Naivete about Temptation and Self-Control: Foundations for Recursive Naive Quasi-Hyperbolic Discounting*

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Abstract

We introduce and characterize a recursive model of dynamic choice that accommodates naivete about present bias. While recursive representations are important for infinite-horizon problems, the commonly used Strotz model of time inconsistency presents well-known technical difficulties that preclude such representations. Our model incorporates costly self-control in the sense of Gul and Pesendorfer (2001) to overcome these hurdles. The important novel condition is an axiom for naivete. We first introduce definitions of absolute and comparative naivete for a simple two-period model and show that they correspond to tight parametric restrictions for the costly self-control representation. We then proceed to study preferences in infinite-horizon environments. Incorporating our definition of absolute naivete as an axiom, we characterize a recursive representation of naive quasi-hyperbolic discounting with self-control for an individual who is jointly overoptimistic about her present-bias factor and her ability to exert self-control. We also study comparative statics for differences in naivete across individuals, and we present an extension of our model where naivete diminishes over time.

Keywords: Naive, sophisticated, self-control, quasi-hyperbolic discounting

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1 Introduction

Naivete about dynamically inconsistent behavior is both plausible and empirically well-documented, and it has important economic consequences. Behavioral models of agents with overoptimistic beliefs about their future decisions are now prevalent tools used across a variety of applications. Naivete is an inherently dynamic phenomenon where today's projections regarding future trade-offs and behavior diverge from the future's actual choices.

Of course, complicated long-run dynamic problems are central in many economic settings that have nothing to do with naivete. Usually such problems are simplified by recursively representing the dynamic choice problem. The development of modern finance or macroeconomics seems unimaginable without the recursive techniques that are now a standard part of the graduate curriculum. Despite the general importance of behavior over time in economics and its particular importance for applications of naivete, a recursive dynamic model of a naive agent making choices over time has not yet been developed. This paper remedies that gap, providing the appropriate environment and conditions to characterize a system of implicit recursive equations that represents naive behavior over an infinite time horizon.

An immediate obstacle to developing a dynamic model of naivete is that the ubiquitous Strotz model of dynamic inconsistency is poorly suited for recursive representations. Even assuming full sophistication, the Strotz model is well-known to be discontinuous and consequently ill-defined for environments with more than two periods of choice (Peleg and Yaari (1973), Gul and Pesendorfer (2005)).[1] This is because a Strotzian agent lacks any self-control to curb future impulses and therefore is highly sensitive to small changes in the characteristics of tempting options. Our approach instead follows Gul and Pesendorfer (2004), Noor (2011), and Krusell, Kuruşçu, and Smith (2010) in considering self-control in a dynamic environment. The moderating effect of even a small amount of self-control circumvents the technical limitations of the Strotz model, by restoring continuity and allowing us to write well-defined recursive formulae for long-run naive behavior. In addition to its methodological benefits, self-control has compelling substantive motivations, as argued in the seminal paper by Gul and Pesendorfer (2001). Therefore, we employ the self-control model to represent dynamic naive choice. Our primary contribution is in extending existing recursive models of sophisticated time-inconsistency to accommodate naivete.

An important foundational step in developing our recursive representation is to formulate appropriate behavioral definitions of naivete. We introduce definitions of absolute and comparative naivete for individuals who can exert costly self-control in the face of

[1]One workaround to finesse this impossibility is to restrict the set of decision problems and preference parameters, e.g., by imposing lower bounds on risk aversion, the present-bias parameter, and uncertainty about future income (Harris and Laibson (2001)). We take a different approach in this paper.

temptation. While definitions of sophistication for self-control preferences have been proposed by Noor (2011) and definitions of absolute and comparative naivete for Strotz preferences have been proposed by Ahn, Iijima, Le Yaouanq, and Sarver (2018),[2] no suitable definitions of naivete for self-control preferences currently exist. Such a definition is interesting because it facilitates better understanding of how self-control and naivete interact. More importantly, such a definition is necessary to develop a recursive model of choice with naivete. Since the definition of naivete proposed in Ahn, Iijima, Le Yaouanq, and Sarver (2018) is sensible only if the agent has no self-control as in the Strotz model and since the Strotz model cannot be used in our infinite-horizon environment, their definition is not useful for our analysis. To understand the economic effects of naivete while still maintaining standard recursive formulations, an alternate definition is required. This paper proposes such a definition. Thus, while our main motivation is not to generalize the definition of Ahn, Iijima, Le Yaouanq, and Sarver (2018), extending that definition turns out to be an essential first step in formulating a recursive model of naivete.

In Section 2, we develop intuition by exploring absolute and comparative naivete in a simple two-stage environment with ex-ante rankings of menus and ex-post choice from menus. These nonparametric definitions of naivete for two-period self-control preferences are not the primary contribution of the paper. Our main contribution is in extending these intuitions to infinite-horizon environments as a foundation for a recursive representation with naivete. In Section 3, we propose a recursive system of equations to represent naive quasi-hyperbolic discounting over time, building on earlier formulations for fully sophisticated choice by Gul and Pesendorfer (2004) and Noor (2011). These equations accommodate an agent who has mispredictions about both her present-bias parameter and her self-control parameter. Incorporating an infinite-horizon version of our definition of absolute naivete as an axiom, we provide a behavioral characterization of the model. To our knowledge, this provides the first recursive model of dynamic naive choice. We then introduce comparative measures of naivete into our recursive model and use them in two ways: First, we apply comparative naivete across agents to develop the appropriate parametric restrictions for comparing different individuals. Second, we apply comparative naivete to a single individual's choices across time to characterize a representation with diminishing naivete. This extension of our main stationary representation is important, since naivete can change over time as an individual gains greater self-understanding. Finally, the model is applied to a simple consumption-savings problem to illustrate how naivete influences consumption choice in a recursive environment.

We conclude in Section 4 by discussing the scope of our proposed definition of naivete with self-control and its relationship to models beyond the linear self-control preferences of Gul and Pesendorfer (2001). We relate our definition to the definition of naivete for

[2]See also the recent theoretical analysis by Freeman (2016) that uses procrastination to uncover naivete within Strotzian models of dynamic inconsistency.

consequentialist behavior proposed by Ahn, Iijima, Le Yaouanq, and Sarver (2018) and show that the two approaches are equivalent for deterministic Strotz preferences. This is perhaps unsurprising, since Strotz preferences are a limit case of self-control preferences and both classes satisfy the independence and set betweenness axioms. Using specific examples, we then argue that our definitions are robust to models that relax independence while maintaining set betweenness, but fail to extend well to other models that drop set betweenness or incorporate stochastic choice.

2 Prelude: A Two-Stage Model

While the main contribution of this paper is in analyzing a recursive model of naivete, we commence our analysis with a two-stage model to develop intuition. Aside from being of some interest in its own right, the two-period definitions of absolute and comparative naivete for the self-control model that are described in this section will serve as a springboard for our analysis of the infinite-horizon model in the next section.

2.1 Primitives

Let C denote a compact and metrizable space of outcomes and $\Delta(C)$ denote the set of lotteries (countably-additive Borel probability measures) over C, with typical elements $p, q, \ldots \in \Delta(C)$. Slightly abusing notation, we identify c with the degenerate lottery $\delta_c \in \Delta(C)$. Endow $\Delta(C)$ with the topology of weak convergence, and let $\mathcal{K}(\Delta(C))$ denote the family of nonempty compact subsets of $\Delta(C)$, with typical elements $x, y, \ldots \in \mathcal{K}(\Delta(C))$. An expected-utility function is a continuous affine function $u : \Delta(C) \to \mathbb{R}$, that is, a continuous function such that, for all lotteries p and q, $u(\alpha p + (1-\alpha)q) = \alpha u(p) + (1-\alpha)u(q)$. We write $u \approx v$ when u is a positive affine transformation of v.

We study a pair of behavioral primitives that capture choice at two different points in time. The first is a preference relation $\succsim$ on $\mathcal{K}(\Delta(C))$. This ranking of menus is assumed to occur in the first period ("ex ante") before the direct experience of temptation but while (possibly incorrectly) anticipating its future occurrence. As such, it allows inferences about the individual's projection of her future behavior. The second is a choice correspondence $\mathcal{C} : \mathcal{K}(\Delta(C)) \rightrightarrows \Delta(C)$ with $\mathcal{C}(x) \subset x$ for all $x \in \mathcal{K}(\Delta(C))$. The behavior encoded in $\mathcal{C}$ occurs in the second period ("ex post") and is taken while experiencing temptation.

These two-stage primitives are a special case of the domain used in Ahn, Iijima, Le Yaouanq, and Sarver (2018) to study naivete without self-control and in Ahn and Sarver

(2013) to study unforeseen contingencies.[3] The identification of naivete and sophistication in our model relies crucially on observing both periods of choice data. Clearly, multiple stages of choice are required to identify time-inconsistent behavior. In addition, the ex-ante ranking of non-singleton option sets is required to elicit beliefs about future choice and hence to identify whether an individual is naive or sophisticated. This combination of ex-ante choice of option sets (or equivalently, commitments) and ex-post choice is also common in the empirical literature that studies time inconsistency and naivete.[4] Perhaps most closely related is a recent experiment by Toussaert (2018) that elicited ex-ante menu preferences and ex-post choices of subjects and found evidence for the self-control model of Gul and Pesendorfer (2001). However, there are some important economic settings, such as consumption-savings problems without commitment devices, in which it is difficult to identify the decision maker's preferences over commitments. Our behavioral conditions do not apply in such settings. In Appendix B we briefly discuss how naivete can be potentially identified in a consumption-savings example.

2.2 Naivete about Temptation with Self-Control

We introduce the following behavioral definitions of sophistication and naivete that account for the possibility of costly self-control.

Definition 1 *An individual is* sophisticated *if, for all lotteries p and q with $\{p\} \succ \{q\}$,*

$$\mathcal{C}(\{p, q\}) = \{p\} \quad \textit{if and only if} \quad \{p, q\} \succ \{q\}.$$

An individual is naive *if, for all lotteries p and q with $\{p\} \succ \{q\}$,*

$$\mathcal{C}(\{p, q\}) = \{p\} \quad \textit{implies} \quad \{p, q\} \succ \{q\}.$$

An individual is strictly naive *if she is naive and not sophisticated.*[5]

This definition of sophistication was introduced by Noor (2011, Axiom 7), and similar conditions were used by Kopylov (2012) and Noor and Takeoka (2015). To our knowledge, the definition of naivete is new. Both definitions admit simple interpretations: An

[3]In these papers the second-stage choice is allowed to be random. While we feel that this is an important consideration when there is uncertainty about future behavior, in this paper we restrict attention to deterministic choice in each period. This restriction is not solely for the sake of exposition: We argue in Section 4 that no definition of naivete can satisfactorily accommodate both self-control and random choice.

[4]Examples include DellaVigna and Malmendier (2006); Shui and Ausubel (2005); Giné, Karlan, and Zinman (2010); Kaur, Kremer, and Mullainathan (2015); Augenblick, Niederle, and Sprenger (2015).

[5]Definition 1 can be stated in terms of non-singleton menus. That is, an individual is *sophisticated* if for all menus x, y such that $\{p\} \succ \{q\}$ for all $p \in y$ and $q \in x$, $\mathcal{C}(x \cup y) \subset y \iff x \cup y \succ x$. An individual is *naive* if for all menus x, y such that $\{p\} \succ \{q\}$ for all $p \in y$ and $q \in x$, $\mathcal{C}(x \cup y) \subset y \implies x \cup y \succ x$.

individual is sophisticated if she correctly anticipates her future choices and exhibits no unanticipated preference reversals, whereas a naive individual may have preference reversals that she fails to anticipate. More concretely, consider both sides of the required equivalence in the definition of sophistication. On the right, a strict preference for $\{p, q\}$ over $\{q\}$ reveals that the individual believes that she will choose the alternative p over q if given the option ex post. On the left, the ex-ante preferred option p is actually chosen. That is, her anticipated and actual choices align. A sophisticated individual correctly forecasts her future choices and therefore strictly prefers to add an ex-ante superior option p to the singleton menu $\{q\}$ if and only if it will be actually chosen over q ex post.

In contrast, a naive individual might exhibit the ranking $\{p, q\} \succ \{q\}$, indicating that she anticipates choosing the ex-ante preferred option p, yet ultimately choose q over p in the second period. Thus a naive individual may exhibit unanticipated preference reversals. However, our definition of naivete still imposes some structure on the relationship between believed and actual choices. Anytime the individual will actually choose in a time-consistent manner ($\{p\} \succ \{q\}$ and $\mathcal{C}(\{p, q\}) = \{p\}$), she correctly predicts her consistent behavior. That is, she does not anticipate preference reversals when there are none. Rather than permitting arbitrary incorrect beliefs for a naive individual, our definition is intended to capture the most pervasive form of naivete that has been documented empirically and used in applications: underestimation of the future influence of temptation. It is also important to note that we use the term "naivete" to mean "weakly naive," since sophistication is included as a special case of our definition naivete. We use the term "strictly naive" to refer to individuals who fail to be sophisticated.[6]

Ahn, Iijima, Le Yaouanq, and Sarver (2018) proposed definitions of sophistication and naivete for individuals who are consequentialist in the sense that they are indifferent between any two menus that share the same anticipated choices, as for example in the case of the Strotz model of changing tastes. Specifically, Ahn, Iijima, Le Yaouanq, and Sarver (2018) classify an individual as naive if $x \succsim \{p\}$ for all x and $p \in \mathcal{C}(x)$, and as sophisticated if $x \sim \{p\}$ for all x and $p \in \mathcal{C}(x)$. In the presence of self-control, these conditions are too demanding. An individual who chooses salad over cake may still strictly prefer to go to a restaurant that does not serve dessert to avoid having to exercise self-control and defeat the temptation to eat cake. That is, costly self-control may decrease the value of a menu that contains tempting options so that $\{p\} \succ x$ for $p \in \mathcal{C}(x)$ is possible for a

[6] Including the boundary case of sophistication as part of the definition of naivete is analogous to the norm of including risk neutrality as the boundary of the family of risk-averse preferences. Our definition could be strengthened to exclude sophistication without materially affecting any of the results in the sequel. Also, note that our definition classifies an individual as strictly naive if she makes *any* unanticipated preference reversals, which is sometimes referred to as "partial naivete" in the literature on time inconsistency. Some papers in this literature reserve the term "naive" for the case of complete ignorance of future time inconsistency. This extreme of complete naivete is the special case of our definition where $\{p, q\} \succ \{q\}$ anytime $\{p\} \succ \{q\}$.

sophisticated, or even a naive, individual. Definition 1 instead investigates the marginal impact of making a new option p available in the ex-ante and ex-post stages. Section 4 analyzes the relationship between these two sets of definitions and shows that Definition 1 is applicable more broadly to preferences both with and without self-control.[7]

With the definition of absolute naivete in hand, we can now address the comparison of naivete across different individuals. Our approach is to compare the number of violations of sophistication: A more naive individual exhibits more unexpected preference reversals than a less naive individual.

Definition 2 *Individual 1 is* more naive *than individual 2 if, for all lotteries p and q,*

$$\big[\{p,q\} \succ_2 \{q\} \text{ and } \mathcal{C}_2(\{p,q\}) = \{q\}\big] \implies \big[\{p,q\} \succ_1 \{q\} \text{ and } \mathcal{C}_1(\{p,q\}) = \{q\}\big].$$

A more naive individual has more instances where she desires the addition of an option ex ante that ultimately goes unchosen ex post. Our interpretation of this condition is that any time individual 2 anticipates choosing the ex-ante superior alternative p over q (as reflected by $\{p,q\} \succ_2 \{q\}$) but in fact chooses q ex post, individual 1 makes the same incorrect prediction. Note that any individual is trivially more naive than a sophisticate: If individual 2 is sophisticated, then it is never the case that $\{p,q\} \succ_2 \{q\}$ and $\mathcal{C}_2(\{p,q\}) = \{q\}$; hence Definition 2 is vacuously satisfied.

As an application of these concepts, consider a two-stage version of the self-control representation of Gul and Pesendorfer (2001).

Definition 3 *A* self-control representation *of $(\succsim, \mathcal{C})$ is a triple $(u, v, \hat{v})$ of expected-utility functions such that the function $U : \mathcal{K}(\Delta(C)) \to \mathbb{R}$ defined by*

$$U(x) = \max_{p\in x}\big[u(p) + \hat{v}(p)\big] - \max_{q\in x}\hat{v}(q)$$

represents $\succsim$ and

$$\mathcal{C}(x) = \operatorname*{argmax}_{p\in x}[u(p) + v(p)].$$

The function u represents the commitment preference. For example, if the individual could commit to food choices in advance, she might rank them solely on the basis of healthiness. The function v reflects how tempting different options are, for example, how strongly she will experience an urge to eat desserts. The function $\hat{v}$ reflects how tempting

[7]However, the definitions proposed in Ahn, Iijima, Le Yaouanq, and Sarver (2018) have the advantage that they are readily extended to random choice driven by uncertain temptations, so long as the individual is consequentialist (exhibits no self-control).

the individual expects each option to be, which might be different from the actual temptation v. The interpretation is that the individual expects to maximize $u(p)$ minus the cost $[\max_{q \in x} \hat{v}(q) - \hat{v}(p)]$ of having to exert self-control to refrain from eating the most tempting option. She therefore anticipates choosing the option that maximizes the compromise $u(p) + \hat{v}(p)$ of the commitment and (anticipated) temptation utility among the available options in menu x. The divergence between u and $u + \hat{v}$ captures the individual's perception of how temptation will influence her future choices. For a potentially naive individual, her actual ex-post choices are not necessarily those anticipated ex ante. Instead, the actual self-control cost associated with choosing p from the menu x is $[\max_{q \in x} v(q) - v(p)]$, where the actual temptation v can differ from anticipated temptation $\hat{v}$. The decision maker's ex-post choices are therefore governed by the utility function $u + v$ rather than $u + \hat{v}$.

