now show how to embed T into T_1^* . We first naively unpack each type $t \in T$ to its implied hierarchy of beliefs on the state space $S \times T$. Let

$$\begin{aligned} \hat{X}_0 &= S \times T \\ &\vdots \\ \hat{X}_{n+1} &= S \times \mathcal{K}(\Delta \hat{X}_n) \\ &\vdots \end{aligned}$$

Let $R_0 = f : T \rightarrow \mathcal{K}(\Delta \hat{X}_0)$. Consider $\mathcal{L}_{(\text{Id}_S; R_n)} : \Delta \hat{X}_n \rightarrow \Delta \hat{X}_{n+1}$. Inductively define $R_{n+1} : \mathcal{K}(\Delta \hat{X}_n) \rightarrow \mathcal{K}(\Delta \hat{X}_{n+1})$ by

$$R_{n+1} = \mathcal{L}_{(\text{Id}_S; R_n)}^{\mathcal{K}}.$$

Any type $t \in T$ identifies a hierarchy $(\hat{A}_1, \hat{A}_2, \dots) \in \prod_{n=0}^{\infty} \mathcal{K}(\Delta \hat{X}_n)$ of joint beliefs on the state space and the type space by setting $\hat{A}_n = [R_{n-1} \circ \dots \circ R_0](t)$. But, these identified hierarchies depend on the particular type space T , while a truly explicit description of beliefs should make no reference to the type structure. We identify each naive hierarchy with a $*$ -hierarchy by recursively marginalizing out notational artifacts involving the type space T from the expression. The rest of this section formalizes and studies this identification.

Let $Q_0 : \mathcal{K}(\Delta \hat{X}_0) \rightarrow \mathcal{K}(\Delta X_0^*)$ be defined by $Q_0(\hat{A}_1) = \text{marg}_S^{\mathcal{K}} \hat{A}_1$. In words, Q_0 takes a set of beliefs on $S \times T$ and integrates out the dependence on the type space T , leaving only a belief on S . We now recursively carry this marginalization to higher orders. Define $Q_{n+1} : \mathcal{K}(\Delta \hat{X}_{n+1}) \rightarrow \mathcal{K}(\Delta X_{n+1}^*)$ by

$$Q_{n+1} = \mathcal{L}_{(\text{Id}_S; Q_n)}^{\mathcal{K}}.$$

In words, Q_n applies the marginalization from the previous level to the second dimension of the next level. We can apply the appropriate marginalization level by level. Then $(\hat{A}_1, \hat{A}_2, \dots)$ identifies the $*$ -hierarchy $(Q_0(\hat{A}_1), Q_1(\hat{A}_2), \dots) \in \prod_{n=0}^{\infty} \mathcal{K}(\Delta X_n^*)$.

Suppose $f : T \rightarrow \mathcal{K}(\Delta(S \times T))$ is continuous. Define $\varphi_{T,f} : T \rightarrow T_0^*$ by $(\varphi_{T,f})_n = Q_{n-1} \circ R_{n-1} \circ \dots \circ R_0$, for $n \geq 1$. Thus φ composes the two procedures: first each type $t \in T$ is taken to a naive hierarchy $(\hat{A}_1, \hat{A}_2, \dots)$ of beliefs on the space $S \times T$; then $(\hat{A}_1, \hat{A}_2, \dots)$ is taken to the space of $*$ -hierarchies by integrating out the type space at each order.

This function $\varphi_{T,f}$ continuously maps each type in T to a coherent $*$ -hierarchy, embedding T into T_1^* . Moreover, the embedding of the previously constructed space T_∞ covers all of T_1^* . Hence T_1^* is isomorphic to T_∞ , demonstrating that T_∞ is indeed universal.

⁵The space T_1^* is mathematically identical to the class of infinite horizon consumption problems considered in Gul and Pesendorfer [18, Appendix A]; we will compare the models explicitly in Section 4.

Proposition 7. *Suppose $f : T \rightarrow \mathcal{K}(\Delta(S \times T))$ is continuous. Then:*

1. $\varphi_{T,f}$ continuously maps T into T_1^* ;
2. $\varphi_{T_\infty,g} : T_\infty \rightarrow T_1^*$ is a homeomorphism.

