Online Appendix to "Hierarchies of Ambiguous Beliefs"

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Abstract

Omitted proofs for results in "Hierarchies of Ambiguous Beliefs" [1] are presented.

B Online appendix

Unless explicitly stated otherwise, references to lemmata, proofs, and propositions are to [1].

B.1 Proof of Proposition 9

Let $\bar{H}_1 = \{(A_1, A_2, \ldots) \in H_1 : |A_n(\cdot|B)| = 1, \forall B \in \mathcal{B}\}$, which is also naturally identified as a subset of $\prod_{n=0}^{\infty} \Delta^{\mathcal{B}} X_n$. The proof of [2, Proposition 1] can be applied verbatim to produce a canonical homeomorphism $\bar{f} : \bar{H}_1 \to \Delta^{\mathcal{B}}(S \times H_0)$. Then $\bar{f}^{\mathcal{K}} : \mathcal{K}(\bar{H}_1) \to \mathcal{K}(\Delta^{\mathcal{B}}(S \times H_0))$ is a homeomorphism by Lemma 2. For each compact $K \subseteq \bar{H}_1$, let $G(K) = (\mathrm{Proj}^{\mathcal{K}}_{\Delta^{\mathcal{B}} X_0}(K), \mathrm{Proj}^{\mathcal{K}}_{\Delta^{\mathcal{B}} X_1}(K), \ldots)$. An obvious modification of the proof of Proposition 4 implies that $f = F \circ \bar{f}^{\mathcal{K}} : H_1 \to \mathcal{K}(\Delta^{\mathcal{B}}(S \times H_0))$ is the desired homeomorphism, where $F = G^{-1} : H_1 \to \mathcal{K}(\bar{H}_1)$.

B.2 Proof of Proposition 10

Using arguments similar to Lemmata 3 and 5, we can demonstrate that $\bar{H}_1$, hence $\bar{H}_m$, is closed. Let $\bar{H}_m = \{h \in H_m : f(h) \in \Delta^{\mathcal{B}}(S \times T_0)$ and $\bar{H} = \bigcap_{m=1}^{\infty} \bar{H}_m$. A slight notational variation of the proof of [2, Proposition 2] implies that the restriction $\bar{f} : \bar{H} \to \Delta^{\mathcal{B}}(S \times H_\infty)$ is a homeomorphism. By Lemma 2, $\bar{f}^{\mathcal{K}} : \mathcal{K}(\bar{H}) \to \mathcal{K}(\Delta^{\mathcal{B}}(S \times H_\infty))$ is also a homeomorphism. The second paragraph of the proof of Proposition 6 can be obviously modified to show that the restriction $f : H_\infty \to \mathcal{K}(\Delta^{\mathcal{B}}(S \times H_\infty))$ is the desired homeomorphism.

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B.3 Proof of Proposition 11

Since $\bar{T}_\infty = T_\infty \cap \bar{T}_0$ is a closed set, Lemma 5 implies each $\mathbf{K}_m(\bar{T}_\infty)$ is a closed set. Arguments completely analogous to those in the proof of Proposition 6 establish that the restriction of $g : T_\infty^{\mathrm{MZ}} \to \Delta(S \times T_\infty^{\mathrm{MZ}})$ is a homeomorphism. Now let

$$\begin{aligned}\bar{T}_1^* &= \{(A_1^*, A_2^*, \ldots) \in T_1^* : |A_n| = 1, \forall n\} \\ \bar{T}_{k+1}^* &= \{t^* \in \bar{T}_1^* : g^*(t^*) \subseteq \Delta(S \times \bar{T}_k^*)\} \\ \bar{T}_\infty^* &= \bigcap_{k=1}^{\infty} \bar{T}_k^*\end{aligned}$$

We will demonstrate that both T_∞^{MZ} and $\Theta_\infty^{\mathrm{MZ}}$ are homeomorphic to $\bar{T}_\infty^*$, hence to each other.

For notational ease, let $\varphi^{\mathrm{MZ}} = \varphi_{T_\infty^{\mathrm{MZ}},g}$ denote the embedding of T_∞^{MZ} into T_∞^*. We begin by showing $\varphi^{\mathrm{MZ}}(T_\infty^{\mathrm{MZ}}) \subseteq \bar{T}_\infty^*$ by induction. Fix $t \in T_\infty^{\mathrm{MZ}}$. Since $g(t) \in \Delta(S \times T_\infty)$, $[Q_0 \circ R_0](t) \in \Delta X_0^*$. By canonicity of g^* and the commutativity established in Lemma 13, this suffices to show $[Q_{n-1} \circ R_{n-1} \circ \cdots \circ R_0](t) \in \Delta X_{n-1}^*$, i.e. that $\varphi^{\mathrm{MZ}}(t) \in \bar{T}_1^*$. Now suppose $\varphi^{\mathrm{MZ}}(T_\infty^{\mathrm{MZ}}) \subseteq \bar{T}_m^*$. Then $\mathcal{L}_{(\mathrm{Id}_S;\varphi^{\mathrm{MZ}})}$ maps $\Delta(S \times T_\infty^{\mathrm{MZ}})$ into $\Delta(S \times T_m^*)$. Since $g^* \circ \varphi^{\mathrm{MZ}} = \mathcal{L}_{(\mathrm{Id}_S;\varphi^{\mathrm{MZ}})} \circ g$ and $g(T_\infty^{\mathrm{MZ}}) = \Delta(S \times T_\infty^{\mathrm{MZ}})$, this implies $\varphi^{\mathrm{MZ}}(T_\infty^{\mathrm{MZ}}) \subseteq \bar{T}_{m+1}^*$.