The following definition offers a structured comparison of two utility functions w and w' and formalizes the a notion of greater congruence with the commitment utility u. Recall that $w \approx w'$ denotes equivalence of expected-utility functions, that is, one is a positive affine transformation of the other.

Definition 4 *Let u, w, w' be expected-utility functions. Then w is* more u-aligned *than w', written as $w \gg_u w'$, if $w \approx \alpha u + (1 - \alpha) w'$ for some $\alpha \in [0, 1]$.*

With this ordering on expected utilities, we can now provide a functional characterization of our definitions of absolute and comparative naivete for the self-control representation. Our result begins with the assumption that the individual has a two-stage self-control representation, which is a natural starting point since the primitive axioms on choice that characterize this representation are already well established.[8] We say a pair $(\succsim, \mathcal{C})$ is *regular* if there exist lotteries p and q such that $\{p\} \succ \{q\}$ and $\mathcal{C}(\{p, q\}) = \{p\}$. Regularity excludes preferences where the choices resulting from actual temptation in the second period are exactly opposed to the commitment preference.

Theorem 1 *Suppose $(\succsim, \mathcal{C})$ is regular and has a self-control representation $(u, v, \hat{v})$. Then the individual is naive if and only if $u + \hat{v} \gg_u u + v$ (and is sophisticated if and only if $u + \hat{v} \approx u + v$).*

If the decision maker is naive, then she believes that her future choices will be closer to her commitment choices. This overoptimism about virtuous future behavior corresponds to a particular alignment of these utility functions:

$$u + \hat{v} \approx \alpha u + (1 - \alpha)(u + v).$$

[8]Specifically, $(\succsim, \mathcal{C})$ has a (two-stage) self-control representation $(u, v, \hat{v})$ if and only if $\succsim$ satisfies the axioms of Gul and Pesendorfer (2001, Theorem 1) and $\mathcal{C}$ satisfies the weak axiom of revealed preference, continuity, and independence.

The individual optimistically believes that her future choices will overweight the commitment preference u. Although the behavioral definition of naivete permits incorrect beliefs, it does place some structure on the relationship between anticipated and actual choices. For example, it excludes situations like a consumer who thinks she will find sweets tempting when in fact she will be tempted by salty snacks. Excluding such orthogonally incorrect beliefs is essential in relating $\hat{v}$ to v and deriving some structure in applications.

Note that our behavioral definition of naivete places restrictions on the utility functions $u + \hat{v}$ and $u + v$ governing anticipated and actual choices, but it does not apply directly to the alignment of the temptation utilities $\hat{v}$ and v themselves. This seems natural since our focus is on naivete about the choices that result from temptation. Example 1 below illustrates the distinction: It is possible for an individual to be overly optimistic about choice, as captured by $u+\hat{v} \gg_u u+v$, while simultaneously being overly pessimistic about how many options she will find tempting, as captured by $v \gg_u \hat{v}$.

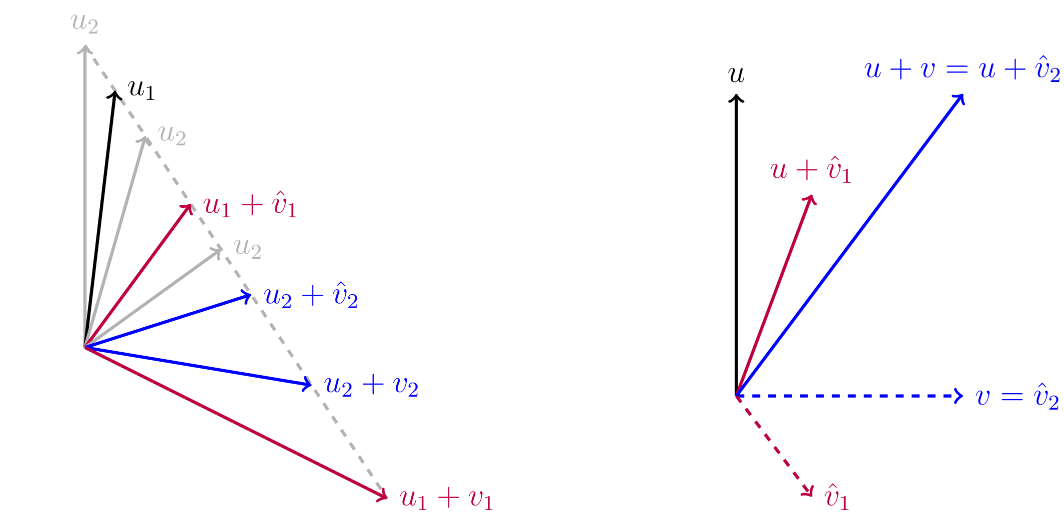
Our behavioral comparison of naivete is necessary and sufficient for linear alignment of the actual and believed utilities across individuals. In particular, the more naive individual has a more optimistic view of her future behavior ($u_1 + \hat{v}_1 \gg_{u_1} u_2 + \hat{v}_2$), while her actual choices deviate further from her commitment preferences ($u_2 + v_2 \gg_{u_1} u_1 + v_1$). We say $(\succsim_1, \mathcal{C}_1)$ and $(\succsim_2, \mathcal{C}_2)$ are *jointly regular* if there exist lotteries p and q such that $\{p\} \succ_i \{q\}$ and $\mathcal{C}_i(\{p, q\}) = \{p\}$ for $i = 1, 2$.

Theorem 2 *Suppose $(\succsim_1, \mathcal{C}_1)$ and $(\succsim_2, \mathcal{C}_2)$ are naive, jointly regular, and have self-control representations $(u_1, v_1, \hat{v}_1)$ and $(u_2, v_2, \hat{v}_2)$. Then individual 1 is more naive than individual 2 if and only if either*

$$u_1 + \hat{v}_1 \gg_{u_1} u_2 + \hat{v}_2 \gg_{u_1} u_2 + v_2 \gg_{u_1} u_1 + v_1,$$

or individual 2 is sophisticated ($u_2 + \hat{v}_2 \approx u_2 + v_2$).

Figure 1a illustrates the conditions in Theorems 1 and 2. Naivete implies that, up to affine transformations, the anticipated compromise between commitment and temptation utility $u_i + \hat{v}_i$ for each individual is a convex combination of the commitment utility u_i and the actual compromise utility $u_i + v_i$. Moreover, if individual 1 is more naive than individual 2, then the "wedge" between the believed and actual utilities governing choices, $u_i + \hat{v}_i$ and $u_i + v_i$, respectively, is smaller for individual 2. These relationships clarify the restrictions that correspond to the statement that individual 2's beliefs about temptation are more accurate than individual 1's. Figure 1a also illustrates several different possible locations of u_2 relative to the other utility functions. There is some freedom in how the commitment utilities of the two individuals are aligned, which permits meaningful comparisons of the degree of naivete of individuals even when they do not have identical



(a) Theorem 2: Alignment of believed and actual utilities implied by comparative naivete.

(b) Example 1: Individual 1 can be more naive than individual 2 even if $\hat{v}_2 \gg_u \hat{v}_1$ ($u_1 = u_2 = u$ and $v_1 = v_2 = v$).

Figure 1. Comparing naivete

ex-ante commitment preferences.[9]

There is an obvious connection between the choices an individual anticipates making and her demand for commitment: If an individual anticipates choosing an ex-ante inferior alternative from a menu, she will exhibit a preference for commitment. However, for self-control preferences, there will also be instances in which an individual desires commitment even though she anticipates choosing the ex-ante superior option from the menu. This occurs when she finds another option in the menu tempting, but expects to resist that temptation. Our comparative measure concerns the relationship between the anticipated and actual choices by individuals; it does not impose restrictions on whether one individual or another is tempted more often. The following example illustrates the distinction.

Example 1 Fix any u and v that are not affine transformations of each other. Let $(u, v, \hat{v}_1)$ and $(u, v, \hat{v}_2)$ be self-control representations of individuals 1 and 2, respectively, where $\hat{v}_1 = (1/3)(v - u)$ and $\hat{v}_2 = v$. Then,

$$u + \hat{v}_1 = \frac{2}{3}u + \frac{1}{3}v \approx \frac{1}{2}u + \frac{1}{2}(u + v).$$

[9] There are, of course, some restrictions on the relationship between u_1 and u_2 in Theorem 2. The assumption that $(\succsim_1, \mathcal{C}_1)$ and $(\succsim_2, \mathcal{C}_2)$ are jointly regular implies there exist lotteries p and q such that $u_i(p) > u_i(q)$ and $(u_i + v_i)(p) > (u_i + v_i)(q)$ for $i = 1, 2$. When individual 2 is strictly naive, this implies that u_2 lies in the arc between $-(u_1 + v_1)$ and $u_2 + \hat{v}_2$ in Figure 1a, which can be formalized as $u_2 + \hat{v}_2 \gg_{u_2} u_2 + v_2 \gg_{u_2} u_1 + v_1$.

Since $\hat{v}_2 = v_2 = v_1 = v$, this implies that the condition in Theorem 2 is satisfied:

$$u + \hat{v}_1 \gg_u u + \hat{v}_2 = u + v_2 = u + v_1.$$

Thus, the two individuals make the same ex-post choices, individual 2 is sophisticated, and individual 1 is naive. In particular, individual 1 is more naive than individual 2, even though her anticipated temptation utility diverges further from her commitment utility than that of individual 2, $\hat{v}_2 \gg_u \hat{v}_1$.[10] Figure 1b illustrates these commitment and temptation utilities. □

The self-control representation has been applied to a variety of settings, including habit formation, social preferences, and non-Bayesian belief updating.[11] Our characterization of absolute and comparative naivete are also applicable to these settings. While naivete in self-control models has been relatively less explored in the literature, we are not the first to formalize its implications. The welfare effects of naivete within a special case of the self-control representation were examined by Heidhues and Kőszegi (2009). In the next section, we illustrate the implications of our definitions for their proposed model.

2.3 Naivete about the Cost of Exerting Self-Control

Heidhues and Kőszegi (2009) proposed the following special case of the self-control representation.

Definition 5 *A* Heidhues-Kőszegi representation *of $(\succsim, \mathcal{C})$ is tuple $(u, \bar{v}, \gamma, \hat{\gamma})$ of expected-utility functions u and $\bar{v}$ and scalars $\gamma, \hat{\gamma} \geq 0$ such that the function $U : \mathcal{K}(\Delta(C)) \to \mathbb{R}$ defined by*

$$U(x) = \max_{p \in x} \left[u(p) + \hat{\gamma}\bar{v}(p) \right] - \max_{q \in x} \hat{\gamma}\bar{v}(q)$$

represents $\succsim$ and

$$\mathcal{C}(x) = \operatorname*{argmax}_{p \in x} [u(p) + \gamma\bar{v}(p)].$$

The Heidhues-Kőszegi representation can be written as a self-control representation $(u, v, \hat{v})$ by taking $v = \gamma\bar{v}$ and $\hat{v} = \hat{\gamma}\bar{v}$. The interpretation of this representation is that the

[10] Gul and Pesendorfer (2001, Theorem 8) characterized a comparative measure of preference for commitment. In the case where individuals 1 and 2 have the same commitment utility u, their results show that $\hat{v}_2 \gg_u \hat{v}_1$ if and only if individual 1 has greater preference for commitment than individual 2: That is, for any menu x, if there exists $y \subset x$ such that $y \succ_2 x$ then there exists $y' \subset x$ such that $y' \succ_1 x$. Their comparative measure could easily be applied in conjunction with ours to impose restrictions on both the relationship between $\hat{v}_1$ and $\hat{v}_2$ and the relationship between $u + \hat{v}_1$ and $u + \hat{v}_2$.

[11] Lipman and Pesendorfer (2013) provide a comprehensive survey.

individual correctly anticipates which alternatives will be tempting but may incorrectly anticipate the magnitude of temptation and hence the cost of exerting self-control. Put differently, temptation may have a greater influence on future choice than the individual realizes, but she will not have any unexpected temptations.

The following proposition characterizes the Heidhues-Kőszegi representation within the class of two-stage self-control representations. We say that $\succsim$ *has no preference for commitment* if $\{p\} \succ \{q\}$ implies $\{p\} \sim \{p, q\}$.

Proposition 1 *Suppose $(\succsim, \mathcal{C})$ is has a self-control representation $(u, v, \hat{v})$, and suppose there exists some pair of lotteries p and q such that $\{p\} \sim \{p, q\} \succ \{q\}$. Then the following are equivalent:*

1. *Either $\succsim$ has no preference for commitment or, for any lotteries p and q,*

$$\{p\} \sim \{p, q\} \succ \{q\} \implies \mathcal{C}(\{p, q\}) = \{p\}.$$

2. *$(\succsim, \mathcal{C})$ has a Heidhues-Kőszegi representation $(u, \bar{v}, \gamma, \hat{\gamma})$.*

To interpret the behavioral condition in this proposition, recall that $\{p\} \sim \{p, q\} \succ \{q\}$ implies that q is *not* more tempting than p. In contrast, $\{p\} \succ \{p, q\} \succ \{q\}$ implies that q is more tempting than p but the individual anticipates exerting self-control and resisting this temptation. Condition 1 in Proposition 1 still permits preference reversals in the latter case, but rules out reversals in the former case. In other words, the individual may hold incorrect beliefs about how tempting an alternative is, but she will never end up choosing an alternative that she does not expect to find tempting at all.[12]

The implications of absolute and comparative naivete for the Heidhues-Kőszegi representation follow as immediate corollaries of Theorems 1 and 2. To simplify the statement of the conditions in this result, we assume that the function $\bar{v}$ is *independent* of u, meaning it is not constant and it is not the case that $\bar{v} \approx u$. Note that this assumption is without loss of generality.[13]

Corollary 1 *Suppose $(\succsim_1, \mathcal{C}_1)$ and $(\succsim_2, \mathcal{C}_2)$ are jointly regular and have Heidhues-Kőszegi representations $(u, \bar{v}, \gamma_1, \hat{\gamma}_1)$ and $(u, \bar{v}, \gamma_2, \hat{\gamma}_2)$, where $\bar{v}$ is independent of u.*

1. *Individual i is naive if and only if $\hat{\gamma}_i \leq \gamma_i$ (and is sophisticated if and only if $\hat{\gamma}_i = \gamma_i$).*

2. *When both individuals are naive, individual 1 is more naive than individual 2 if and only if either $\hat{\gamma}_1 \leq \hat{\gamma}_2 \leq \gamma_2 \leq \gamma_1$ or individual 2 is sophisticated ($\hat{\gamma}_2 = \gamma_2$).*

[12] The exception is the case where $\succsim$ has no preference for commitment. In this case, the individual anticipates no temptation whatsoever ($\hat{\gamma} = 0$), yet may in fact be tempted ($\gamma > 0$).

[13] If $(u, \bar{v}, \gamma, \hat{\gamma})$ is a Heidhues-Kőszegi representation of $(\succsim, \mathcal{C})$ and $\bar{v}$ is not independent of u, there is an equivalent representation $(u, \bar{v}', 0, 0)$, where $\bar{v}'$ is an arbitrary non-constant function with $\bar{v}' \not\approx u$.

3 Infinite Horizon

The main contribution of this paper is formulating a recursive model that parsimoniously encodes behavior in all time periods through a finite system of equations while also accommodating the possibility of naivete regarding future behavior.

3.1 Primitives

We represent the environment recursively. Let C be a compact metric space for consumption in each period. Gul and Pesendorfer (2004) prove there exists a space Z that is homeomorphic to $\mathcal{K}(\Delta(C \times Z))$, the family of compact subsets of $\Delta(C \times Z)$. Each menu $x \in Z$ represents a continuation problem. We study choices over $\Delta(C \times Z)$. For notational ease, we identify each degenerate lottery with its sure outcome, that is, we write (c, x) for the degenerate lottery $\delta_{(c,x)}$ returning (c, x) with probability one. To understand the domain, consider a deterministic $(c, x) \in C \times Z$. The first component c represents current consumption, while the second component $x \in Z$ represents a future continuation problem. Therefore preferences over (c, x) capture how the decision maker trades off immediate consumption against future flexibility.

At each period $t = 1, 2, \ldots$, the individual's behavior is summarized by a preference relation $\succsim_t$ on $\Delta(C \times Z)$.[14] The dependence of behavior on the date t allows for the possibility that sophistication can vary over time. In Sections 3.2, 3.3, and 3.4, we study preferences that are stationary (time-invariant), so $p \succsim_t q \iff p \succsim_{t+1} q$. This is an understandably common assumption, as it allows for a fully recursive representation of behavior, which can help facilitate applications of the model in finance or macroeconomics. In Section 3.5, we relax stationarity to allow for increasing sophistication over time.