Since $\varphi_{T_\infty,g}$ is a homeomorphism from T_∞ to T_1^* , the composition $g^* = \mathcal{L}_{(\text{Id}_S; \varphi_{T_\infty,g})}^{\mathcal{K}} \circ g \circ \phi_{T_\infty,g}^{-1}$ defines a homeomorphism from T_1^* to $\mathcal{K}(\Delta(S \times T_1^*))$. Moreover, g^* canonically satisfies $P_n \circ g^* = \text{Proj}_{\mathcal{K}(\Delta X_{n-1}^*)}$. Then $\varphi_{T,f}$ preserves the hierarchies implicitly coded in T because $g^* \circ \varphi_{T,f} = \mathcal{L}_{(\text{Id}_S; \varphi_{T,f})}^{\mathcal{K}} \circ f$ from T to $\mathcal{K}(S \times T_1^*)$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
 T & \xrightarrow{\varphi_{T,f}} & T_1^* \\
 \downarrow f & & \downarrow g^* \\
 \mathcal{K}(\Delta(S \times T)) & \xrightarrow{\mathcal{L}_{(\text{Id}_S; \varphi_{T,f})}^{\mathcal{K}}} & \mathcal{K}(\Delta(S \times T_1^*))
 \end{array}$$

Finally, $\varphi_{T,f}$ is the unique map from T to T_1^* with this property. The following result can be proven by adapting the argument outlined by [30, p. 312].

Proposition 8. *If $\phi : T \rightarrow T_1^*$ satisfies $g^* \circ \phi = \mathcal{L}_{(\text{Id}_S; \phi)}^{\mathcal{K}} \circ f$, then $\phi = \varphi_{T,f}$.*

4 Extensions and related literature

Here, we outline relationships between our space and others in the literature. Proofs can be found in an [online appendix](#) linked to the author's website.

4.1 Conditional probability systems

Simple probabilities are generally inadequate as descriptions of beliefs in extensive form games. For example, a fully specified strategy must assign actions on subtrees that may occur with probability zero. In an important paper, Battigalli and Siniscalchi [4], henceforth BS, consider hierarchies of conditional probability systems, providing a formal language for epistemic analysis of dynamic games [4, 5, 6]. Our model easily extends to such systems. To our knowledge, this extension is the first universal space designed to address interactive ambiguity for dynamic games.

We briefly review the model of BS; their original paper gives a comprehensive discussion. Suppose X is a compact Polish space, with Borel σ -algebra \mathcal{A} , and \mathcal{B} is a nonempty countable collection of clopen subsets of X , which are interpreted as conditioning events.

Definition 3. A conditional probability system (CPS) on (X, \mathcal{B}) is a mapping $\mu(\cdot|\cdot) : \mathcal{A} \times \mathcal{B} \rightarrow [0, 1]$ such that:

1. For all $B \in \mathcal{B}$, $\mu(B|B) = 1$;
2. For all $B \in \mathcal{B}$, $\mu(\cdot|B) \in \Delta X$;
3. For all $A \in \mathcal{A}$ and $B, C \in \mathcal{B}$, if $A \subseteq B \subseteq C$, then $\mu(A|B)\mu(B|C) = \mu(A|C)$.

Let $\Delta^{\mathcal{B}}X$ denote the space of all CPS's on (X, \mathcal{B}) . Since $\mu(\cdot|B) \in \Delta X$, we can identify $\Delta^{\mathcal{B}}X \subset [\Delta X]^{\mathcal{B}}$ and endow $\Delta^{\mathcal{B}}X$ with the relative product topology and its Borel σ -algebra. BS prove that $\Delta^{\mathcal{B}}X$ is compact Polish whenever X is compact Polish.

As before, S is compact Polish, but is now equipped with a countable collection \mathcal{B} of conditioning hypotheses. Let

$$\begin{array}{ll}
X_0 = S & \mathcal{B}_0 = \mathcal{B} \\
\vdots & \vdots \\
X_{n+1} = X_n \times \mathcal{K}(\Delta^{\mathcal{B}_n} X_n) & \mathcal{B}_{n+1} = \{B \times \mathcal{K}(\Delta^{\mathcal{B}_n} X_n) : B \in \mathcal{B}_n\} \\
\vdots & \vdots
\end{array}$$

The right column projectively carries each basic conditioning event B to its cylinders $\mathcal{C}_n(B) = B \times \prod_{m=0}^{n-1} \mathcal{K}(\Delta^{\mathcal{B}_m} X_m)$ in higher order spaces. Since each $B_n \in \mathcal{B}_n$ is identified with a unique cylinder $\mathcal{C}_n(B)$, we abuse notation and write $\Delta^{\mathcal{B}}X_n$ for $\Delta^{\mathcal{B}_n} X_n$. An ambiguous conditional type is a hierarchy of sets of CPS's $(A_1, A_2, \dots) \in \prod_{n=0}^{\infty} \mathcal{K}(\Delta^{\mathcal{B}} X_n) = H_0$. For any set $A_n \in \mathcal{K}(\Delta^{\mathcal{B}} X_{n-1})$, let $A_n(\cdot|B) = \{\mu_n(\cdot|B) : \mu_n \in A_n\}$ for all $B \in \mathcal{B}$.⁶

The following is a simple generalization of the definition of coherence by BS: a hierarchy is coherent if every induced conditional hierarchy is coherent in the sense of Definition 1.