Now, fix $(\bar{A}_1^*, \bar{A}_2^*, \ldots) \in \bar{T}_\infty^*$. Since g is onto $\Delta(S \times \bar{T}_\infty)$, we have $[Q_0 \circ R_0](\tilde{T}_\infty) \supseteq \mathrm{Proj}_{\mathcal{K}(X_0^*)}(\bar{T}_1^*)$. By canonicity of g^* and Lemma 13, this implies $[Q_{n-1} \circ R_{n-1} \circ \cdots \circ R_0](\tilde{T}_\infty) \supseteq \mathrm{Proj}_{\mathcal{K}(X_{n-1}^*)(\bar{T}_n^*)}$. Let $D_n = \{t \in T_\infty : [Q_{n-1} \circ R_{n-1} \circ \cdots \circ R_0](t) = \bar{A}_{n+1}^*\}$. Each D_n is closed. Since

$$[Q_{n-1} \circ R_{n-1} \circ \cdots \circ R_0](\bar{T}_\infty) \supseteq \mathrm{Proj}_{\mathcal{K}(X_{n-1}^*)}(\bar{T}_n^*) \supseteq \mathrm{Proj}_{\mathcal{K}(X_{n-1}^*)}(\bar{T}_\infty^*),$$

each D_n is nonempty. By coherence, $\bigcap_{m \le n} D_m = D_n$, so $\{D_n\}$ has the finite intersection property. So select any $t^* \in \bigcap_{n=1}^\infty D_n$ and t^* satisfies $\varphi^{\mathrm{MZ}}(t^*) = (\bar{A}_1^*, \bar{A}_2^*, \ldots)$. Thus φ^{MZ} surjectively maps $\bar{T}_\infty$ onto $\bar{T}_\infty^*$.

Since g is canonical and injective, the argument at the end of the proof of Proposition 7 implies φ^{MZ} is injective. Thus $\varphi^{\mathrm{MZ}} : \bar{T}_\infty \to \bar{T}_\infty^*$ is a continuous bijection between compact sets, thus $\bar{T}_\infty$ and $\bar{T}_\infty^*$ are homeomorphic. The same argument, with some notational changes, proves $\varphi_{\Theta_\infty^{\mathrm{MZ}},g^{\mathrm{MZ}}} : \Theta_\infty^{\mathrm{MMP}} \to \bar{T}_\infty^*$ is also a homeomorphism. Hence $\Theta_\infty^{\mathrm{MMP}}$ and $\bar{T}_\infty$ are homeomorphic to each other.

B.4 Proof of Proposition 12

Since $S \times T_\infty$ is separable, the set of Dirac measures $\delta(S \times T_\infty)$ is a closed subset of $\Delta(S \times T_\infty)$ [3, Theorem 14.8]. Then $\mathcal{K}(\delta(S \times T_\infty))$ is a closed subset of $\mathcal{K}(\Delta(S \times T_\infty))$. To see this, consider any convergent sequence of sets $K_i \in \mathcal{K}(\delta(S \times T_\infty))$ with $K_i \to K$. Pick any point $x \in K$. Since $x \in \lim K_i$, there must exist a sequence of selections $x_i \in K_i$ such that $x_i \to x$. But, since $x_i \in \delta(S \times T_\infty)$, which is a closed set, we have $x \in \delta(S \times T_\infty)$. Thus $K \in \mathcal{K}(\delta(S \times T_\infty))$. Therefore $T^{\mathrm{MMP}} = g^{-1}(\mathcal{K}(\delta(S \times T_\infty))$ is the continuous preimage of a closed set, hence closed. Finally, Lemma 5 implies that $T_\infty^{\mathrm{MMP}} = \mathbf{CK}(T^{\mathrm{MMP}})$ is closed.

Arguments completely analogous to the proof of Proposition 6 implies the restriction $g : T_\infty^{\text{MMP}} \to \mathcal{K}(\delta(S \times T_\infty^{\text{MMP}}))$ is a homeomorphism. Recalling that g^* is the canonical homeomorphism from $T_1^* \to \mathcal{K}(\Delta(S \times T_1^*))$, let

$$\begin{aligned}
\tilde{T}_1^* &= \{(A_1^*, A_2^*, \ldots) \in T_1^* : A_n^* \in \mathcal{K}(\delta(X_{n-1}^*)), \forall n\}; \\
\tilde{T}_{k+1}^* &= \{t^* \in \tilde{T}_1^* : g^*(t^*) \subseteq \Delta(S \times \tilde{T}_k^*)\}; \\
\tilde{T}_\infty^* &= \bigcap_{k=1}^{\infty} \tilde{T}_k^*
\end{aligned}$$

Now arguments identical to those in the proof of Proposition 11, with appropriate replacements of notation, establish that both $\tilde{T}_\infty$ and $\tilde{\Theta}_\infty$ are homeomorphic to $\tilde{T}_\infty^*$, hence to each other.

References

[1] D. S. Ahn, Hierarchies of ambiguous beliefs, J. Econ. Theory.

[2] P. Battigalli, M. Siniscalchi, Hierarchies of conditional beliefs and interactive epistemology in dynamic games, J. Econ. Theory 88 (1999) 188–230.

[3] C. D. Aliprantis, K. C. Border, Infinite Dimensional Analysis, 2nd Edition, Springer, New York, 1999.