In this domain, the decision maker's choices reveal both preferences over today's options and preferences over tomorrow's menus, which allows us to differentiate between sophisticated and naive beliefs using an approach similar to the one employed in the two-stage setting. For example, suppose $(c, \{p\}) \succ_t (c, \{q\})$. This means that the consumer strictly prefers to commit to p for tomorrow than to q, keeping today's consumption constant. Moreover, if $(c, \{p, q\}) \succ_t (c, \{q\})$, then she believes she will in fact select p over q tomorrow. Now, suppose that $q \succ_{t+1} p$. Then the consumer exhibits an unanticipated preference reversal, and hence she is not sophisticated. Note that such behavior is possible even if her preferences are time-invariant (i.e., $q \succ_t p$). Thus, stationarity does *not* imply dynamic consistency or sophistication.

[14] Alternatively, we could take a choice correspondence as primitive and impose rationalizability as an axiom as in Noor (2011).

3.2 Stationary Quasi-Hyperbolic Discounting

Recall that a self-control representation consists of commitment utility U and (actual and perceived) temptation utilities $(V, \hat{V})$. Using the dynamic structure of our domain, we can sharpen these into more precise functional forms. In particular, we exclude static temptations over immediate consumption, like eating chocolate instead of salad, and focus solely on temptations related to intertemporal trade-offs: better options today versus future opportunities.

Specifically, we introduce a recursive version of the $(\beta, \hat{\beta}, \delta)$ quasi-hyperbolic discounting model of O'Donoghue and Rabin (2001) that incorporates self-control. A leading application of their $(\beta, \hat{\beta}, \delta)$ model is procrastination on a single project like the decision to enroll in a 401(k). However, many economic decisions are not one-time stopping problems but perpetual ones, such as how much to contribute each period to the 401(k) after enrollment. To our knowledge, the $(\beta, \hat{\beta}, \delta)$ model has not yet been applied in such recursive infinite-horizon settings, and our model is a step toward bridging that gap.

Definition 6 *A* naive quasi-hyperbolic discounting representation *of* $\{\succsim_t\}_{t\in\mathbb{N}}$ *consists of continuous functions* $u : C \to \mathbb{R}$ *and* $U, \hat{V}, V : \Delta(C \times Z) \to \mathbb{R}$ *satisfying the following system of equations:*

$$
\begin{aligned}
U(p) &= \int_{C\times Z} (u(c) + \delta \hat{W}(x))\, dp(c, x) \\
V(p) &= \gamma \int_{C\times Z} (u(c) + \beta\delta \hat{W}(x))\, dp(c, x) \\
\hat{V}(p) &= \hat{\gamma} \int_{C\times Z} (u(c) + \hat{\beta}\delta \hat{W}(x))\, dp(c, x) \\
\hat{W}(x) &= \max_{q\in x}(U(q) + \hat{V}(q)) - \max_{q\in x} \hat{V}(q)
\end{aligned}
$$

and such that, for all $t \in \mathbb{N}$,

$$p \succsim_t q \iff U(p) + V(p) \geq U(q) + V(q),$$

where $\beta, \hat{\beta} \in [0, 1]$, $0 < \delta < 1$, *and* $\gamma, \hat{\gamma} \geq 0$ *satisfy*

$$\frac{1 + \hat{\gamma}\hat{\beta}}{1 + \hat{\gamma}} \geq \frac{1 + \gamma\beta}{1 + \gamma}. \tag{1}$$

To illustrate the tension between the commitment utility U and temptation utility V, consider the choice over deterministic consumption streams, where the only nontrivial flexibility is in the first period. Observe that

$$
\begin{aligned}
U(p)+V(p) &= \int_{C\times Z}\Big((1+\gamma)u(c)+(1+\gamma\beta)\delta\hat{W}(x)\Big)\,dp(c,x)\\
&= (1+\gamma)\int_{C\times Z}\left(u(c)+\frac{1+\gamma\beta}{1+\gamma}\delta\hat{W}(x)\right)dp(c,x).
\end{aligned}
$$

For a deterministic consumption stream $(c_t, c_{t+1}, \dots)$, the indirect utility is simple:

$$\hat{W}(c_{t+1}, c_{t+2}, \dots) = U(c_{t+1}, c_{t+2}, \dots) = \sum_{i=1}^{\infty}\delta^{i-1}u(c_{t+i}).$$

Thus, choice at period t of a deterministic consumption stream from a menu of such streams is made to maximize

$$U(c_t, c_{t+1}, \dots) + V(c_t, c_{t+1}, \dots) = u(c_t) + \frac{1+\gamma\beta}{1+\gamma}\sum_{i=1}^{\infty}\delta^{i}u(c_{t+i}). \tag{2}$$

When $\beta < 1$, there is disagreement between the temptation utility V and the commitment utility U in the form of a bias toward current consumption, and the parameter γ measures the intensity of this temptation. The individual may also hold incorrect beliefs about her future behavior: In period $t-1$, she anticipates that she will maximize Equation (2) under the perceived present bias parameter $\hat{\beta}$ and perceived strength of temptation $\hat{\gamma}$.

More generally, the individual believes she will maximize $U+\hat{V}$ in future periods even though she chooses to maximize $U+V$ today. Naivete could be generated by incorrect beliefs about either β or γ. For example, suppose $\beta < 1$ and $\gamma > 0$, so the individual has a nontrivial temptation to increase current consumption. If $\hat{\beta} = \beta$ and $\hat{\gamma} < \gamma$, then the individual correctly anticipates the nature of her temptations but incorrectly anticipates their intensity, and she is therefore overoptimistic about her future choices. Alternatively, if $\hat{\beta} > \beta$ and $\hat{\gamma} = \gamma$, then the individual underestimates her degree of present bias even though she accurately anticipates the strength of her temptation, and she is again overoptimistic about her future choices. In general, she could have incorrect beliefs about both parameters simultaneously. As we observed in Equation (2), the combined influence of the parameters β and γ on consumption decisions is determined entirely by the ratio $\frac{1+\gamma\beta}{1+\gamma}$, and therefore requiring that this value be less than its anticipated value $\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}$ as in Equation (1) gives us the appropriate statistical comparison for determining that the individual is naive about her present bias. This intuition will be confirmed momentarily by our axiomatic analysis.[15]

[15] Note that if the individual were instead future-biased ($\beta, \hat{\beta} > 1$) then naivete would be captured by reversing the inequality in Equation (1), so that $\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}} \leq \frac{1+\gamma\beta}{1+\gamma}$. In general, naivete requires anticipating that this ratio that determines the influence of present or future bias on choices is closer to 1 than it is

It should also be noted that Equation (1) does not require that $\beta \leq \hat{\beta}$ and $\gamma \geq \hat{\gamma}$. It is possible, for example, to overestimate the degree of present bias by having β slightly larger than $\hat{\beta}$, yet to underestimate the intensity of temptation γ so dramatically that this inequality is still satisfied, and hence the individual is naive. This is closely related to the observation made previously in Example 1.[16]

The benchmark case of $\hat{\beta} = \beta$ and $\hat{\gamma} = \gamma$ corresponds to the (β, δ) model of quasi-hyperbolic discounting with self-control from Gul and Pesendorfer (2005) and Krusell, Kuruşçu, and Smith (2010), which is a special case of a model characterized axiomatically by Noor (2011).[17]

Definition 7 *A* sophisticated quasi-hyperbolic discounting representation *of* $\{\succsim_t\}_{t\in\mathbb{N}}$ *consists of continuous functions* $u : C \to \mathbb{R}$ *and* $U, V : \Delta(C \times Z) \to \mathbb{R}$ *satisfying the following system of equations:*

$$U(p) = \int_{C\times Z} (u(c) + \delta W(x))\, dp(c, x)$$
$$V(p) = \gamma \int_{C\times Z} (u(c) + \beta\delta W(x))\, dp(c, x)$$
$$W(x) = \max_{q\in x}(U(q) + V(q)) - \max_{q\in x} V(q)$$

and such that, for all $t \in \mathbb{N}$,

$$p \succsim_t q \iff U(p) + V(p) \geq U(q) + V(q),$$

where $0 \leq \beta \leq 1$, $0 < \delta < 1$, *and* $\gamma \geq 0$.

The relationship between the self-control and Strotzian models in the dynamic case is similar to their relationship in the two-period model, but now with additional structure. As $\gamma \to \infty$, the representation in Definition 6 converges to the Strotzian version of $(\beta, \hat{\beta}, \delta)$ quasi-hyperbolic discounting with the same parameters.[18] However, there are technical

in actuality.

[16] It is easy to see that the parameters γ and β in our representation are not separately identified. These parameters can be replaced with any γ' and β' that satisfy $\frac{1+\gamma\beta}{1+\gamma} = \frac{1+\gamma'\beta'}{1+\gamma'}$ without altering preferences. Therefore, by Equation (1), if a naive quasi-hyperbolic discounting representation of the preferences exists, then there also exists a representation with $\beta \leq \hat{\beta}$ and $\gamma \geq \hat{\gamma}$.

[17] This is a special case of what Noor (2011) refers to as "quasi-hyperbolic self-control" (see his Definition 2.2 and Theorems 4.5 and 4.6). His representation permits the static felicity function in the expression for V to be another function v and it allows $\beta > 1$.

[18] In fact, when preferences are restricted to full commitment streams, Equation (2) shows that the observed choices of the quasi-hyperbolic self-control model over budget sets of consumption streams can be rationalized by a normalized quasi-hyperbolic Strotzian representation with present bias factor $\frac{1+\gamma\beta}{1+\gamma}$. This fraction converges to β as $\gamma \to \infty$.

difficulties in developing even sophisticated versions of Strotzian models with infinite horizons, as observed by Peleg and Yaari (1973) and Gul and Pesendorfer (2005). While admitting the Strotz model as a limit case, allowing just a touch of self-control through a positive but finite γ allows for recursive formulations and makes the self-control model amenable to applications, as in Gul and Pesendorfer (2004) and Krusell, Kuruşçu, and Smith (2010). Alternate perturbations can also recover continuity; for example, Harris and Laibson (2013) introduce random duration of the "present" time period towards which the agent is tempted to transfer consumption.

3.3 Characterization

The naive version of the quasi-hyperbolic model is new, so its foundations obviously do not yet exist. Related axiomatizations of sophisticated dynamic self-control do exist, for example, Gul and Pesendorfer (2004) and Noor (2011), and we maintain some of their conditions. Recall that (c, x) refers to the degenerate lottery $\delta_{(c,x)}$. Mixtures of menus are defined pointwise: $\lambda x + (1-\lambda)y = \{\lambda p + (1-\lambda)q : p \in x, q \in y\}$. The first six axioms are standard in models of dynamic self-control and appear in Gul and Pesendorfer (2004) and Noor (2011).

Axiom 1 (Weak Order) $\succsim_t$ *is a complete and transitive binary relation.*

Axiom 2 (Continuity) *The sets* $\{p : p \succsim_t q\}$ *and* $\{p : q \succsim_t p\}$ *are closed.*

Axiom 3 (Independence) $p \succ_t q$ *implies* $\lambda p + (1-\lambda)r \succ_t \lambda q + (1-\lambda)r$.

Axiom 4 (Set Betweenness) $(c, x) \succsim_t (c, y)$ *implies* $(c, x) \succsim_t (c, x \cup y) \succsim_t (c, y)$.

Axiom 5 (Indifference to Timing) $\lambda(c, x) + (1-\lambda)(c, y) \sim_t (c, \lambda x + (1-\lambda)y)$.

Axiom 6 (Separability) $\frac{1}{2}(c, x) + \frac{1}{2}(c', y) \sim_t \frac{1}{2}(c, y) + \frac{1}{2}(c', x)$ *and* $(c'', \{\frac{1}{2}(c, x) + \frac{1}{2}(c', y)\}) \sim_t (c'', \{\frac{1}{2}(c, y) + \frac{1}{2}(c', x)\})$.

These first six axioms guarantee that preferences over continuation problems, defined by $(c, x) \succsim_t (c, y)$, can be represented by a self-control representation $(U_t, \hat{V}_t)$. For this section, we restrict attention to stationary preferences. The following stationarity axiom links behavior across time periods and implies the same $(U, \hat{V})$ can be used to represent preferences over continuation problems in every period.

Axiom 7 (Stationarity) $p \succsim_t q$ *if and only if* $p \succsim_{t+1} q$.

The next two axioms are novel and provide more structure on the temptation utility V. Before introducing them, some notation is required. For any $p \in \Delta(C \times Z)$, let p^1 denote the marginal distribution over C and p^2 denote the marginal distribution over Z. For any marginal distributions p^1 and q^2, let $p^1 \times q^2$ denote their product distribution. In particular, $p^1 \times p^2$ is the measure that has the same marginals on C and Z as p, but removes any correlation between the two dimensions. The prior axioms make any correlation irrelevant, so $p \sim_t p^1 \times p^2$. Considering marginals is useful because it permits the replacement of a stream's marginal distribution over continuation problems, holding fixed the marginal distribution over current consumption.

Axiom 8 (Present Bias) *If $q \succ_t p$ and $(c, \{p\}) \succsim_t (c, \{q\})$, then $p \succ_t p^1 \times q^2$.*

In many dynamic models without present bias, an individual prefers p to q in the present if and only if she holds the same ranking when committing for some future period:

$$p \succsim_t q \iff (c, \{p\}) \succsim_t (c, \{q\}). \tag{3}$$

Clearly, this condition is not satisfied by an individual who is present biased, as the prototypical experiment on present bias finds preference reversals occur with temporal distancing. Axiom 8 relaxes this condition: Equation (3) can be violated by preferring q to p today while preferring p to q when committing for the future, but only if q offers better immediate consumption and p offers better future consumption—this is the essence of present bias. Thus replacing the marginal distribution p^2 over continuation values with the marginal q^2 makes the lottery strictly worse, as formalized in our axiom.

The next axiom rules out temptations when there is no intertemporal trade-off. As a consequence, all temptations involve rates of substitution across time, and do not involve static temptations at a single period.

Axiom 9 (No Temptation by Atemporal Choices) *If $p^1 = q^1$ or $p^2 = q^2$, then $(c, \{p, q\}) \succsim_t (c, \{p\})$.*

Correctly anticipating all future choices corresponds to the sophistication condition defined previously in Section 2.2. The following conditions directly apply the definitions for sophistication and naivete introduced in the two-period model to the projection of preferences on future menus. Some subtleties do arise in extending the two-stage definitions of naivete to general environments. In particular, the analogue of a "commitment" in an infinite horizon is not obvious, especially when considering a recursive representation. For example, the notion of a commitment as a singleton choice set in the subsequent period (i.e., an alternative of the form $(c, \{p\})$) is arguably too weak in a recursive representation because such a choice set may still include nontrivial choices at later future dates: It fixes a single lottery over continuation problems in its second component $\{p\} \in Z$, but leaves open what the choice from that period onward will be, since p is itself a lottery over $C \times Z$. Instead, the appropriate analogue of a commitment should fully specify static consumption levels at all dates, that is, a commitment is an element of $\Delta(C^{\mathbb{N}})$. It is important to observe that $\Delta(C^{\mathbb{N}})$ is a strict subset of $\Delta(C \times Z)$.

The following definitions extend the concepts from the two-period model, using elements in $\Delta(C^{\mathbb{N}})$ as the fully committed streams of consumption levels.

Axiom 10 (Sophistication) *For all* $p, q \in \Delta(C^{\mathbb{N}})$ *with* $(c, \{p\}) \succ_t (c, \{q\})$,

$$p \succ_{t+1} q \quad \textit{if and only if} \quad (c, \{p, q\}) \succ_t (c, \{q\}).$$

Axiom 11 (Naivete) *For all* $p, q \in \Delta(C^{\mathbb{N}})$ *with* $(c, \{p\}) \succ_t (c, \{q\})$,

$$p \succ_{t+1} q \quad \textit{implies} \quad (c, \{p, q\}) \succ_t (c, \{q\}).$$

In words, if the alternative chosen in the subsequent period is the same one that the individual would select if she could commit today, then that choice is correctly anticipated. The converse also holds under sophistication, but not necessarily under naivete. Under naivete, the individual may incorrectly anticipate choosing the alternative in the future that is more desirable from today's perspective.

In the two-period model, there is only one immediate future choice period. In the dynamic model, there are many periods beyond $t + 1$. Therefore, Axiom 11 may appear too weak because it only implicates conjectures at period t regarding choices in period $t+1$, but leaves open the possibility of naive conjectures regarding choices in some period $t + \tau$ with $\tau > 1$. For example, one might consider the following, stronger definition of naivete: For every $\tau \geq 1$ and $p, q \in \Delta(C^{\mathbb{N}})$,

$$(\underbrace{c, \ldots, c}_{\tau \text{ periods}}, \{p, q\}) \succ_t (\underbrace{c, \ldots, c}_{\tau \text{ periods}}, \{q\})$$

whenever

$$(\underbrace{c, \ldots, c}_{\tau \text{ periods}}, \{p\}) \succ_t (\underbrace{c, \ldots, c}_{\tau \text{ periods}}, \{q\}) \quad \text{and} \quad p \succ_{t+\tau} q.$$

It turns out that the other axioms that are invoked in our stationary representation will render these additional restrictions for $\tau > 1$ redundant: Together with the other axioms used in our representation theorem, this stronger condition is implied by Axiom 11.

The following representation result characterizes sophisticated and naive stationary quasi-hyperbolic discounting. We say a profile of preference relations $\{\succsim_t\}_{t\in\mathbb{N}}$ is *nontrivial* if, for every $t \in \mathbb{N}$, there exist $c, c' \in C$ and $x \in Z$ such that $(c, x) \succ_t (c', x)$.