Definition 4. An ambiguous conditional type $(A_1, A_2, \dots) \in H_0$ is *coherent* if

$$A_n(\cdot|\mathcal{C}_{n-1}(B)) = \text{marg}_{\mathcal{K}_{X_{n-1}}}^{\mathcal{K}} A_{n+1}(\cdot|\mathcal{C}_n(B)).$$

Let H_1 denote the space of all coherent ambiguous conditional types.

Proposition 9. *There exists a homeomorphism $f : H_1 \rightarrow \mathcal{K}(\Delta^{\mathcal{B}}(S \times H_0))$ that is canonical in the following sense:*

$$[\text{marg}_{\mathcal{K}_{X_{n-1}}}^{\mathcal{K}} f(h)](\cdot|\mathcal{C}_{\infty}(B)) = [\text{Proj}_{\mathcal{K}(\Delta^{\mathcal{B}} X_{n-1})}(h)](\cdot|\mathcal{C}_{n-1}(B)).$$

Let $H_{m+1} = \{h \in H_1 : [f(h)](\cdot|B) \subseteq \Delta(S \times H_m), \forall B \in \mathcal{B}\}$ and $H_{\infty} = \bigcap_{k=1}^{\infty} H_k$. Then H_{∞} corresponds to common knowledge of coherence and closes the model.

⁶Here, we have slightly expanded the model of BS to allow for sets of CPS's; beliefs are expressed as $\mathcal{K}(\Delta^{\mathcal{B}}X)$ rather than as $\Delta^{\mathcal{B}}X$. Another approach might allow for conditional sets of probability measures, which could be formally expressed as subsets of $[\mathcal{K}(\Delta X)]^{\mathcal{B}}$. We chose the former strategy because the generalization Bayes rule, the third part of Definition 3, for sets of beliefs is unclear and an active area of inquiry even for updating problems on non-null sets.

Proposition 10. *There exists a canonical homeomorphism $g : H_\infty \rightarrow \mathcal{K}(\Delta^{\mathcal{B}}(S \times H_\infty))$.*

Finally, the arguments in BS and in Section 3 can be modified to demonstrate that H_∞ is universal: if H admits a continuous function $f : H \rightarrow \mathcal{K}(\Delta^{\mathcal{B}}(S \times H))$, then H is embedded in H_∞ in a unique beliefs-preserving manner.

4.2 Standard hierarchies of beliefs

MZ and BD construct a space $\Theta_\infty^{\text{MZ}}$ representing all coherent unambiguous hierarchies where $X_0^{\text{MZ}} = S$ and $X_{n+1}^{\text{MZ}} = X_n^{\text{MZ}} \times \Delta X_n^{\text{MZ}}$. Any standard type space Θ equipped with a continuous function $f : \Theta \rightarrow \Delta(S \times \Theta)$ defines an ambiguous type space, by naturally translating $f : \Theta \rightarrow \mathcal{K}(\Delta(S \times \Theta))$. Then Θ is embedded as a subset of our universal space T_∞ . In particular, the standard universal type space $\Theta_\infty^{\text{MZ}}$ is subset of T_∞ .

We can identify the embedding with explicit epistemic conditions. Assuming probabilistic sophistication, restricting attention to singleton beliefs in \bar{T}_∞ , does not guarantee that i knows that j 's beliefs are unambiguous. But common knowledge of probabilistic sophistication does work and identifies the embedding exactly. Let $T_\infty^{\text{MZ}} = \mathbf{CK}(S \times \bar{T}_\infty)$, those types with common knowledge of coherence and of probabilistic sophistication, which reduces the model to the standard case.

Proposition 11. *T_∞^{MZ} is homeomorphic to $\Theta_\infty^{\text{MZ}}$.*

4.3 Hierarchies of compact possibilities

Mariotti, Meier, and Piccione [30], hereafter MMP, consider compact continuous possibility structures, consisting of a space Θ and a continuous map $f : \Theta \rightarrow \mathcal{K}(S \times \Theta)$. Such types are obviously related to partitioned representations of knowledge and possible states. Let $X_0^{\text{MMP}} = S$ and $X_{n+1}^{\text{MMP}} = X_n^{\text{MMP}} \times \mathcal{K}(X_n^{\text{MMP}})$. The space of MMP-hierarchies is $\Theta_0^{\text{MMP}} = \prod_{n=0}^\infty \mathcal{K}(X_n^{\text{MMP}})$. An MMP-hierarchy $(A_1^{\text{MMP}}, A_2^{\text{MMP}}, \dots)$ is MMP-coherent if, for all $n \geq 1$:

$$A_n^{\text{MMP}} = \text{Proj}_{X_{n-1}^{\text{MMP}}} A_{n+1}^{\text{MMP}}.$$

MMP construct a universal space $\Theta_\infty^{\text{MMP}}$ homeomorphic to $\mathcal{K}(S \times \Theta_\infty^{\text{MMP}})$.

Since a marginal measure is the image measure induced by a projection function, MMP-coherence on projections naturally translates to our definition. Let $\delta : X \rightarrow \Delta X$ continuously map each $x \in X$ to its Dirac measure $\delta(x)$. Observe that $\text{marg}_X \circ \delta = \delta \circ \text{Proj}_X$. A possibility hierarchy $(A_1^{\text{MMP}}, A_2^{\text{MMP}}, \dots)$ is MMP-coherent if and only if the induced hierarchy of ambiguous beliefs $(\delta^{\mathcal{K}}(A_1^{\text{MMP}}), \delta^{\mathcal{K}}(A_2^{\text{MMP}}), \dots)$ is coherent. Then any possibility structure Θ defines an ambiguous type structure, by $\delta^{\mathcal{K}} \circ f : \Theta \rightarrow \mathcal{K}(S \times \Theta) \rightarrow \mathcal{K}(\Delta(S \times \Theta))$, and hence embedded in T_∞ . In particular, MMP's universal space $\Theta_\infty^{\text{MMP}}$ is embedded.

This embedding can be made more explicit. Let $T_\infty^{\text{MMP}} = \{t \in T_\infty : g(t) \subseteq \delta(S \times T_\infty)\}$, the set of resolute types with Dirac beliefs. Let $T_\infty^{\text{MMP}} = \mathbf{CK}(S \times T_\infty^{\text{MMP}})$, the set of types with common knowledge of resoluteness.

Proposition 12. T_∞^{MMP} is homeomorphic to $\Theta_\infty^{\text{MMP}}$.

As MMP carefully explain, the standard type space of MZ and BD cannot be embedded as a possibility structure; nor does the opposite embedding exist. By working with both possibility and probability, we encompass both spaces and hope this provides some unifying perspective.

4.4 Infinite horizon consumption problems

Gul and Pesendorfer [18, Appendix A] examine a similar technical structure to model recursive consumption problems, extending a construction by Epstein and Zin [16]. They construct a space of recursive consumption problems Z homeomorphic to $\mathcal{K}(\Delta(S \times Z))$. The space Z is not obviously homeomorphic to T_∞ , since there are spaces T which are homeomorphic to $\mathcal{K}(\Delta(S \times T))$, but topologically distinct from T_∞ . Nor is it obvious that Z is universal and captures all recursive formulations. However, their Z is exactly T_1^* , the space of coherent $*$ -hierarchies of beliefs. Then, corollary to Proposition 7, Z is indeed homeomorphic to T_∞ , hence embeds all recursive consumption problems.

Besides the interpretative difference, our construction is formally distinct: we set $X_{n+1} = X_n \times \mathcal{K}(\Delta X_{n+1})$, replacing S with X_n . This replacement allows separate components at $n + 1$ regarding others' beliefs at lower orders. Removing these independent lower order components precludes a formal expression of common or m -level knowledge of coherence, while our ambient type space is designed to highlight its implications. Therefore, we do not consider our construction subsidiary, as the results of BD similarly do not follow from those in Epstein and Zin [16].

4.5 Hierarchies of preferences

The seminal work of Epstein and Wang [15], henceforth EW, provides another notion of type which deviates from the standard model. Their departure is much more fundamental than ours, dropping the dependence on probability and belief entirely. Instead, types are hierarchies of *preferences* over acts, rather than hierarchies of beliefs. Using this novel space, [14] provides epistemic foundations for game theoretic equilibria.

By assuming particular forms of preference, one can relate subsets of T_∞ to subsets of the EW space of preference hierarchies. As EW point out, common knowledge of Choquet expected utility [32] with a fixed vNM index identifies a space T^C homeomorphic to the family $\mathcal{C}(S \times T^C)$ of regular capacities on $S \times T^C$. Under common knowledge of maxmin expected utility [17] with a fixed vNM index, one can similarly derive a space T^M homeomorphic to the family $\mathcal{K}_C(\Delta(S \times T^M))$ of convex compact sets of probabilities on $S \times T^M$. Since $\mathcal{K}_C(\Delta(S \times T^M)) \subset \mathcal{K}(\Delta(S \times T^M))$, the universality result Proposition 7 implies that T^M can be embedded as a subset of T_∞ .⁷

The examples assume a specific class of preferences with a fixed vNM index is commonly known, which might be unnatural in games. For example, many models specify uncertainty by varying

⁷While often natural, convexity is sometimes restrictive. For example, sets of Dirac measures are not convex, so T^M does not embed the possibility structures of MMP.

players' payoffs with states of the world. Such state dependence confounds a player's belief and vNM index, precluding any meaningful identification of common knowledge of the latter. We do not know of an approach linking general preferences to compact sets of beliefs, nor a particular class of preferences that can derive the existence of a set T homeomorphic to the entirety of $\mathcal{K}(\Delta(S \times T))$. We therefore consider the two approaches as loosely orthogonal, but with at least some specific intersections as outlined above.