Theorem 3

1. *A profile of nontrivial relations $\{\succsim_t\}_{t\in\mathbb{N}}$ satisfies Axioms 1–10 if and only if it has a sophisticated quasi-hyperbolic discounting representation $(u, \gamma, \beta, \delta)$.*
2. *A profile of nontrivial relations $\{\succsim_t\}_{t\in\mathbb{N}}$ satisfies Axioms 1–9 and 11 if and only if it has a naive quasi-hyperbolic discounting representation $(u, \gamma, \hat{\gamma}, \beta, \hat{\beta}, \delta)$.*

3.4 Comparatives

We now study the comparison of naivete in infinite-horizon settings. The following definition adapts our comparative from the two-period setting to the current dynamic environment. Recalling earlier intuition, a more naive individual has more instances where today at period t she anticipates making the more virtuous choice tomorrow at period $t+1$ (captured by the relation $(c, \{p, q\}) \succ_t^1 (c, \{q\})$), yet in reality makes the less virtuous choice at $t+1$ (captured by the relation $q \succ_{t+1}^1 p$).

Definition 8 *Individual 1 is* more naive *than individual 2 if, for all $p, q \in \Delta(C^{\mathbb{N}})$,*

$$\big[(c, \{p, q\}) \succ_t^2 (c, \{q\}) \text{ and } q \succ_{t+1}^2 p\big] \implies \big[(c, \{p, q\}) \succ_t^1 (c, \{q\}) \text{ and } q \succ_{t+1}^1 p\big].$$

The following theorem characterizes comparative naivete for individuals who have quasi-hyperbolic discounting representations. Recall that if individual 2 is sophisticated, i.e., $\frac{1+\hat{\gamma}^2\hat{\beta}^2}{1+\hat{\gamma}^2} = \frac{1+\gamma^2\beta^2}{1+\gamma^2}$, then individual 1 is trivially more naive. Otherwise, if individual 2 is strictly naive, then our comparative measure corresponds to a natural ordering of the present bias factors.

We say $\{\succsim_t^1\}_{t\in\mathbb{N}}$ and $\{\succsim_t^2\}_{t\in\mathbb{N}}$ are *jointly nontrivial* if, for every $t \in \mathbb{N}$, there exist $c, c' \in C$ and $x \in Z$ such that $(c, x) \succ_t^i (c', x)$ for $i = 1, 2$. Joint nontriviality ensures that both u^1 and u^2 are non-constant and that they agree on the ranking $u^i(c) > u^i(c')$ for some pair of consumption alternatives.

Theorem 4 *Suppose $\{\succsim_t^1\}_{t\in\mathbb{N}}$ and $\{\succsim_t^2\}_{t\in\mathbb{N}}$ are jointly nontrivial and admit naive quasi-hyperbolic discounting representations. Then individual 1 is more naive than individual 2 if and only if either individual 2 is sophisticated or $u^1 \approx u^2$, $\delta^1 = \delta^2$, and*

$$\frac{1+\hat{\gamma}^1\hat{\beta}^1}{1+\hat{\gamma}^1} \geq \frac{1+\hat{\gamma}^2\hat{\beta}^2}{1+\hat{\gamma}^2} \geq \frac{1+\gamma^2\beta^2}{1+\gamma^2} \geq \frac{1+\gamma^1\beta^1}{1+\gamma^1}.$$

3.5 Extension: Diminishing Naivete

In this section we relax the stationarity assumption (Axiom 7) used in Theorem 3. There are many ways to formulate a non-stationary model, but motivated by recent research emphasizing individuals' learning about their self-control over time, we consider the following representation.[19]

Definition 9 *A* quasi-hyperbolic discounting representation with diminishing naivete *of $\{\succsim_t\}_{t\in\mathbb{N}}$ consists of continuous functions $u : C \to \mathbb{R}$ and $U_t, \hat{V}_t, V_t : \Delta(C \times Z) \to \mathbb{R}$ for each t satisfying the following system of equations:*

$$\begin{aligned}
U_t(p) &= \int_{C\times Z} (u(c) + \delta \hat{W}_t(x))\, dp(c,x) \\
V_t(p) &= \gamma \int_{C\times Z} (u(c) + \beta\delta \hat{W}_t(x))\, dp(c,x) \\
\hat{V}_t(p) &= \hat{\gamma}_t \int_{C\times Z} (u(c) + \hat{\beta}_t\delta \hat{W}_t(x))\, dp(c,x) \\
\hat{W}_t(x) &= \max_{q\in x}(U_t(q) + \hat{V}_t(q)) - \max_{q\in x} \hat{V}_t(q)
\end{aligned}$$

and such that, for all $t \in \mathbb{N}$,

$$p \succsim_t q \iff U_t(p) + V_t(p) \geq U_t(q) + V_t(q),$$

where $\beta, \hat{\beta}_t \in [0,1]$, $0 < \delta < 1$, and $\gamma, \hat{\gamma}_t \geq 0$ satisfy

$$\frac{1+\hat{\gamma}_t\hat{\beta}_t}{1+\hat{\gamma}_t} \geq \frac{1+\hat{\gamma}_{t+1}\hat{\beta}_{t+1}}{1+\hat{\gamma}_{t+1}} \geq \frac{1+\gamma\beta}{1+\gamma}.$$

In this formulation, the individual's beliefs change to become more accurate over time, as expressed by the last inequality in the definition. The following axiom states that the individual's period-t self is more naive than her period-$(t+1)$ self, that is, she becomes progressively less naive about her future behavior over time.

Axiom 12 (Diminishing Naivete) *For all $p, q \in \Delta(C^{\mathbb{N}})$,*

$$\left[(c,\{p,q\}) \succ_{t+1} (c,\{q\}) \text{ and } q \succ_{t+2} p\right] \implies \left[(c,\{p,q\}) \succ_t (c,\{q\}) \text{ and } q \succ_{t+1} p\right]$$

We will focus in this section on preference profiles that maintain the same actual present bias over time. The only variation over time is in the increasing accuracy of beliefs

[19] Kaur, Kremer, and Mullainathan (2015) find evidence that sophistication about self-control improves over time. Ali (2011) analyzes a Bayesian individual who updates her belief about temptation strength over time.

about present bias in future periods.[20] We therefore impose the following stationarity axiom for preferences over commitment streams of consumption.

Axiom 13 (Commitment Stationarity) *For $p, q \in \Delta(C^{\mathbb{N}})$,*

$$p \succsim_t q \iff p \succsim_{t+1} q.$$

Relaxing Axiom 7 (Stationarity) and instead using Axioms 12 and 13, we obtain the following characterization result for the quasi-hyperbolic discounting model with diminishing naivete.

Theorem 5 *A profile of nontrivial relations $\{\succsim_t\}_{t\in\mathbb{N}}$ satisfies Axioms 1–6, 8–9, and 11–13 if and only if it has a quasi-hyperbolic discounting representation with diminishing naivete $(u, \gamma, \hat{\gamma}_t, \beta, \hat{\beta}_t, \delta)_{t\in\mathbb{N}}$.*

In the diminishing naivete representation, although beliefs about temptation can change over time, the agent is myopic in the sense that she does not anticipate these future changes in beliefs. That is, she does not entertain the possibility that her future selves at periods $s > t$ may have beliefs $\hat{\beta}_s$ and $\hat{\gamma}_s$ that differ from the current beliefs $\hat{\beta}_t$ and $\hat{\gamma}_t$. More complicated alternative models are of course possible, such as allowing partial anticipation of future changes in beliefs, or permitting changes in beliefs to depend on the sequence of past choice sets that have been experienced rather than just on the passage of time. Our representation result is intended as a simple and parsimonious first cut at axiomatic analysis of the issue of changing naivete.

3.6 Application: Consumption-Savings Problem

As a simple exercise in the recursive environment, we apply our stationary naive quasi-hyperbolic discounting representation to a consumption-savings problem. The felicity function is a CRRA utility, that is,

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \text{for } \sigma \neq 1 \\ \log c & \text{for } \sigma = 1, \end{cases}$$

where $\sigma > 0$ is the coefficient of relative risk aversion. Let $R > 0$ denote the gross interest rate.

[20]More general representations are also possible. In the proof of Theorems 4 and 5 in Appendix A.5, we first characterize a more general representation in Proposition 6 in which both actual and anticipated present bias can vary over time.

Slightly abusing notation, let $\hat{W}(m)$ denote the anticipated continuation value as a function of wealth $m \geq 0$. It obeys

$$\hat{W}(m) = \max_{\hat{c}\in[0,m]} \Big[(1+\hat{\gamma})u(\hat{c}) + \delta(1+\hat{\gamma}\hat{\beta})\hat{W}(R(m-\hat{c}))\Big] \\ - \hat{\gamma} \max_{\tilde{c}\in[0,m]} \Big[u(\tilde{c}) + \delta\hat{\beta}\hat{W}(R(m-\tilde{c}))\Big]. \quad (4)$$

The consumption choice at m is given by

$$c(m) \in \operatorname*{argmax}_{c\in[0,m]} \left[u(c) + \delta\frac{1+\gamma\beta}{1+\gamma}\hat{W}(R(m-c))\right].$$

In the following proposition, we focus on a solution to this problem in which the value function takes the same isoelastic form as u, which implies the consumption policy is a linear function of current wealth. We do not know whether there exist solutions that do not have this form. However, the restriction seems natural in this exercise, since a solution of this form is uniquely optimal under the benchmark case of exponential discounting (i.e., $\frac{1+\gamma\beta}{1+\gamma} = \frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}} = 1$).[21]

Proposition 2 *Assume that $(1+\hat{\gamma}\hat{\beta})\delta R^{1-\sigma} < 1$.[22] Then there exist unique $A > 0$ and $B \in \mathbb{R}$ such that*

$$\hat{W}(m) = Au(m) + B$$

is a solution to Equation (4). Moreover, the optimal policy c for this value function satisfies $c(m) = \lambda m$ for some $\lambda \in (0,1)$, and:

1. *If $\sigma < 1$, then A is increasing and λ is decreasing in $\hat{\beta}$.*
2. *If $\sigma = 1$, then A and λ are constant in $\hat{\beta}$.*
3. *If $\sigma > 1$, then A is decreasing and λ is increasing in $\hat{\beta}$.*

In all cases, λ is decreasing in β.

While increasing β always leads to a lower current consumption level $c(m)$, the effect of increasing $\hat{\beta}$ depends on the value of σ. Intuitively, increasing $\hat{\beta}$ leads the individual

[21] The optimality of linear consumption rules also implies that an analyst who only observes consumption-savings choices in this environment cannot determine whether an agent is naive or sophisticated. We briefly discuss in Appendix B how naivete can potentially be tested with consumption-savings data in the context of a finite-horizon example. More importantly, the representation and comparative statics results for our recursive representation (Theorems 3 and 4) show that naivete can be distinguished from sophistication in richer environments that include commitment devices.

[22] This assumption is used to guarantee the unique existence of a solution.

to believe that she will over-consume by less in all future periods. When $\sigma < 1$, this increases her anticipated continuation value $\hat{W}(m)$ for all wealth levels and also increases its marginal value $\hat{W}'(m)$ for all wealth levels. When $\sigma > 1$, increasing $\hat{\beta}$ again increases her anticipated continuation value but now lowers its derivative. Finally, in the case of $\sigma = 1$, increasing $\hat{\beta}$ increases the anticipated continuation value but has no effect on its derivative.

As an analogy, it may be helpful to observe that current consumption moves in the same direction in response to an increase in $\hat{\beta}$ as it does in response to an increase in the gross interest rate R. Recall that, under standard exponential discounting, as R becomes higher, current consumption increases if $\sigma > 1$, is constant if $\sigma = 1$, and decreases if $\sigma < 1$. This is because a higher interest rate generates two conflicting forces: The first is the intertemporal substitution effect that makes current consumption lower, and the second is the income effect that raises current consumption. The first effect dominates when the intertemporal elasticity of substitution $1/\sigma$ is higher than 1, and the second effect dominates if $1/\sigma$ is less than 1. Although the mechanisms through which $\hat{\beta}$ and R impact current consumption are slightly different, in both cases, the impact of a change in the parameter on the derivative of the continuation value depends on the value of σ.

4 Connections and Impossibilities

Ahn, Iijima, Le Yaouanq, and Sarver (2018) consider naivete in a class of Strotz preferences where the individual's ex-post choice maximizes the temptation utility v, rather than the compromise utility $u + v$ as in the self-control model. That paper proposes a different definition of naivete than the one in this paper. A natural question is how either definition would work for the other environment. To facilitate this, we introduce a deterministic version of the two-stage Strotz model.

For any expected-utility function w, let $B_w(x)$ denote the set of w-maximizers in x, that is, $B_w(x) = \operatorname{argmax}_{p \in x} w(p)$.

Definition 10 *A* Strotz representation *of* $(\succsim, \mathcal{C})$ *is a triple* $(u, v, \hat{v})$ *of expected-utility functions such that the function* $U : \mathcal{K}(\Delta(C)) \to \mathbb{R}$ *defined by*

$$U(x) = \max_{p \in B_{\hat{v}}(x)} u(p)$$

represents $\succsim$ *and*

$$\mathcal{C}(x) = B_u(B_v(x)).$$

The following are the definitions of naivete and sophistication for Strotz preferences from Ahn, Iijima, Le Yaouanq, and Sarver (2018), adapted to the current domain.

Definition 11 *An individual is* Strotz sophisticated *if $x \sim \{p\}$ for all menus x and for all $p \in \mathcal{C}(x)$. An individual is* Strotz naive *if $x \succsim \{p\}$ for all menus x and for all $p \in \mathcal{C}(x)$.*

The definition of Strotz naivete is too restrictive in the case of self-control preferences. The following result shows the exact implications of this definition for the self-control representation.

Proposition 3 *Suppose $(\succsim, \mathcal{C})$ is regular and has a self-control representation $(u, v, \hat{v})$ such that $\hat{v}$ is non-constant. Then the individual is Strotz naive (Definition 11) if and only if $\hat{v} \gg_u u + v$.*

One interesting implication of Proposition 3 is that the Heidhues–Koszegi representation in Definition 3 can never be Strotz naive, and hence it requires alternate definitions like those provided in this paper for nonparametric foundations.

It is important to note that the case of $\hat{v} \approx u + v$ does not correspond to being Strotz sophisticated. In fact, Strotz sophistication automatically fails whenever there are lotteries p, q such that $\{p\} \succ \{p, q\} \succ \{q\}$ because there is no selection in $x = \{p, q\}$ that is indifferent to x.

Although the implications of Strotz naivete are too strong when applied to the self-control representation, in the converse direction the definition of naivete proposed in this paper is suitable for Strotz representations. This is because Strotz representations are a limit case of self-control representations. To see this, parameterize a family of representations $(u, \gamma v, \gamma \hat{v})$ and take γ to infinity. Then the vectors v and $\hat{v}$ dominate the smaller u vector in determining actual and anticipated choice. Moreover, since choices are driven almost entirely by temptation, the penalty for self-control diminishes since no self-control is actually exerted. Given appropriate continuity in the limit, our definitions of naivete for self-control representations should therefore also have the correct implications for Strotz representations. Indeed they do.

Proposition 4 *Suppose $(\succsim, \mathcal{C})$ is regular and has a Strotz representation $(u, v, \hat{v})$ such that v and $\hat{v}$ are non-constant. Then, the following are equivalent:*

1. *The individual is naive (resp. sophisticated)*
2. *The individual is Strotz naive (resp. Strotz sophisticated)*
3. *$\hat{v} \gg_u v$ (resp. $\hat{v} \approx v$)*

In this paper, we considered linear self-control costs. In principle, the cost function could be nonlinear, as in the following representation proposed by Noor and Takeoka

(2010). Such preferences maintain the set betweenness axiom of Gul and Pesendorfer (2001) (formally, $x \succsim y$ implies $x \succsim x \cup y \succsim y$) but violate independence, and therefore provide some insight into how robust the definition of naivete is to preferences beyond Strotz and self-control models.[23] Fix expected-utility functions u and v. A *cost function* is a continuous function $c : \Delta(C) \times v(\Delta(C)) \to \mathbb{R}_+$ that is weakly increasing in its second argument and satisfies (i) if $c(p, v(q)) > 0$ then $v(p) < v(q)$, and (ii) if $u(p) > u(q)$ and $v(p) < v(q)$ then $c(p, v(q)) > 0$.

Definition 12 *A* general self-control representation *of* $(\succsim, \mathcal{C})$ *is a triple* $(u, v, \hat{v}, c, \hat{c})$ *of expected-utility functions* $u, v, \hat{v}$ *and cost functions* $c, \hat{c}$ *such that* $\succsim$ *is represented by*

$$U(x) = \max_{p \in x} \Big[u(p) - \hat{c}(p, \max_{q \in x} \hat{v}(q)) \Big]$$

and

$$\mathcal{C}(x) = \operatorname*{argmax}_{p \in x} \Big[u(p) - c(p, \max_{q \in x} v(q)) \Big].$$

The cost function is not assumed to be linear, so this model nests others like Fudenberg and Levine (2006), Noor and Takeoka (2015), Masatlioglu, Nakajima, and Ozdenoren (2019), and Grant, Hsieh, and Liang (2018) (subject to allowing for a discontinuous cost).