By working only with preferences, EW cannot distinguish the existence of ambiguity from its resolution; by working only with beliefs, neither can we. We take the ambiguity as fixed, but remain agnostic on how this ambiguity is resolved by preference. On the other hand, EW take the preference as fixed, but remain agnostic on the exact nature of ambiguity. We believe both approaches have merits in particular applications.

A Appendix

A.1 Proof of Lemma 1

(1) Observe $\mu(g^{-1}(f^{-1}(E))) = \mu((g \circ f)^{-1}(E))$.

(2) Suppose μ_n weakly converges to $\mu \in \Delta X$. Fix any continuous and bounded $g : Y \rightarrow \mathbb{R}$. Then $\int_Y g d\mathcal{L}_f(\mu_n) = \int_X g \circ f d\mu_n \rightarrow \int_X g \circ f d\mu = \int_Y g d\mathcal{L}_f(\mu)$; convergence follows because μ_n weakly converges to μ and $g \circ f : X \rightarrow \mathbb{R}$ is continuous and bounded. So $\mathcal{L}_f(\mu_n)$ weakly converges to $\mathcal{L}_f(\mu)$.

(3) Suppose f is injective. Then $f^{-1}(f(D)) = D$ for any $D \subseteq X$. Suppose μ, μ' are distinct probability measures on X . There exists some Borel set $D \subseteq X$ such that $\mu(D) \neq \mu'(D)$. Measurable injections between Polish spaces preserve Borel measurability, so $f(D)$ is a Borel subset of Y [1, Theorem 10.28]. Then $\mu(f^{-1}(f(D))) \neq \mu'(f^{-1}(f(D)))$, so $[\mathcal{L}_f(\mu)](f(D)) \neq [\mathcal{L}_f(\mu')](f(D))$. Thus $\mathcal{L}_f(\mu) \neq \mathcal{L}_f(\mu')$.

(4) Suppose f is surjective. Fix $\nu \in \Delta Y$. Let $M_E = \{\mu \in \Delta X : \mu(f^{-1}(E)) = \nu(E)\}$, for any nonempty Borel set $E \subseteq Y$. Each M_E is a closed subset of ΔX [1, Corollary 14.6]. We now demonstrate the finite intersection property for the family $\{M_E : \text{Borel } E \subseteq Y\}$. Consider any finite family E_1, \dots, E_m of Borel subsets of Y . Eliminate intersections through the standard construction: let $E'_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$ and $E'_{m+1} = Y \setminus \bigcup_{i=1}^m E_i$. Because E'_1, \dots, E'_{m+1} constitutes a partition of Y and ν is additive, $\mu(f^{-1}(E'_k)) = \nu(E'_k)$ for all $k = 1, \dots, m+1$ implies $\mu(f^{-1}(E_k)) = \nu(E_k)$ for all $k = 1, \dots, m$. We lose no generality by assuming that each E'_k is nonempty, since $\mu(f^{-1}(\emptyset)) = \nu(\emptyset) = 0$. Since f is surjective, $f^{-1}(E)$ is nonempty for any nonempty $E \subseteq Y$. A probability measure μ such that $\mu(f^{-1}(E'_k)) = \nu(E'_k)$ for all $k = 1, \dots, m+1$ can be obviously constructed as a convex combination of $m+1$ point masses on selections $x_k \in f^{-1}(E'_k)$ from each k . The collection $\{M_E : \text{Borel } E \subseteq Y\}$ is a collection of closed subsets with the finite intersection property and ΔX is compact. Then any $\mu \in \bigcap_{\text{Borel } E} M_E \neq \emptyset$ satisfies $\mathcal{L}_f(\mu) = \nu$.

A.2 Proof of Lemma 2

The first, third, and fourth claims follow immediately from definitions. The second claim is proven more generally in [30]; we provide a proof for the metric definition we use here. Fix $\varepsilon > 0$. Let d, e respectively refer to the metrics on X, Y and let d_h, e_h refer to their induced Hausdorff metrics. By continuity of f , there exists $\gamma > 0$ such that $e(f(x), f(x')) < \varepsilon$ whenever $d(x, x') < \gamma$. Take $K, K' \in \mathcal{K}(X)$ with $d_h(K, K') < \gamma$. By definition of d_h , $\max_{x \in K} \min_{x' \in K'} d(x, x') < \gamma$. Fix $x \in K$ and let $x'_* \in \arg \min_{x' \in K'} d(x, x')$, which exists since K is compact and d is continuous. Then $d(x, x'_*) < \gamma$. By selection of γ , $e(f(x), f(x'_*)) < \varepsilon$. Then $\min_{x' \in K'} e(f(x), f(x')) \leq e(f(x), f(x'_*)) < \varepsilon$. Since $x \in K$ was arbitrary, this implies $\max_{x \in K} \min_{x' \in K'} e(f(x), f(x')) < \varepsilon$. Similarly, $\max_{x' \in K'} \min_{x \in K} e(f(x), f(x')) < \varepsilon$. Thus $e_h(f^\mathcal{K}(K), f^\mathcal{K}(K')) < \varepsilon$, therefore $f^\mathcal{K}$ is continuous.

A.3 Proof of Proposition 7

We begin by demonstrating the following result.

Lemma 13. $Q_n = P_{n+1} \circ Q_{n+1} \circ R_{n+1}$, for all $n \geq 0$, i.e. the following diagram commutes:

$$\begin{array}{ccccccc}
 T & \xrightarrow{R_0} & \mathcal{K}(\Delta \hat{X}_0) & \xrightarrow{R_1} & \mathcal{K}(\Delta \hat{X}_1) & \xrightarrow{R_2} & \mathcal{K}(\Delta \hat{X}_2) \xrightarrow{R_3} \dots \\
 & & \downarrow Q_0 & & \downarrow Q_1 & & \downarrow Q_2 \\
 & & \mathcal{K}(\Delta X_0^*) & \xleftarrow{P_1} & \mathcal{K}(\Delta X_1^*) & \xleftarrow{P_2} & \mathcal{K}(\Delta X_2^*) \xleftarrow{P_3} \dots
 \end{array}$$

Proof. The proof is by induction on n . First:

$$\begin{aligned}
 P_1 \circ Q_1 \circ R_1 &= \text{marg}_S^\mathcal{K} \circ \mathcal{L}_{(\text{Id}_S; Q_0)}^\mathcal{K} \circ \mathcal{L}_{(\text{Id}_S; R_0)}^\mathcal{K} \\
 &= \mathcal{L}_{\text{Proj}_S}^\mathcal{K} \circ \mathcal{L}_{(\text{Id}_S; Q_0)}^\mathcal{K} \circ \mathcal{L}_{(\text{Id}_S; R_0)}^\mathcal{K} \\
 &= \mathcal{L}_{\text{Proj}_S \circ (\text{Id}_S; Q_0) \circ (\text{Id}_S; R_0)}^\mathcal{K}, \text{ by Lemmata 1 and 2} \\
 &= \mathcal{L}_{\text{Proj}_S}^\mathcal{K} = \text{marg}_S = Q_0
 \end{aligned}$$

Now, suppose $Q_{n-1} = P_n \circ Q_n \circ R_n$. Then:

$$\begin{aligned}
 P_{n+1} \circ Q_{n+1} \circ R_{n+1} &= \mathcal{L}_{(\text{Id}_S; P_n)}^\mathcal{K} \circ \mathcal{L}_{(\text{Id}_S; Q_n)}^\mathcal{K} \circ \mathcal{L}_{(\text{Id}_S; R_n)}^\mathcal{K} \\
 &= \mathcal{L}_{(\text{Id}_S; P_n) \circ (\text{Id}_S; P_n) \circ (\text{Id}_S; R_n)}^\mathcal{K}, \text{ by Lemmata 1 and 2} \\
 &= \mathcal{L}_{(\text{Id}_S; P_n \circ Q_n \circ R_n)}^\mathcal{K} \\
 &= \mathcal{L}_{(\text{Id}_S; Q_{n-1})}^\mathcal{K}, \text{ by hypothesis} \\
 &= Q_n
 \end{aligned}$$

□

We now prove the first claim. $R_0 = f$ is continuous by assumption. Lemma 2 implies $Q_0 = \text{marg}_S^{\mathcal{K}}$ is continuous. Inductively, $R_{n+1} = \mathcal{L}_{(\text{Id}_S; R_n)}^{\mathcal{K}}$ and $Q_{n+1} = \mathcal{L}_{(\text{Id}_S; R_n)}^{\mathcal{K}}$ are continuous by Lemmata 1 and 2. Because each component of $\varphi_{T,f}$ is a composition of continuous functions, $\varphi_{T,f}$ is continuous. Coherence of $\varphi_{T,f}(t)$ follows immediately from Lemma 13.

We now prove the second claim. We begin by proving that $\varphi_{T_\infty, g}$ is surjective. Obviously $R_0 = g$ is homeomorphic. Now suppose R_n is homeomorphic. Then (Id_S, R_n) is homeomorphic, so $R_{n+1} = \mathcal{L}_{(\text{Id}_S, R_n)}$ is homeomorphic by Lemma 1. So each R_n is onto. Similarly, each Q_n is onto. Fix some *-hierarchy $(A_1^*, A_2^*, \dots) \in T_1^*$. Let $D_n = \{t \in T_\infty : [Q_{n-1} \circ R_{n-1} \circ \dots \circ R_0](t) = A_{n+1}^*\}$. D_n is the inverse image of the singleton family $\{A_{n+1}^*\}$, which is trivially closed in the Hausdorff metric, hence itself closed. By the commutativity established in Lemma 13 and the coherence of T_1^* , $D_n \subseteq \bigcap_{m \leq n} D_m$. But D_n is nonempty, because $Q_{n-1} \circ R_{n-1} \circ \dots \circ R_0$ is onto, so $\bigcap_{m \leq n} D_m$ is nonempty. Thus $\{D_n\}$ is a family of closed subsets of T_∞ with the finite intersection property and T_∞ is compact, so there exists some $t \in \bigcap_{n=1}^\infty D_n$. But then $\varphi_{T_\infty, g}(t) = (A_1^*, A_2^*, \dots)$. So $\varphi_{T_\infty, g}$ is onto.