Given such a representation, define binary relations over $\Delta(C)$ by

$$p \succ_0^* q \iff u(p) > u(q),$$

$$p \succ_1^* q \iff u(p) - c(p, \max_{r \in \{p,q\}} v(r)) > u(q) - c(q, \max_{r \in \{p,q\}} v(r)),$$

$$p \mathrel{\hat{\succ}_1^*} q \iff u(p) - \hat{c}(p, \max_{r \in \{p,q\}} \hat{v}(r)) > u(q) - \hat{c}(q, \max_{r \in \{p,q\}} \hat{v}(r)),$$

which capture the commitment ranking, actual ex-post ranking, and anticipated ex-post ranking of lotteries. For any lotteries p, q with $\{p\} \succ \{q\}$, observe that

$$\{p, q\} \succ \{q\} \iff p \mathrel{\hat{\succ}_1^*} q. \tag{5}$$

This yields the following straightforward observation.

Proposition 5 *Suppose* $(\succsim, \mathcal{C})$ *has a general self-control representation* $(u, v, \hat{v}, c, \hat{c})$*. Then the individual is naive (resp. sophisticated) if and only if* $(\succ_0^* \cap \succ_1^*) \subseteq (\succ_0^* \cap \hat{\succ}_1^*)$ *(resp.* $(\succ_0^* \cap \succ_1^*) = (\succ_0^* \cap \hat{\succ}_1^*)$*).*

[23] We thank an anonymous referee for suggesting this line of inquiry.

That is, naive agents perceive that $\hat{\succ}_1^*$ is more aligned with $\succ_0^*$ than $\succ_1^*$ is, here measured through the occurrence of shared comparisons. When each of these relations admits a linear representation, such as $u + \hat{v}$ and $u + v$ in the (linear) self-control model, this condition is equivalent to u-alignment under the regularity condition $\succ_0^* \cap \succ_1^* \neq \emptyset$.

The analysis in this section suggests that the definitions of naivete and sophistication proposed in this paper might work well for *deterministic* preferences that satisfy *set betweenness*. Proposition 4 shows that our definitions work well regardless of whether individuals can exert self-control (as in the self-control representation) or not (as in the Strotz representation). Proposition 5 shows that our definitions still yield sensible interpretations for preferences that violate independence but maintain set betweenness. However, our definitions are not immediately suitable for models that violate set betweenness or involve random choice.[24] For example, our definition of naivete does not yield reasonable parametric restrictions in models with multiple simultaneous temptations or stochastic temptations, such as Dekel, Lipman, and Rustichini (2009), Stovall (2010), or Dekel and Lipman (2012), or in the model of perfectionism proposed by Kopylov (2012).

In contrast, Ahn, Iijima, Le Yaouanq, and Sarver (2018) propose a single definition of naivete that is suitable for both Strotz representations and the more general class of *random* Strotz representations considered in Dekel and Lipman (2012), which may violate set betweenness. In addition to the general benefits of using random choice to capture population heterogeneity, a nondegenerate belief about future behavior seems less extreme for a naive agent than a resolute but incorrect belief. Thus, an important advantage of the definition in Ahn, Iijima, Le Yaouanq, and Sarver (2018) is that it generalizes to random Strotz representations where actual and anticipated choices are allowed to be stochastic.

On the other hand, the benefit of the definition in this paper is that it is suitable both for deterministic self-control representations and for deterministic Strotz representations. This begs the question of whether a single definition of naivete exists that can be applied across a general class of models including both random Strotz and self-control representations (and possibly other preferences that relax set betweenness). This is unfortunately impossible. It can be shown that there is no suitable definition of naivete or sophistication that can be applied to both consequentialist and non-consequentialist models once random choice is permitted. This is due to an important identification issue that is introduced when both self-control and randomness are allowed.[25] Therefore, when attempting to differentiate naivete from sophistication (or overly pessimistic beliefs), a fundamental limitation is that one can allow for either randomness or self-control, but one cannot simultaneously accommodate both phenomena.

[24]The latter can serve as examples of the former, since models of random temptation can (although do not necessarily) violate set betweenness.

[25]An explicit example illustrating this issue can be found in an earlier working version of this paper (Ahn, Iijima, and Sarver (2018)) or can be requested from the authors.

A Proofs

A.1 Preliminaries

The following lemma will be used repeatedly in the proofs of our main results. In the case of finite C, the lemma easily follows from Lemma 3 in Dekel and Lipman (2012), who also noted the connection to the Harsanyi Aggregation Theorem. Our analysis of dynamic representations defined on infinite-horizon decision problems necessitates infinite outcome spaces.

Lemma 1 *Let u, w, w' be expected-utility functions defined on $\Delta(C)$ such that u and w' are not ordinally opposed.*[26] *Then the following are equivalent:*

1. *For all lotteries p and q, $\big[u(p) > u(q)$ and $w'(p) > w'(q)\big] \implies w(p) > w(q)$*
2. *There exist scalars $a, b \geq 0$ and $c \in \mathbb{R}$ such that $a + b > 0$ and $w = au + bw' + c$*
3. *$w \gg_u w'$*

Proof: The direction 1 $\Rightarrow$ 2 follows from the affine aggregation result shown in Proposition 2 of De Meyer and Mongin (1995). The direction 2 $\Rightarrow$ 3 follows by $w \approx \alpha u + (1-\alpha)w'$ for $\alpha = a/(a+b) \in [0,1]$. The direction 3 $\Rightarrow$ 1 is clear from the definition of $\gg_u$. ∎

A.2 Proof of Theorem 1

Sufficiency: To establish sufficiency, suppose the individual is naive. Then, for any lotteries p and q,

$$\begin{aligned}\big[u(p) > u(q) \text{ and } (u+v)(p) > (u+v)(q)\big] &\implies \mathcal{C}(\{p,q\}) = \{p\} \succ \{q\} \\ &\implies \{p,q\} \succ \{q\} \qquad \text{(by naivete)} \\ &\implies (u+\hat{v})(p) > (u+\hat{v})(q).\end{aligned}$$

Regularity requires that u and $u + v$ not be ordinally opposed. Therefore, Lemma 1 implies $u + \hat{v} \gg_u u + v$.

If in addition the individual is sophisticated, then an analogous argument leads to

$$\big[u(p) > u(q) \text{ and } (u+\hat{v})(p) > (u+\hat{v})(q)\big] \implies (u+v)(p) > (u+v)(q)$$

for any lotteries p and q, which ensures $u + v \gg_u u + \hat{v}$ by Lemma 1. Thus $u + v \approx u + \hat{v}$.

[26] That is, there exist lotteries p and q such that both $u(p) > u(q)$ and $w'(p) > w'(q)$.

Necessity: To establish necessity, suppose $u+\hat{v} \approx \alpha u+(1-\alpha)(u+v)$ for $\alpha \in [0,1]$ and take any lotteries p and q. Then

$$\begin{aligned}
\big[\{p\} \succ \{q\} \text{ and } \mathcal{C}(\{p,q\}) = \{p\}\big] &\implies \big[u(p) > u(q) \text{ and } (u+v)(p) > (u+v)(q)\big] \\
&\implies \big[u(p) > u(q) \text{ and } (u+\hat{v})(p) > (u+\hat{v})(q)\big] \\
&\implies \{p,q\} \succ \{q\},
\end{aligned}$$

and thus the individual is naive. If in addition $u+v \approx u+\hat{v}$, then one can analogously show

$$\big[\{p\} \succ \{q\} \text{ and } \{p,q\} \succ \{q\}\big] \implies \mathcal{C}(\{p,q\}) = \{p\},$$

and thus the individual is sophisticated.

A.3 Proof of Theorem 2

We first make an observation that will be useful later in the proof. Since each individual is assumed to be naive, Theorem 1 implies $u_i + \hat{v}_i \approx \alpha_i u_i + (1-\alpha_i)(u_i+v_i)$ for some $\alpha_i \in [0,1]$, and consequently, for any lotteries p and q,

$$\big[(u_i+\hat{v}_i)(p) > (u_i+\hat{v}_i)(q) \text{ and } (u_i+v_i)(q) > (u_i+v_i)(p)\big] \implies u_i(p) > u_i(q).$$

Therefore, for any lotteries p and q,

$$\begin{aligned}
&\big[(u_i+\hat{v}_i)(p) > (u_i+\hat{v}_i)(q) \text{ and } (u_i+v_i)(q) > (u_i+v_i)(p)\big] \\
&\iff \big[u_i(p) > u_i(q) \text{ and } (u_i+\hat{v}_i)(p) > (u_i+\hat{v}_i)(q) \text{ and } (u_i+v_i)(q) > (u_i+v_i)(p)\big] \\
&\iff \big[\{p,q\} \succ_i \{q\} \text{ and } \mathcal{C}_i(\{p,q\}) = \{q\}\big].
\end{aligned} \tag{6}$$

Sufficiency: Suppose individual 1 is more naive than individual 2. By Equation (6), this can equivalently be stated as

$$\begin{aligned}
&\big[(u_2+\hat{v}_2)(p) > (u_2+\hat{v}_2)(q) \text{ and } (u_2+v_2)(q) > (u_2+v_2)(p)\big] \\
&\qquad \implies \big[(u_1+\hat{v}_1)(p) > (u_1+\hat{v}_1)(q) \text{ and } (u_1+v_1)(q) > (u_1+v_1)(p)\big].
\end{aligned}$$

If individual 2 is sophisticated then the conclusion of the theorem is trivially satisfied, so suppose not. Then individual 2 must be strictly naive, and hence there must exist lotteries p and q such that $(u_2+\hat{v}_2)(p) > (u_2+\hat{v}_2)(q)$ and $(u_2+v_2)(q) > (u_2+v_2)(p)$. Thus the functions $(u_2+\hat{v}_2)$ and $-(u_2+v_2)$ are not ordinally opposed. Therefore, by Lemma 1, there exist scalars $a, \hat{a}, b, \hat{b} \geq 0$ and $c, \hat{c} \in \mathbb{R}$ such that

$$\begin{aligned}
u_1 + \hat{v}_1 &= \hat{a}(u_2+\hat{v}_2) - \hat{b}(u_2+v_2) + \hat{c}, \\
-(u_1+v_1) &= a(u_2+\hat{v}_2) - b(u_2+v_2) + c.
\end{aligned}$$

Taking b times the first expression minus $\hat{b}$ times the second, and taking a times the first expression minus $\hat{a}$ times the second yields the following:

$$\begin{aligned} b(u_1+\hat{v}_1)+\hat{b}(u_1+v_1) &= (\hat{a}b-a\hat{b})(u_2+\hat{v}_2)+(b\hat{c}-\hat{b}c), \\ a(u_1+\hat{v}_1)+\hat{a}(u_1+v_1) &= (\hat{a}b-a\hat{b})(u_2+v_2)+(a\hat{c}-\hat{a}c). \end{aligned} \tag{7}$$

Claim 1 *Since* $(\succsim_1, \mathcal{C}_1)$ *and* $(\succsim_2, \mathcal{C}_2)$ *are jointly regular,* $\hat{a}b > a\hat{b}$*. In particular,* $\hat{a} > 0$*,* $b > 0$*, and* $\frac{b}{\hat{b}+b} > \frac{a}{\hat{a}+a}$.

Proof: Joint regularity requires there exist lotteries p and q such that $u_i(p) > u_i(q)$ and $(u_i+v_i)(p) > (u_i+v_i)(q)$ for $i = 1, 2$. Since both individuals are naive, by Theorem 1 this also implies $(u_i+\hat{v}_i)(p) > (u_i+\hat{v}_i)(q)$. Thus

$$\begin{aligned} \hat{a}(u_2+\hat{v}_2)(p)-\hat{b}(u_2+v_2)(p) &= (u_1+\hat{v}_1)(p)-\hat{c} \\ &> (u_1+\hat{v}_1)(q)-\hat{c} = \hat{a}(u_2+\hat{v}_2)(q)-\hat{b}(u_2+v_2)(q), \\ a(u_2+\hat{v}_2)(q)-b(u_2+v_2)(q) &= -(u_1+v_1)(q)-c \\ &> -(u_1+v_1)(p)-c = a(u_2+\hat{v}_2)(p)-b(u_2+v_2)(p). \end{aligned}$$

Rearranging terms, these equations imply

$$\begin{aligned} \hat{a}(u_2+\hat{v}_2)(p-q) &> \hat{b}(u_2+v_2)(p-q) \\ b(u_2+v_2)(p-q) &> a(u_2+\hat{v}_2)(p-q). \end{aligned}$$

Multiplying these inequalities, and using the fact that $(u_2+\hat{v}_2)(p-q) > 0$ and $(u_2+v_2)(p-q) > 0$ by the regularity inequalities for individual 2, we have $\hat{a}b > a\hat{b}$. This implies $(\hat{a}+a)b > a(\hat{b}+b)$, and hence $\frac{b}{\hat{b}+b} > \frac{a}{\hat{a}+a}$. ■

By Claim 1, Equation (7) implies

$$\begin{aligned} u_2+\hat{v}_2 &\approx \hat{\alpha}(u_1+\hat{v}_1)+(1-\hat{\alpha})(u_1+v_1), \\ u_2+v_2 &\approx \alpha(u_1+\hat{v}_1)+(1-\alpha)(u_1+v_1), \end{aligned}$$

where

$$\hat{\alpha} = \frac{b}{\hat{b}+b} > \frac{a}{\hat{a}+a} = \alpha.$$

Since $u_1+\hat{v}_1$ is itself an affine transformation of a convex combination of u_1 and u_1+v_1, we have

$$u_1+\hat{v}_1 \gg_{u_1} u_2+\hat{v}_2 \gg_{u_1} u_2+v_2 \gg_{u_1} u_1+v_1,$$

as claimed.

Necessity: If individual 2 is sophisticated, then trivially individual 1 is more naive than individual 2. Consider now the case where individual 2 is strictly naive and

$$u_1 + \hat{v}_1 \gg_{u_1} u_2 + \hat{v}_2 \gg_{u_1} u_2 + v_2 \gg_{u_1} u_1 + v_1,$$

which can equivalently be stated as

$$\begin{aligned} u_2 + \hat{v}_2 &\approx \hat{\alpha}(u_1 + \hat{v}_1) + (1 - \hat{\alpha})(u_1 + v_1), \\ u_2 + v_2 &\approx \alpha(u_1 + \hat{v}_1) + (1 - \alpha)(u_1 + v_1), \end{aligned}$$

for $\hat{\alpha} > \alpha$. Then, for any lotteries p and q,

$$\begin{aligned} &\big[(u_2 + \hat{v}_2)(p) > (u_2 + \hat{v}_2)(q) \text{ and } (u_2 + v_2)(q) > (u_2 + v_2)(p)\big] \\ \implies\ &\hat{\alpha}(u_1 + \hat{v}_1)(p - q) + (1 - \hat{\alpha})(u_1 + v_1)(p - q) \\ &\qquad > 0 > \alpha(u_1 + \hat{v}_1)(p - q) + (1 - \alpha)(u_1 + v_1)(p - q) \\ \implies\ &\big[(u_1 + \hat{v}_1)(p) > (u_1 + \hat{v}_1)(q) \text{ and } (u_1 + v_1)(q) > (u_1 + v_1)(p)\big]. \end{aligned}$$

By Equation (6), this condition is equivalent to individual 1 being more naive than 2.

A.4 Proof of Proposition 1

Proof of 2 $\Rightarrow$ 1: The relation $\succsim$ has no preference for commitment when $\hat{\gamma} = 0$. Otherwise, when $\hat{\gamma} > 0$, $\{p\} \sim \{p, q\} \succ \{q\}$ is equivalent to $u(p) > u(q)$ and $\bar{v}(p) \geq \bar{v}(q)$. Thus $(u + \gamma\bar{v})(p) > (u + \gamma\bar{v})(q)$ for any $\gamma \geq 0$, and hence $\mathcal{C}(\{p, q\}) = \{p\}$.

Proof of 1 $\Rightarrow$ 2: If $\succsim$ has no preference for commitment, let $\bar{v} = v$, $\gamma = 1$, and $\hat{\gamma} = 0$. In the alternative case where $\succsim$ has a preference for commitment (so $\hat{v}$ is non-constant and $\hat{v} \not\approx u$), condition 1 requires that for any p and q,

$$\begin{aligned} [u(p) > u(q) \text{ and } \hat{v}(p) \geq \hat{v}(q)] &\iff \{p\} \sim \{p, q\} \succ \{q\} \\ &\implies \mathcal{C}(\{p, q\}) = \{p\} \\ &\iff (u + v)(p) > (u + v)(q). \end{aligned} \tag{8}$$

We assumed there exist some pair of lotteries p and q such that $\{p\} \sim \{p, q\} \succ \{q\}$. Therefore, u and $\hat{v}$ are not ordinally opposed. Thus, by Lemma 1, $u + v \gg_u \hat{v}$. That is, $u + v \approx \alpha u + (1 - \alpha)\hat{v}$ for some $0 \leq \alpha \leq 1$.

Note that $u \not\approx \hat{v}$ since $\succsim$ has a preference for commitment, and $u \not\approx -\hat{v}$ since the two functions are not ordinally opposed. Therefore, there must exist lotteries p and q such that $u(p) > u(q)$ and $\hat{v}(p) = \hat{v}(q)$. By Equation (8), this implies $(u + v)(p) > (u + v)(q)$. Hence $u + v \not\approx \hat{v}$, that is, $\alpha > 0$. We therefore have $u + v \approx u + \frac{1-\alpha}{\alpha}\hat{v}$. Let $\bar{v} = \hat{v}$, $\gamma = \frac{1-\alpha}{\alpha}$, and $\hat{\gamma} = 1$.