Next, $\varphi_{T_\infty, g}$ is continuous and injective. Let $\mathcal{K}_n = \text{Proj}_{\mathcal{K}(\Delta X_{n-1})}(T_\infty)$ and

$$\mathcal{K}^n = [\text{Proj}_{\mathcal{K}(\Delta X_0)}, \dots, \text{Proj}_{\mathcal{K}(\Delta X_{n-1})}](T_\infty).$$

Define $\tau_n : \mathcal{K}^n \rightarrow \mathcal{K}_n$ by $\tau_n : (A_1, \dots, A_n) \mapsto A_n$. By iterated application of coherence, τ_n is homeomorphic. Let $f_1 : \mathcal{K}_1 \rightarrow \mathcal{K}(\Delta S)$ be defined by $f_1 = \text{Id}_{\mathcal{K}(\Delta S)}$. Clearly, f_1 is continuously injective. By canonicity of g , $f_1 \circ \text{Proj}_{\mathcal{K}(\Delta S)} = \text{marg}_S \circ g = Q_0 \circ R_0 = (\varphi_{T_\infty, g})_1$.

$$\begin{aligned} Q_n \circ R_n \circ \dots \circ R_0 &= \mathcal{L}_{(\text{Id}_S, Q_{n-1})}^{\mathcal{K}} \circ \mathcal{L}_{(\text{Id}_S, R_{n-1})}^{\mathcal{K}} \circ \dots \circ \mathcal{L}_{(\text{Id}_S, R_0)}^{\mathcal{K}} \circ g \\ &= \mathcal{L}_{(\text{Id}_S, Q_{n-1} \circ R_{n-1} \circ \dots \circ R_0)}^{\mathcal{K}} \circ g \\ &= \mathcal{L}_{(\text{Id}_S, f_n \circ \text{Proj}_{\mathcal{K}(\Delta X_{n-1})})}^{\mathcal{K}} \circ g \\ &= \mathcal{L}_{(\text{Id}_S, f_n \circ \tau_n \circ [\text{Proj}_{\mathcal{K}(\Delta X_0)}, \dots, \text{Proj}_{\mathcal{K}(\Delta X_0)}])}^{\mathcal{K}} \circ g \\ &= \mathcal{L}_{(\text{Id}_S, f_n \circ \tau_n)}^{\mathcal{K}} \circ \mathcal{L}_{(\text{Id}_S, [\text{Proj}_{\mathcal{K}(\Delta X_0)}, \dots, \text{Proj}_{\mathcal{K}(\Delta X_0)}])}^{\mathcal{K}} \circ g \\ &= \mathcal{L}_{(\text{Id}_S, f_n \circ \tau_n)}^{\mathcal{K}} \circ \mathcal{L}_{[\text{Proj}_S, \text{Proj}_{\mathcal{K}(\Delta X_0)}, \dots, \text{Proj}_{\mathcal{K}(\Delta X_0)}]}^{\mathcal{K}} \circ g \\ &= \mathcal{L}_{(\text{Id}_S, f_n \circ \tau_n)}^{\mathcal{K}} \circ \mathcal{L}_{\text{Proj}_{X_n}}^{\mathcal{K}} \circ g \\ &= \mathcal{L}_{(\text{Id}_S, f_n \circ \tau_n)}^{\mathcal{K}} \circ \text{marg}_{X_n}^{\mathcal{K}} \circ g, \text{ as } g \text{ is canonical} \\ &= \mathcal{L}_{(\text{Id}_S, f_n \circ \tau_n)}^{\mathcal{K}} \circ \text{Proj}_{\mathcal{K}(\Delta X_{n+1})}. \end{aligned}$$

Thus $f_{n+1} = \mathcal{L}_{(\text{Id}_S, f_n \circ \tau_n)}^{\mathcal{K}}$ is a continuous injection satisfying $f_n \circ \text{Proj}_{\mathcal{K}(\Delta X_n)} = (\varphi_{T_\infty, g})_n$.

Now, suppose $\varphi_{T_\infty, g}(t) = \varphi_{T_\infty, g}(t')$. Since each f_n is injective, it follows that $\text{Proj}_{\mathcal{K}(\Delta X_n)}(t) = \text{Proj}_{\mathcal{K}(\Delta X_n)}(t')$ for all n , i.e. that $t = t'$. Thus $\varphi_{T_\infty, g}$ is injective. Being a continuous bijection between compact sets, $\varphi_{T_\infty, g}$ is a homeomorphism.

References

- [1] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis*. Springer, New York, 2nd edition, 1999.
- [2] W. Armbruster and W. Böge. Bayesian game theory. In O. Moeschlin and D. Pallaschke, editors, *Game Theory and Related Topics*. North-Holland, 1979.
- [3] Sophie Bade. Nash equilibrium in games with incomplete preferences. Working paper, New York University, 2003.
- [4] Pierpaolo Battigalli and Marciano Siniscalchi. Hierarchies of conditional beliefs and interactive epistemology in dynamic games. *J. Econ. Theory*, 88:188–230, 1999.
- [5] Pierpaolo Battigalli and Marciano Siniscalchi. Strong belief and forward induction reasoning. *J. Econ. Theory*, 106:356–391, 2002.
- [6] Pierpaolo Battigalli and Marciano Siniscalchi. Rationalization and incomplete information. *Adv. in Theor. Econ.*, 3(1):Article 3, 2003.
- [7] Truman F. Bewley. Knightian decision theory, part I. Cowles Foundation Discussion Paper #807, 1986.
- [8] W. Böge and T. Eisele. On solutions of Bayesian games. *Int. J. Game Theory*, 8:193–215, 1979.
- [9] Adam Brandenburger and Eddie Dekel. Common knowledge with probability 1. *J. Math. Econ.*, 16:237–245, 1987.
- [10] Adam Brandenburger and Eddie Dekel. Hierarchies of beliefs and common knowledge. *J. Econ. Theory*, 59:189–198, 1993.
- [11] Adam Brandenburger, Amanda Friedenberg, and H. Jerome Keisler. Admissibility in games. Working paper, Stern School of Business, Olin School of Business, and University of Wisconsin–Madison, 2004.
- [12] Alfredo Di Tillio. Subjective expected utility in games. Working paper, University of Texas at Austin and Bocconi University, 2004.
- [13] Daniel Ellsberg. Risk, ambiguity, and the Savage axioms. *Quart. J. of Econ.*, 75:643–669, 1961.
- [14] Larry G. Epstein. Preference, rationalizability and equilibrium. *J. Econ. Theory*, 73:1–29, 1997.
- [15] Larry G. Epstein and Tan Wang. “Beliefs about beliefs” without probabilities. *Econometrica*, 64:1343–1373, 1996.
- [16] Larry G. Epstein and Stanley Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57:937–969, 1989.
- [17] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with a non-unique prior. *J. Math. Econ.*, 18:141–153, 1989.
- [18] Faruk Gul and Wolfgang Pesendorfer. Self-control and the theory of consumption. *Econometrica*, 72:119–158, 2004.
- [19] J. C. Harsanyi. Games with incomplete information played by Bayesian players, I–III. *Management Sci.*, 14:159–182, 320–334, 486–502, 1967–1968.
- [20] Atsushi Kajii and Takashi Ui. Trade with heterogeneous multiple priors. KIER Working paper #601, Kyoto University, 2004.
- [21] Atsushi Kajii and Takashi Ui. Incomplete information games with multiple priors. *Japanese Econ. Rev.*, 56:332–351, 2005.
- [22] Atsushi Kajii and Takashi Ui. Agreeable bets with multiple priors. *J. Econ. Theory*, 128:299–305, 2006.
- [23] Peter Klibanoff. Uncertainty, decision, and normal form games. Working paper, Kellogg Graduate School of Management, Northwestern University, 1996.

- [24] Dan Levin and Emre Ozdenoren. Auctions with uncertain numbers of bidders. *J. Econ. Theory*, 118:229–251, 2004.
- [25] Kin Chung Lo. Equilibrium in beliefs under uncertainty. *J. Econ. Theory*, 71:443–484, 1996.
- [26] Kin Chung Lo. Sealed bid auctions with uncertainty averse bidders. *Econ. Theory*, 12:1–20, 1998.
- [27] Kin Chung Lo. Extensive form games with uncertainty averse players. *Games Econ. Behav.*, 28:256–270, 1999.
- [28] Kin Chung Lo. Correlated equilibrium under uncertainty. *Math. Soc. Sci.*, 44:183–209, 2002.
- [29] Giuseppe Lopomo, Luca Rigotti, and Chris Shannon. Uncertainty in mechanism design. Working paper, University of California, Berkeley and Fuqua School of Business, Duke University, 2004.
- [30] Thomas Mariotti, Martin Meier, and Michele Piccione. Hierarchies of beliefs for compact possibility models. *J. Math. Econ.*, 41:303–324, 2005.
- [31] Jean-François Mertens and Shmuel Zamir. Formulation of Bayesian analysis for games with incomplete information. *Int. J. Game Theory*, 14:1–29, 1985.
- [32] David Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.