A.5 Proof of Theorems 3 and 5

We begin by proving a general representation result using the following weaker form of stationarity.

Axiom 14 (Weak Commitment Stationarity) *For* $p, q \in \Delta(C^{\mathbb{N}})$,

$$(c, \{p\}) \succsim_t (c, \{q\}) \iff (c, \{p\}) \succsim_{t+1} (c, \{q\}).$$

Axiom 14 permits the actual present bias to vary over time. After proving the following general result, we add Axiom 7 (Stationarity) to prove Theorem 3, and we add Axioms 12 (Diminishing Naivete) and 13 (Commitment Stationarity) to prove Theorem 5.

Proposition 6 *A profile of nontrivial relations* $\{\succsim_t\}_{t\in\mathbb{N}}$ *satisfies Axioms 1–6, 8–9, 11, and 14 if and only if there exist continuous functions* $u : C \to \mathbb{R}$ *and* $U_t, \hat{V}_t, V_t : \Delta(C\times Z) \to \mathbb{R}$ *satisfying the following system of equations:*

$$\begin{aligned}
U_t(p) &= \int_{C\times Z} (u(c) + \delta \hat{W}_t(x))\, dp(c,x) \\
V_t(p) &= \gamma_t \int_{C\times Z} (u(c) + \beta_t\delta \hat{W}_t(x))\, dp(c,x) \\
\hat{V}_t(p) &= \hat{\gamma}_t \int_{C\times Z} (u(c) + \hat{\beta}_t\delta \hat{W}_t(x))\, dp(c,x) \\
\hat{W}_t(x) &= \max_{q\in x}(U_t(q) + \hat{V}_t(q)) - \max_{q\in x} \hat{V}_t(q)
\end{aligned}$$

and such that, for all $t \in \mathbb{N}$,

$$p \succsim_t q \iff U_t(p) + V_t(p) \geq U_t(q) + V_t(q),$$

where $\beta_t, \hat{\beta}_t \in [0,1]$, $0 < \delta < 1$, *and* $\gamma_t, \hat{\gamma}_t \geq 0$ *satisfy*

$$\frac{1+\hat{\gamma}_t\hat{\beta}_t}{1+\hat{\gamma}_t} \geq \frac{1+\gamma_{t+1}\beta_{t+1}}{1+\gamma_{t+1}}. \tag{9}$$

Moreover, $\{\succsim_t\}_{t\in\mathbb{N}}$ *also satisfies Axiom 10 if and only if Equation (9) holds with equality.*

A.5.1 Proof of Proposition 6

We only show the sufficiency of the axioms. Axioms 1–3 imply there exist continuous functions $f_t : C \times Z \to \mathbb{R}$ for $t \in \mathbb{N}$ such that

$$p \succsim_t q \iff \int f_t(c,x)\, dp(c,x) \geq \int f_t(c,x)\, dq(c,x).$$

The first part of Axiom 6 (Separability) implies that f is separable, so

$$f_t(c,x) = f_t^1(c) + f_t^2(x)$$

for some continuous functions f_t^1 and f_t^2. In addition, Axiom 5 (Indifference to Timing) implies f_t is linear in the second argument: $\lambda f_t(c,x)+(1-\lambda)f_t(c,y) = f_t(c,\lambda x+(1-\lambda)y)$. Equivalently,

$$\lambda f_t^2(x) + (1-\lambda)f_t^2(y) = f_t^2(\lambda x + (1-\lambda)y).$$

Next, Axiom 14 (Weak Commitment Stationarity) implies that, for any $p,q \in \Delta(C^{\mathbb{N}})$,

$$f_t^2(\{p\}) \geq f_t^2(\{q\}) \iff f_{t+1}^2(\{p\}) \geq f_{t+1}^2(\{q\}).$$

By the linearity of f_t^2, this implies that, for any $t,t' \in \mathbb{N}$, the restrictions of f_t^2 and $f_{t'}^2$ to deterministic consumption streams in $C^{\mathbb{N}}$ are identical up to a positive affine transformation. Therefore, by taking an affine transformation of each f_t, we can without loss of generality assume that $f_t^2(\{p\}) = f_{t'}^2(\{p\})$ for all $t,t' \in \mathbb{N}$ and for all $p \in \Delta(C^{\mathbb{N}})$.

Define a preference $\succsim_t^*$ over Z by $x \succsim_t^* y$ if and only if $f_t^2(x) \geq f_t^2(y)$ or, equivalently, $(c,x) \succsim_t (c,y)$. Note that this induced preference does not depend on the choice of c by separability. Axioms 1-5 imply that the induced preference over menus Z satisfies Axioms 1-4 in Gul and Pesendorfer (2001). Specifically, the linearity of f_t^2 in the menu (which we obtained using the combination of Axioms 3 and 5) implies that $\succsim_t^*$ satisfies the independence axiom for mixtures of menus (Gul and Pesendorfer, 2001, Axiom 3). Their other axioms are direct translations of ours. Thus, for each $t \in \mathbb{N}$, there exist continuous and linear functions U_t, $\hat{V}_t : \Delta(C \times Z) \to \mathbb{R}$ such that

$$x \succsim_t^* y \iff \max_{p\in x}(U_t(p) + \hat{V}_t(p)) - \max_{q\in x}\hat{V}_t(q) \geq \max_{p\in y}(U_t(p) + \hat{V}_t(p)) - \max_{q\in y}\hat{V}_t(q).$$

Since both f_t^2 and this self-control representation are linear in menus, they must be the same up to an affine transformation. Taking a common affine transformation of U_t and $\hat{V}_t$ if necessary, we therefore have

$$f_t^2(x) = \max_{p\in x}(U_t(p) + \hat{V}_t(p)) - \max_{q\in x}\hat{V}_t(q). \tag{10}$$

By Equation (10), $f_t^2(\{p\}) = U_t(p)$ for all $p \in \Delta(C \times Z)$. Thus the second part of Axiom 6 (Separability) implies that U_t is separable, so

$$U_t(c,x) = u_t^1(c) + u_t^2(x) \tag{11}$$

for some continuous functions u_t^1 and u_t^2.

Claim 2 *There exist scalars $\theta_{t,i}^u, \alpha_{t,i}^u$ for $i = 1,2$ with $\theta_{t,2}^u \geq \theta_{t,1}^u > 0$ such that $u_t^i = \theta_{t,i}^u f_t^i + \alpha_{t,i}^u$.*

Proof: Axiom 8 (Present Bias) ensures that (i) $u_t^1 \approx f_t^1$ and (ii) $u_t^2 \approx f_t^2$. To show (i), take any p, q such that $p^2 = q^2$ and $f_t^1(q^1) > f_t^1(p^1)$.[27] Then $q \succ_t p$ and $p \sim_t p^1 \times q^2$, which implies $(c, \{q\}) \succ_t (c, \{p\})$ by Axiom 8. Thus $u_t^1(q^1) > u_t^1(p^1)$, and the claim follows since f_t^1 is non-constant (by nontriviality). To show (ii), take any p, q such that $p^1 = q^1$ and $f_t^2(q^2) > f_t^2(p^2)$. Then $q \succ_t p$ and $p \prec_t p^1 \times q^2$, which implies $(c, \{q\}) \succ_t (c, \{p\})$ by Axiom 8. Thus $u_t^2(q^2) > u_t^2(p^2)$, and the claim follows since f_t^2 is non-constant (by Equation (10) and u_t^1 non-constant).

Thus we can write $u_t^i = \theta_{t,i}^u f_t^i + \alpha_{t,i}^u$ for some constants $\theta_{t,i}^u, \alpha_{t,i}^u$ with $\theta_{t,i}^u > 0$ for $i = 1, 2$. Finally, toward a contradiction, suppose that $\theta_{t,2}^u < \theta_{t,1}^u$. Then, since f_t^1 and f_t^2 are non-constant, we can take p, q such that $f_t^1(p^1) > f_t^1(q^1)$, $f_t^2(p^2) < f_t^2(q^2)$, and

$$\frac{\theta_{t,2}^u}{\theta_{t,1}^u} < \frac{f_t^1(p^1) - f_t^1(q^1)}{f_t^2(q^2) - f_t^2(p^2)} < 1.$$

The first inequality implies $\theta_{t,1}^u f_t^1(q^1) + \theta_{t,2}^u f_t^2(q^2) < \theta_{t,1}^u f_t^1(p^1) + \theta_{t,2}^u f_t^2(p^2)$, and hence $U_t(p) > U_t(q)$ or, equivalently, $(c, \{p\}) \succ_t (c, \{q\})$. The second inequality implies $f_t^1(p^1) + f_t^2(p^2) < f_t^1(q^1) + f_t^2(q^2)$, and hence $q \succ_t p$. Axiom 8 therefore requires that $p \succ_t p^1 \times q^2$. However, since $f_t^2(p^2) < f_t^2(q^2)$, we have $p^1 \times q^2 \succ_t p$, a contradiction. Thus we must have $\theta_{t,2}^u \geq \theta_{t,1}^u$. ■

Claim 3 *For all $t, t' \in \mathbb{N}$, $\theta_{t,2}^u = \theta_{t',2}^u \in (0, 1)$ and $u_t^1(c) + \alpha_{t,2}^u = u_{t'}^1(c) + \alpha_{t',2}^u$ for all $c \in C$.*

Proof: Note that by Equations (10) and (11) and Claim 2, for any $(c_0, c_1, c_2, \dots) \in C^{\mathbb{N}}$,[28]

$$\begin{aligned} f_t^2(c_0, c_1, c_2, \dots) &= U_t(c_0, c_1, c_2, \dots) \\ &= u_t^1(c_0) + u_t^2(c_1, c_2, \dots) \\ &= u_t^1(c_0) + \alpha_{t,2}^u + \theta_{t,2}^u f_t^2(c_1, c_2, \dots). \end{aligned} \tag{12}$$

Following the same approach as Gul and Pesendorfer (2004, page 151), we show $\theta_{t,2}^u < 1$ using continuity. Fix any $c \in C$ and let $x^c = \{(c, c, c, \dots)\} = \{(c, x^c)\}$. Fix any other consumption stream $y = \{(c_0, c_1, c_2, \dots)\} \in Z$ such that $f_t^2(y) \neq f_t^2(x^c)$. Let $y^1 = \{(c, y)\}$ and define y^n inductively by $y^n = \{(c, y^{n-1})\}$. Then $y^n \to z^c$, and therefore by continuity,

$$f_t^2(y^n) - f_t^2(x^c) = \left(\theta_{t,2}^u\right)^n \left(f_t^2(y) - f_t^2(x^c)\right) \to 0,$$

which requires that $\theta_{t,2}^u < 1$.

Recall that $f_t^2(\{p\}) = f_{t'}^2(\{p\})$ for all $t, t' \in \mathbb{N}$ and for all $p \in \Delta(C^{\mathbb{N}})$ or, equivalently, $U_t|_{\Delta(C^{\mathbb{N}})} = U_{t'}|_{\Delta(C^{\mathbb{N}})}$. Therefore, by Equation (12), we must have $\theta_{t,2}^u = \theta_{t',2}^u$ and $u_t^1(c) + \alpha_{t,2}^u =$

[27] We write $f_t^1(p^1)$ to denote $\int f_t^1(c)\, dp^1(c)$, and write $f_t^2(p^2)$ to denote $\int f_t^2(x)\, dp^2(x)$. We adopt similar notational conventions for u_t^1 and u_t^2.

[28] Our notation here is slightly informal. More precisely, for any $(c_0, c_1, c_2, \dots) \in C^{\mathbb{N}}$, there exists $x^i \in Z$ for $i = 0, 1, 2, \dots$ such that $x^i = \{(c_i, x^{i+1})\}$. To simplify notation, we write $(c_0, c_1, c_2, \dots)$ to indicate the menu $x^0 = \{(c_0, x^1)\} = \{(c_0, \{(c_1, x^2)\})\} = \cdots$.

$u^1_{t'}(c) + \alpha^u_{t',2}$ for all $c \in C$, as claimed. ■

To begin constructing the representation, set $\delta \equiv \theta^u_{t,2} \in (0,1)$ and

$$u(c) \equiv u^1_t(c) + \alpha^u_{t,2} = \theta^u_{t,1} f^1_t(c) + \alpha^u_{t,1} + \alpha^u_{t,2}.$$

Claim 3 ensures that δ and u are well-defined, as they do not depend on the choice of t. Set $\hat{W}_t(x) \equiv f^2_t(x)$ and hence, by Equation (11) and Claim 2,

$$U_t(c,x) = \theta^u_{t,1} f^1_t(c) + \alpha^u_{t,1} + \theta^u_{t,2} f^2_t(x) + \alpha^u_{t,2} = u(c) + \delta \hat{W}_t(x),$$

so the first displayed equation in Proposition 6 is satisfied.

By Claim 2, we have $0 < \theta^u_{t,1}/\theta^u_{t,2} \leq 1$. Therefore, there exist $\gamma_t \geq 0$ and $\beta_t \in [0,1]$ such that

$$\frac{1+\gamma_t\beta_t}{1+\gamma_t} = \frac{\theta^u_{t,1}}{\theta^u_{t,2}}.$$

Note that there are multiple values of γ_t and β_t that satisfy this equality, so these parameters are not individually identified from preferences. Next, defining V_t as in the second displayed equation in Proposition 6, we have

$$\begin{aligned}(U_t + V_t)(c,x) &= (1+\gamma_t)u(c) + (1+\gamma_t\beta_t)\delta\hat{W}_t(x) \\ &= (1+\gamma_t)\Big(u(c) + \frac{1+\gamma_t\beta_t}{1+\gamma_t}\delta\hat{W}_t(x)\Big) \\ &= (1+\gamma_t)\Big(\theta^u_{t,1} f^1_t(c) + \theta^u_{t,1} f^2_t(x) + \alpha^u_{t,1} + \alpha^u_{t,2}\Big),\end{aligned}$$

which is a positive affine transformation of $f_t(c,x)$. Thus

$$p \succsim_t q \iff U_t(p) + V_t(p) \geq U_t(q) + V_t(q),$$

The next claims are used to establish the desired form for $\hat{V}_t$.

Claim 4 *The function $\hat{V}_t$ is separable for all t, so $\hat{V}_t(c,x) = \hat{v}^1_t(c) + \hat{v}^2_t(x)$.*

Proof: It suffices to show that correlation does affect the value assigned to a lottery p by the function $\hat{V}_t$. That is, we only need to show $\hat{V}_t(p) = \hat{V}_t(p^1 \times p^2)$ for all lotteries p.[29] We will show that non-equality leads to a contradiction of Axiom 9 (No Temptation by Atemporal Choices) by considering two cases. For now, restrict attention to lotteries in the set

$$A = \Big\{p \in \Delta(C \times Z) : \min_{c\in C} u^1_t(c) < u^1_t(p^1) < \max_{c \in C} u^1_t(c)\Big\}.$$

[29] To see that this condition is sufficient for separability, fix any $\bar{c} \in C$ and $\bar{x} \in Z$, and define $\hat{v}^1_t(c) \equiv \hat{V}_t(c,\bar{x})$ and $\hat{v}^2_t(x) \equiv \hat{V}_t(\bar{c},x) - \hat{V}_t(\bar{c},\bar{x})$. For any (c,x), let $p = \frac{1}{2}\delta_{(c,x)} + \frac{1}{2}\delta_{(\bar{c},\bar{x})}$ and $q = \frac{1}{2}\delta_{(c,\bar{x})} + \frac{1}{2}\delta_{(\bar{c},x)}$. Then $p^1 \times p^2 = q^1 \times q^2$, so $\hat{V}_t(p) = \hat{V}_t(q)$. Thus $\hat{V}_t(c,x) + \hat{V}_t(\bar{c},\bar{x}) = \hat{V}_t(c,\bar{x}) + \hat{V}_t(\bar{c},x)$ or, equivalently, $\hat{V}_t(c,x) = \hat{v}^1_t(c) + \hat{v}^2_t(x)$.

Case (i): $\hat{V}_t(p) > \hat{V}_t(p^1 \times p^2)$. By the continuity of $\hat{V}_t$, there exists $q^1 \in \Delta(C)$ such that $u_t^1(q^1) > u_t^1(p^1)$ and $\hat{V}_t(p) > \hat{V}_t(q^1 \times p^2)$. The first inequality implies $U_t(q^1 \times p^2) > U_t(p)$. By the self-control representation in Equation (10), this implies $(c, \{q^1 \times p^2\}) \succ_t (c, \{q^1 \times p^2, p\})$, in violation of Axiom 9.

Case (ii): $\hat{V}_t(p) < \hat{V}_t(p^1 \times p^2)$. By the continuity of $\hat{V}_t$, there exists $q^1 \in \Delta(C)$ such that $u_t^1(q^1) < u_t^1(p^1)$ and $\hat{V}_t(p) < \hat{V}_t(q^1 \times p^2)$. The first inequality implies $U_t(p) > U_t(q^1 \times p^2)$. By the self-control representation in Equation (10), this implies $(c, \{p\}) \succ_t (c, \{p, q^1 \times p^2\})$, in violation of Axiom 9.

We have now shown that $\hat{V}_t(p) = \hat{V}_t(p^1 \times p^2)$ for all $p \in A$. Since u_t^1 is non-constant, A is dense in $\Delta(C \times Z)$. By the continuity of $\hat{V}_t$, we therefore have $\hat{V}_t(p) = \hat{V}_t(p^1 \times p^2)$ for all $p \in \Delta(C \times Z)$. ■

Claim 5 *There exist scalars $\theta_{t,i}^v \geq 0$ and $\alpha_{t,i}^v \in \mathbb{R}$ for $i = 1, 2$ such that $\hat{v}_t^1 = \theta_{t,1}^v u + \alpha_{t,1}^v$ and $\hat{v}_t^2 = \theta_{t,2}^v \delta \hat{W}_t + \alpha_{t,2}^v$.*

Proof: Axiom 9 (No Temptation by Atemporal Choices) ensures that (i) $\hat{v}_t^1 \approx u$ or $\hat{v}_t^1$ is constant, and (ii) $\hat{v}_t^2 \approx \delta \hat{W}_t$ or $\hat{v}_t^2$ is constant. To show (i), take any p, q with $p^2 = q^2$ and $u(p^1) > u(q^1)$. Then $u_t^1(p^1) > u_t^2(q^1)$ and hence $(c, \{p\}) \sim_t (c, \{p, q\}) \succ_t (c, \{q\})$ by Axiom 9, which requires that $\hat{v}_t^1(p^1) \geq \hat{v}_t^1(q^1)$. Since u is non-constant, the desired claim follows. Part (ii) is analogously shown by taking any p, q with $p^1 = q^1$ and $\delta \hat{W}_t(p^2) > \delta \hat{W}_t(q^2)$. Then $u_t^2(p^2) > u_t^2(q^2)$ and hence $(c, \{p\}) \sim_t (c, \{p, q\}) \succ_t (c, \{q\})$ by Axiom 9, which implies $\hat{v}_t^2(p^2) \geq \hat{v}_t^2(q^2)$. ■

By Equation (10), changing $\hat{V}_t$ by the addition of a scalar does not alter the function f_t^2. Therefore, we can without loss of generality assume that $\alpha_{t,1}^v = \alpha_{t,2}^v = 0$. We next characterize the implications of naivete and sophistication.

Claim 6

1. *If $\{\succsim_t\}_{t \in \mathbb{N}}$ satisfies Axiom 11 (Naivete), then* $\frac{1+\gamma_{t+1}\beta_{t+1}}{1+\gamma_{t+1}} \leq \frac{1+\theta_{t,2}^v}{1+\theta_{t,1}^v} \leq 1$.

2. *If $\{\succsim_t\}_{t \in \mathbb{N}}$ satisfies Axiom 10 (Sophistication), then* $\frac{1+\gamma_{t+1}\beta_{t+1}}{1+\gamma_{t+1}} = \frac{1+\theta_{t,2}^v}{1+\theta_{t,1}^v} \leq 1$.

Proof: To prove 1, note that for all $p, q \in \Delta(C^{\mathbb{N}})$,

$$\begin{aligned}
&\big[U_t(p) > U_t(q) \text{ and } (U_{t+1} + V_{t+1})(p) > (U_{t+1} + V_{t+1})(q)\big] \\
&\qquad\qquad \implies \big[(c, \{p\}) \succ_t (c, \{q\}) \text{ and } p \succ_{t+1} q\big] \\
&\qquad\qquad \implies (c, \{p, q\}) \succ_t (c, \{q\}) \qquad\qquad \text{(by Axiom 11)} \\
&\qquad\qquad \implies (U_t + \hat{V}_t)(p) > (U_t + \hat{V}_t)(q).
\end{aligned}$$

Since U_t and $U_{t+1}+V_{t+1}$ both rank constant consumption streams $(c, c, c, \dots) \in C^{\mathbb{N}}$ in accordance with u, they are not ordinally opposed on $\Delta(C^{\mathbb{N}})$. Therefore, by Lemma 1, there exists $\alpha \in [0, 1]$ such that

$$(U_t + \hat{V}_t)\big|_{\Delta(C^{\mathbb{N}})} \approx (\alpha U_t + (1-\alpha)(U_{t+1} + V_{t+1}))\big|_{\Delta(C^{\mathbb{N}})}.$$

Thus, for any $(c_0, c_1, c_2, \dots) \in C^{\mathbb{N}}$,

$$\begin{aligned}(U_t + \hat{V}_t)(c_0, c_1, c_2, \dots) &= (u + \hat{v}_t^1)(c_0) + (\delta \hat{W}_t + \hat{v}_t^2)(c_1, c_2, \dots) \\ &= (1 + \theta^v_{t,1}) u(c_0) + (1 + \theta^v_{t,2}) \delta \hat{W}_t(c_1, c_2, \dots)\end{aligned}$$

must be a positive affine transformation of

$$\begin{aligned}&(\alpha U_t + (1-\alpha)(U_{t+1} + V_{t+1}))(c_0, c_1, c_2, \dots) \\ &\quad = (\alpha u + (1-\alpha)(1 + \gamma_{t+1})u)(c_0) + (\alpha \delta \hat{W}_t + (1-\alpha)(1 + \gamma_{t+1}\beta_{t+1})\delta \hat{W}_{t+1})(c_1, c_2, \dots) \\ &\quad = (\alpha + (1-\alpha)(1 + \gamma_{t+1}))u(c_0) + (\alpha + (1-\alpha)(1 + \gamma_{t+1}\beta_{t+1}))\delta \hat{W}_t(c_1, c_2, \dots),\end{aligned}$$

where the last equality follows because $\hat{W}_t$ and $\hat{W}_{t+1}$ agree on $C^{\mathbb{N}}$. This is only possible if

$$\frac{1 + \theta^v_{t,2}}{1 + \theta^v_{t,1}} = \frac{\alpha + (1-\alpha)(1 + \gamma_{t+1}\beta_{t+1})}{\alpha + (1-\alpha)(1 + \gamma_{t+1})},$$

which implies

$$\frac{1 + \gamma_{t+1}\beta_{t+1}}{1 + \gamma_{t+1}} \leq \frac{1 + \theta^v_{t,2}}{1 + \theta^v_{t,1}} \leq 1.$$

To prove 2, note that we now have stronger restrictions on the utility functions in the representation: For all $p, q \in \Delta(C^{\mathbb{N}})$,

$$\begin{aligned}&\big[U_t(p) > U_t(q) \text{ and } (U_{t+1} + V_{t+1})(p) > (U_{t+1} + V_{t+1})(q)\big] \\ &\quad \iff \big[(c, \{p\}) \succ_t (c, \{q\}) \text{ and } p \succ_{t+1} q\big] \\ &\quad \iff \big[(c, \{p\}) \succ_t (c, \{q\}) \text{ and } (c, \{p, q\}) \succ_t (c, \{q\})\big] \qquad \text{(by Axiom 10)} \\ &\quad \iff \big[U_t(p) > U_t(q) \text{ and } (U_t + \hat{V}_t)(p) > (U_t + \hat{V}_t)(q)\big].\end{aligned}$$

Applying Lemma 1 twice, we obtain

$$(U_t + \hat{V}_t)\big|_{\Delta(C^{\mathbb{N}})} \approx (U_{t+1} + V_{t+1})\big|_{\Delta(C^{\mathbb{N}})},$$

which implies

$$\frac{1 + \gamma_{t+1}\beta_{t+1}}{1 + \gamma_{t+1}} = \frac{1 + \theta^v_{t,2}}{1 + \theta^v_{t,1}} \leq 1,$$

as claimed. ■

Set $\hat{\gamma}_t = \theta^v_{t,1} \geq 0$. If $\hat{\gamma}_t = 0$, then set $\hat{\beta}_t \equiv 0$. Otherwise, set $\hat{\beta}_t \equiv \theta^v_{t,2}/\theta^v_{t,1} = \theta^v_{t,2}/\hat{\gamma}_t$. By Claim 6, $\theta^v_{t,2} \leq \theta^v_{t,1}$ and therefore $\hat{\beta}_t \in [0, 1]$. In addition, we have $\hat{\gamma}_t \hat{\beta}_t = \theta^v_{t,2}$ in both the case of

$\hat{\gamma}_t = 0$ and $\hat{\gamma}_t > 0$. Thus

$$\hat{V}_t(c,x) = \theta^v_{t,1} u(c) + \theta^v_{t,2} \delta \hat{W}_t(x) = \hat{\gamma}_t u(c) + \hat{\gamma}_t \hat{\beta}_t \delta \hat{W}_t(x),$$

so the third displayed equation in Proposition 6 is satisfied. Note also that by Claim 6,

$$\frac{1 + \hat{\gamma}_t \hat{\beta}_t}{1 + \hat{\gamma}_t} = \frac{1 + \theta^v_{t,2}}{1 + \theta^v_{t,1}} \geq \frac{1 + \gamma_{t+1}\beta_{t+1}}{1 + \gamma_{t+1}},$$

with equality if $\{\succsim_t\}_{t \in \mathbb{N}}$ satisfies Axiom 10 (Sophistication). This completes the proof of Proposition 6.

A.5.2 Proof of Theorem 3

We only show the sufficiency of the axioms.

Part 2: The assumptions in this part of the theorem are the same as in Proposition 6, except that Axiom 14 (Weak Commitment Stationarity) is replaced with the stronger condition of Axiom 7 (Stationarity). The profile of relations $\{\succsim_t\}_{t \in \mathbb{N}}$ therefore has a representation $(u, \gamma_t, \hat{\gamma}_t, \beta_t, \hat{\beta}_t, \delta)_{t \in \mathbb{N}}$ as in Proposition 6, with the additional condition that for any $p, q \in \Delta(C \times Z)$ and any $t, t' \in \mathbb{N}$,

$$U_t(p) + V_t(p) \geq U_t(q) + V_t(q) \iff U_{t'}(p) + V_{t'}(p) \geq U_{t'}(q) + V_{t'}(q).$$

Therefore, for any fixed $t \in \mathbb{N}$, setting $(u, \gamma, \hat{\gamma}, \beta, \hat{\beta}, \delta) = (u, \gamma_t, \hat{\gamma}_t, \beta_t, \hat{\beta}_t, \delta)$ and $(U, \hat{V}, V, \hat{W}) = (U_t, \hat{V}_t, V_t, \hat{W}_t)$ gives a naive quasi-hyperbolic discounting representation for $\{\succsim_t\}_{t \in \mathbb{N}}$.

Part 1: By replacing Axiom 11 (Naivete) with the more restrictive Axiom 10 (Sophistication), Proposition 6 implies that

$$\frac{1 + \hat{\gamma}\hat{\beta}}{1 + \hat{\gamma}} = \frac{1 + \hat{\gamma}_t \hat{\beta}_t}{1 + \hat{\gamma}_t} = \frac{1 + \gamma_{t+1}\beta_{t+1}}{1 + \gamma_{t+1}} = \frac{1 + \gamma\beta}{1 + \gamma}.$$

It is therefore without loss of generality to set $\gamma = \hat{\gamma}$ and $\beta = \hat{\beta}$, giving a sophisticated quasi-hyperbolic discounting representation $(u, \gamma, \beta, \delta)$.

A.5.3 Proof of Theorem 5

We only show the sufficiency of the axioms. Note first that if $\{\succsim_t\}_{t \in \mathbb{N}}$ satisfies Axioms 1–3 and 5, then Axiom 13 (Commitment Stationarity) implies Axiom 14 (Weak Commitment Stationarity). Therefore, the profile of preferences has a representation $(u, \gamma_t, \hat{\gamma}_t, \beta_t, \hat{\beta}_t, \delta)_{t \in \mathbb{N}}$ as in Proposition 6. By Axiom 13, for any $p, q \in \Delta(C^{\mathbb{N}})$ and $t, t' \in \mathbb{N}$,

$$U_t(p) + V_t(p) \geq U_t(q) + V_t(q) \iff U_{t'}(p) + V_{t'}(p) \geq U_{t'}(q) + V_{t'}(q).$$

Thus, for any $(c_0, c_1, c_2, \dots) \in C^{\mathbb{N}}$,

$$(U_t + V_t)(c_0, c_1, c_2, \dots) = (1 + \gamma_t)u(c_0) + (1 + \gamma_t\beta_t)\sum_{i=1}^{\infty} \delta^i u(c_i)$$

must be a positive affine transformation of

$$(U_{t'} + V_{t'})(c_0, c_1, c_2, \dots) = (1 + \gamma_{t'})u(c_0) + (1 + \gamma_{t'}\beta_{t'})\sum_{i=1}^{\infty} \delta^i u(c_i),$$

which is only possible if

$$\frac{1 + \gamma_t\beta_t}{1 + \gamma_t} = \frac{1 + \gamma_{t'}\beta_{t'}}{1 + \gamma_{t'}}.$$

Thus it is without loss of generality to assume that $\gamma \equiv \gamma_t = \gamma_{t'}$ and $\beta \equiv \beta_t = \beta_{t'}$ for all $t, t' \in \mathbb{N}$.

We prove that the individual's beliefs become more accurate over time by mapping into an appropriate version of the two-period environment from Section 2 and applying the comparative naivaté result from Theorem 2.

We construct preferences over menus $\mathcal{K}(\Delta(C^{\mathbb{N}})) \subset Z$ and choice correspondences from these menus as follows: For each time period $t \in \mathbb{N}$, define an induced preference $\succsim_t^*$ over $\mathcal{K}(\Delta(C^{\mathbb{N}}))$ by $x \succsim_t^* y$ if and only if $\hat{W}_t(x) \geq \hat{W}_t(y)$ or, equivalently, $(c, x) \succsim_t (c, y)$. Define an induced choice function from menus in $\mathcal{K}(\Delta(C^{\mathbb{N}}))$ by

$$\mathcal{C}_{t+1}(x) \equiv \operatorname*{argmax}_{p \in x} \big[U_{t+1}(p) + V_{t+1}(p)\big] = \operatorname*{argmax}_{p \in x} \big[U_t(p) + V_t(p)\big],$$

where the second inequality follows because $\gamma_t = \gamma_{t+1}$ and $\beta_t = \beta_{t+1}$ imply $U_t(p) = U_{t+1}(p)$ and $V_t(p) = V_{t+1}(p)$ for all $p \in \Delta(C^{\mathbb{N}})$. By construction, $(U_t, V_t, \hat{V}_t)$ (more precisely, the restrictions of these functions to $\Delta(C^{\mathbb{N}})$) is a self-control representation for $(\succsim_t^*, \mathcal{C}_{t+1})$.

Note that, for any $p, q \in \Delta(C^{\mathbb{N}})$,

$$\big[(c, \{p, q\}) \succ_t (c, \{q\}) \text{ and } q \succ_{t+1} p\big] \iff \big[\{p, q\} \succ_t^* \{q\} \text{ and } \mathcal{C}_{t+1}(\{p, q\}) = \{q\}\big].$$

Therefore, Axiom 12 (Diminishing Naivete) is equivalent to $(\succsim_t^*, \mathcal{C}_{t+1})$ being more naive than $(\succsim_{t+1}^*, \mathcal{C}_{t+2})$ according to Definition 2 for all $t \in \mathbb{N}$. In addition, nontriviality implies there exist $c, c' \in C$ such that $u(c) > u(c')$. Letting $p = \delta_{(c,c,c,\dots)}$ and $q = \delta_{(c',c',c',\dots)}$, this implies $U_t(p) > U_t(q)$ and $V_t(p) > V_t(q)$. Thus $\{p\} \succ_t^* \{q\}$ and $\mathcal{C}_{t+1}(\{p, q\}) = \{p\}$ for all $t \in \mathbb{N}$, so the joint regularity condition from Theorem 2 is satisfied. We can therefore apply the theorem to conclude that, for every $t \in \mathbb{N}$, either

$$U_t + \hat{V}_t \gg_{U_t} U_{t+1} + \hat{V}_{t+1} \gg_{U_t} U_{t+1} + V_{t+1} \gg_{U_t} U_t + V_t,$$

or the individual is sophisticated at $t+1$:

$$U_{t+1} + \hat{V}_{t+1} \approx U_{t+1} + V_{t+1}.$$

Note that it should be understood in these expressions that we are referring to the restrictions of these functions to $\Delta(C^{\mathbb{N}}) \subset \Delta(C \times Z)$. Following similar arguments to those used to prove Claim 6 in the proof of Proposition 6, these conditions translate immediately to the following: Either

$$\frac{1+\hat{\gamma}_t\hat{\beta}_t}{1+\hat{\gamma}_t} \geq \frac{1+\hat{\gamma}_{t+1}\hat{\beta}_{t+1}}{1+\hat{\gamma}_{t+1}} \geq \frac{1+\gamma_{t+1}\beta_{t+1}}{1+\gamma_{t+1}} \geq \frac{1+\gamma_t\beta_t}{1+\gamma_t}$$

or

$$\frac{1+\hat{\gamma}_{t+1}\hat{\beta}_{t+1}}{1+\hat{\gamma}_{t+1}} = \frac{1+\gamma_{t+1}\beta_{t+1}}{1+\gamma_{t+1}}.$$

Since $\gamma_t = \gamma$ and $\beta_t = \beta$ for all $t \in \mathbb{N}$, in either case we have

$$\frac{1+\hat{\gamma}_t\hat{\beta}_t}{1+\hat{\gamma}_t} \geq \frac{1+\hat{\gamma}_{t+1}\hat{\beta}_{t+1}}{1+\hat{\gamma}_{t+1}} \geq \frac{1+\gamma\beta}{1+\gamma}.$$

This completes the proof.

A.6 Proof of Theorem 4

Similar to the proof of Theorem 5, this proof consists of mapping the recursive environment into an appropriate version the two-period environment from Section 2 and applying the comparative naivete result in Theorem 2.

We construct preferences over menus $\mathcal{K}(\Delta(C^{\mathbb{N}})) \subset Z$ and choice correspondences from these menus as follows: For individuals $i = 1, 2$, take $U^i, \hat{V}^i, V^i, \hat{W}^i$ as in the naive quasi-hyperbolic discounting representations. Define an induced preference $\succsim_i^*$ over $\mathcal{K}(\Delta(C^{\mathbb{N}}))$ by $x \succsim_i^* y$ if and only if $\hat{W}^i(x) \geq \hat{W}^i(y)$ or, equivalently, $(c, x) \succsim_t^i (c, y)$. Define an induced choice function from menus in $\mathcal{K}(\Delta(C^{\mathbb{N}}))$ by

$$\mathcal{C}_i(x) \equiv \operatorname*{argmax}_{p \in x} \big[U^i(p) + V^i(p)\big].$$

By construction, $(U^i, V^i, \hat{V}^i)$ (more precisely, the restrictions of these functions to $\Delta(C^{\mathbb{N}})$) is a self-control representation for $(\succsim_i^*, \mathcal{C}_i)$.

Note that, for any $p, q \in \Delta(C^{\mathbb{N}})$,

$$\big[(c, \{p, q\}) \succ_t^i (c, \{q\}) \text{ and } q \succ_{t+1}^i p\big] \iff \big[\{p, q\} \succ_i^* \{q\} \text{ and } \mathcal{C}_i(\{p, q\}) = \{q\}\big].$$

Thus $\{\succsim_t^1\}_{t \in \mathbb{N}}$ is more naive than $\{\succsim_t^2\}_{t \in \mathbb{N}}$ according to Definition 8 if and only if $(\succsim_1^*, \mathcal{C}_1)$ is more naive than $(\succsim_2^*, \mathcal{C}_2)$ according to Definition 2. In addition, joint nontriviality implies there exist $c, c' \in C$ such that $u^i(c) > u^i(c')$ for $i = 1, 2$. Letting $p = \delta_{(c,c,c,\dots)}$ and $q = \delta_{(c',c',c',\dots)}$, this implies $U^i(p) > U^i(q)$ and $V^i(p) > V^i(q)$. Thus $\{p\} \succ_i^* \{q\}$ and $\mathcal{C}_i(\{p, q\}) = \{p\}$ for $i = 1, 2$, so the joint regularity condition from Theorem 2 is satisfied. We can therefore apply the theorem to conclude that individual 1 is more naive than individual 2 if and only if either individual 2 is sophisticated or

$$U^1 + \hat{V}^1 \gg_{U^1} U^2 + \hat{V}^2 \gg_{U^1} U^2 + V^2 \gg_{U^1} U^1 + V^1. \tag{13}$$

Note that it should be understood in this expression that we are referring to the restrictions of these functions to $\Delta(C^{\mathbb{N}}) \subset \Delta(C \times Z)$.

The proof is completed by showing that Equation (13) is equivalent to the conditions in the statement of the theorem. To see this, note first that $U^2 + V^2 \gg_{U^1} U^1 + V^1$ (restricted to $\Delta(C^{\mathbb{N}})$) if and only if there exists $\alpha \in [0,1]$ such that

$$(U^2 + V^2)\big|_{\Delta(C^{\mathbb{N}})} \approx (\alpha U^1 + (1-\alpha)(U^1 + V^1))\big|_{\Delta(C^{\mathbb{N}})}.$$

Thus, for any $(c_0, c_1, c_2, \dots) \in C^{\mathbb{N}}$,

$$(U^2 + V^2)(c_0, c_1, c_2, \dots) = (1 + \gamma^2)u^2(c_0) + (1 + \gamma^2\beta^2)\sum_{i=1}^{\infty}(\delta^2)^i u^2(c_i)$$

must be a positive affine transformation of

$$\begin{aligned}&(\alpha U^1 + (1-\alpha)(U^1 + V^1))(c_0, c_1, c_2, \dots)\\&\quad = \big(\alpha + (1-\alpha)(1+\gamma^1)\big)u^1(c_0) + \big(\alpha + (1-\alpha)(1+\gamma^1\beta^1)\big)\sum_{i=1}^{\infty}(\delta^1)^i u^1(c_i).\end{aligned}$$

This is equivalent to $u^1 \approx u^2$, $\delta^1 = \delta^2$, and

$$\frac{1+\gamma^2\beta^2}{1+\gamma^2} = \frac{\alpha + (1-\alpha)(1+\gamma^1\beta^1)}{\alpha + (1-\alpha)(1+\gamma^1)} \geq \frac{1+\gamma^1\beta^1}{1+\gamma^1}.$$

By analogous arguments, since $u^1 \approx u^2$ and $\delta^1 = \delta^2$,

$$\begin{aligned}U^2 + \hat{V}^2 \gg_{U^1} U^2 + V^2 &\iff \frac{1+\hat{\gamma}^2\hat{\beta}^2}{1+\hat{\gamma}^2} \geq \frac{1+\gamma^2\beta^2}{1+\gamma^2},\\ U^1 + \hat{V}^1 \gg_{U^1} U^2 + \hat{V}^2 &\iff \frac{1+\hat{\gamma}^1\hat{\beta}^1}{1+\hat{\gamma}^1} \geq \frac{1+\hat{\gamma}^2\hat{\beta}^2}{1+\hat{\gamma}^2}.\end{aligned}$$

This completes the proof.

A.7 Proof of Proposition 2

Under any value function of the form $\hat{W}(m) = Au(m) + B$ such that $A > 0$ and $B \in \mathbb{R}$, one can explicitly solve

$$\operatorname*{argmax}_{c\in[0,m]}\Big[(1+\hat{\gamma})u(c) + \delta(1+\hat{\gamma}\hat{\beta})\hat{W}(R(m-c))\Big] = \frac{1}{1 + (\delta' \frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{1/\sigma}}m$$

$$\underset{c\in[0,m]}{\operatorname{argmax}}\left[u(c)+\delta\hat{\beta}\hat{W}(R(m-c))\right]=\frac{1}{1+(\delta'\hat{\beta}A)^{1/\sigma}}m$$

where we define $\delta' := \delta R^{1-\sigma} < 1$. Consider first the case $\sigma \neq 1$. Then any such a value function needs to satisfy

$$\begin{aligned}
\hat{W}(m) &= \max_{\hat{c}\in[0,m]}\left[(1+\hat{\gamma})u(\hat{c})+\delta(1+\hat{\gamma}\hat{\beta})\hat{W}(R(m-\hat{c}))\right]-\hat{\gamma}\max_{\tilde{c}\in[0,m]}\left[u(\tilde{c})+\delta\hat{\beta}\hat{W}(R(m-\tilde{c}))\right]\\
&= (1+\hat{\gamma})\frac{1}{\left(1+(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{1/\sigma}\right)^{1-\sigma}}\frac{m^{1-\sigma}}{1-\sigma}\\
&\quad+\delta(1+\hat{\gamma}\hat{\beta})\left(A\frac{(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{1-\sigma/\sigma}}{\left(1+(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{1/\sigma}\right)^{1-\sigma}}\frac{(Rm)^{1-\sigma}}{1-\sigma}+B\right)\\
&\quad-\hat{\gamma}\frac{1}{\left(1+(\delta'\hat{\beta}A)^{1/\sigma}\right)^{1-\sigma}}\frac{m^{1-\sigma}}{1-\sigma}-\hat{\gamma}\delta\hat{\beta}\left(A\frac{(\delta'\hat{\beta}A)^{1-\sigma/\sigma}}{\left(1+(\delta'\hat{\beta}A)^{1/\sigma}\right)^{1-\sigma}}\frac{(Rm)^{1-\sigma}}{1-\sigma}+B\right)\\
&= (1+\hat{\gamma})\frac{1+(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{1/\sigma}}{\left(1+(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{1/\sigma}\right)^{1-\sigma}}\frac{m^{1-\sigma}}{1-\sigma}-\hat{\gamma}\frac{1+(\delta'\hat{\beta}A)^{1/\sigma}}{\left(1+(\delta'\hat{\beta}A)^{1/\sigma}\right)^{1-\sigma}}\frac{m^{1-\sigma}}{1-\sigma}+\delta B
\end{aligned}$$

for all $m > 0$, and thus A is a solution to the equation

$$A=(1+\hat{\gamma})\left(1+\left(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A\right)^{1/\sigma}\right)^{\sigma}-\hat{\gamma}\left(1+(\delta'\hat{\beta}A)^{1/\sigma}\right)^{\sigma}, \tag{14}$$

and $B = 0$. Let $g(A)$ denote the right-hand side of Equation (14).

The case of $\sigma = 1$ can be solved analogously to obtain Equation (14), and the value of B is uniquely obtained from the value of A.

Claim 7 *Equation (14) has a unique solution $A^* \in \mathbb{R}_{++}$.*

Proof: The derivative of g is

$$g'(A)=(1+\hat{\gamma})\left(1+\left(\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}\delta'A\right)^{-1/\sigma}\right)^{\sigma-1}\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}-\hat{\gamma}\left(1+(\hat{\beta}\delta'A)^{-1/\sigma}\right)^{\sigma-1}\delta'\hat{\beta}. \tag{15}$$

Note that under $\sigma < 1$, the first (resp. second) term of the right-hand side of Equation (15) is increasing (resp. decreasing) in A. Thus, an upper-bound of $g'(A)$ under $\sigma < 1$ is given by

$$\begin{aligned}
&\lim_{A\to\infty}\left[\left(1+\left(\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}\delta'A\right)^{-1/\sigma}\right)^{\sigma-1}\delta'(1+\hat{\gamma}\hat{\beta})\right]-\lim_{A\to 0}\left[\left(1+(\hat{\beta}\delta'A)^{-1/\sigma}\right)^{\sigma-1}\delta'\hat{\gamma}\hat{\beta}\right]\\
&\qquad=\delta'(1+\hat{\gamma}\hat{\beta})<1.
\end{aligned}$$

The second-order derivative of g is

$$g''(A) = (1+\hat{\gamma})\left(1+\left(\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}\delta' A\right)^{-1/\sigma}\right)^{\sigma-2}\left(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}\right)^{\frac{\sigma-1}{\sigma}} A^{-\frac{1+\sigma}{\sigma}}\frac{1-\sigma}{\sigma} \\ -\hat{\gamma}\left(1+(\delta'\hat{\beta}A)^{-1/\sigma}\right)^{\sigma-2}(\delta'\hat{\beta})^{\frac{\sigma-1}{\sigma}} A^{-\frac{1+\sigma}{\sigma}}\frac{1-\sigma}{\sigma},$$

which is proportional to

$$\left(\frac{1+\hat{\gamma}}{\hat{\gamma}}\left[\frac{1+(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{\frac{-1}{\sigma}}}{1+(\delta'\hat{\beta}A)^{\frac{-1}{\sigma}}}\right]^{\sigma-2}\left[\frac{\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}}{\hat{\beta}}\right]^{\frac{\sigma-1}{\sigma}}-1\right)\frac{1-\sigma}{\sigma}A^{-\frac{1+\sigma}{\sigma}}. \tag{16}$$

When $\sigma = 1$, Equation (16) is equal to 0. When $\sigma > 1$, the sign of Equation (16) is negative, as it can be written as

$$\left(\frac{1+\hat{\gamma}}{\hat{\gamma}}\underbrace{\left[\frac{(\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}})^{\frac{1}{\sigma}}+(\delta' A)^{\frac{-1}{\sigma}}}{\hat{\beta}^{\frac{1}{\sigma}}+(\delta' A)^{\frac{-1}{\sigma}}}\right]^{\sigma-1}}_{\geq 1}\underbrace{\left[\frac{1+(\delta'\hat{\beta}A)^{\frac{-1}{\sigma}}}{1+(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A)^{\frac{-1}{\sigma}}}\right]}_{\geq 1}-1\right)\frac{1-\sigma}{\sigma}A^{-\frac{1+\sigma}{\sigma}}.$$

Thus g is concave under $\sigma > 1$.

We now prove the existence of a unique solution A to Equation (14). First, observe that $\lim_{A\to\infty} g'(A) = \delta' < 1$ by Equation (15). Thus, under any σ, there exist $\epsilon > 0$ and $\bar{A}$ such that $g'(A) \leq 1-\epsilon$ at all $A \geq \bar{A}$. This implies that $A > g(A)$ for all A sufficiently large. Given that $g(0) = 1$, the existence of a solution A^* is guaranteed by continuity of g.

If $\sigma < 1$, then since $g'(A) < 1$ for all A, there cannot be another solution. If $\sigma \geq 1$, then since $g(0) = 1$ and g is concave, $g'(A^*) < 1$ at the smallest solution A^*. By concavity $g'(A) < 1$ for all $A \geq A^*$ as well, and thus there cannot be another solution. ■

The above observation implies that there exists a unique value function $\hat{W}$ that has the form of $\hat{W}(m) = Au(m) + B$, where $A > 0$ and $B \in \mathbb{R}$. Below we prove the comparative statics results.

Claim 8 *The unique solution A^* to Equation (14) is increasing in $\hat{\beta}$ if $\sigma < 1$, decreasing in $\hat{\beta}$ if $\sigma > 1$, and constant in $\hat{\beta}$ if $\sigma = 1$.*

Proof: As we have shown in the proof of the previous claim, $g'(A^*) < 1$. Thus, by the implicit function theorem, the unique solution A^* is increasing (resp. decreasing) in $\hat{\beta}$ if the value of $g(A)$ at each $A > 0$ is increasing (resp. decreasing) in $\hat{\beta}$. The derivative of $g(A)$ with respect to $\hat{\beta}$ is

$$\hat{\gamma}\delta' A\left[\left(1+\left(\delta'\frac{1+\hat{\gamma}\hat{\beta}}{1+\hat{\gamma}}A\right)^{-1/\sigma}\right)^{\sigma-1}-\left(1+(\delta'\hat{\beta}A)^{-1/\sigma}\right)^{\sigma-1}\right],$$

which is positive if $\sigma<1$, negative if $\sigma>1$, and zero if $\sigma=1$. ■

The actual consumption level is given by

$$c(m)=\underset{c\in[0,m]}{\operatorname{argmax}}\left[\frac{c^{1-\sigma}}{1-\sigma}+\delta\frac{1+\gamma\beta}{1+\gamma}\hat{W}(R(m-c))\right]=\frac{1}{1+(\delta'\frac{1+\gamma\beta}{1+\gamma}A^*)^{1/\sigma}}m.$$

Thus $\lambda=\frac{1}{1+(\delta'\frac{1+\gamma\beta}{1+\gamma}A^*)^{1/\sigma}}$, which is decreasing in $\hat{\beta}$ under $\sigma<1$, increasing in $\hat{\beta}$ under $\sigma>1$, and constant in $\hat{\beta}$ under $\sigma=1$. Furthermore, it is decreasing in β under any σ.

A.8 Proof of Proposition 3

For any lotteries p and q

$$\begin{aligned}\left[u(p)>u(q)\text{ and }(u+v)(p)>(u+v)(q)\right] &\implies \mathcal{C}(\{p,q\})=\{p\}\succ\{q\}\\ &\implies \{p,q\}\succsim\{p\}\succ\{q\} \qquad \text{(Strotz naivete)}\\ &\implies \hat{v}(p)\geq\hat{v}(q).\end{aligned}\tag{17}$$

Regularity requires that u and $u+v$ not be ordinally opposed. As a result, the following stronger condition is implied:[30]

$$\left[u(p)\geq u(q)\text{ and }(u+v)(p)\geq(u+v)(q)\right]\implies\hat{v}(p)\geq\hat{v}(q).$$

By Proposition 1 from De Meyer and Mongin (1995), this condition implies that $\hat{v}=au+b(u+v)+c$ for some $a,b\geq0$ and $c\in\mathbb{R}$. Since $\hat{v}$ is assumed to be non-constant, we can further conclude that $a+b>0$ and hence $\hat{v}\approx\alpha u+(1-\alpha)(u+v)$ for some $\alpha\in[0,1]$.

To show necessity, note that $\hat{v}\gg_u u+v$ implies

$$\max_{p\in x}[u(p)+\hat{v}(p)]-\max_{q\in x}\hat{v}(q)\geq\max_{p\in B_{\hat{v}}(x)}u(p)\geq\max_{p\in B_{u+v}(x)}u(p),$$

which implies $x\succsim\{p\}$ for all $p\in\mathcal{C}(x)$.

[30]Formally, regularity implies there exist lotteries p^* and q^* such that $u(p^*)>u(q^*)$ and $(u+v)(p^*)>(u+v)(q^*)$. Suppose $u(p)\geq u(q)$ and $(u+v)(p)\geq(u+v)(q)$, and let $p^\alpha\equiv\alpha p^*+(1-\alpha)p$ and $q^\alpha\equiv\alpha q^*+(1-\alpha)q$. Then $u(p^\alpha)>u(q^\alpha)$ and $(u+v)(p^\alpha)>(u+v)(q^\alpha)$ for any $\alpha\in(0,1]$ by the linearity of expected-utility functions. Equation (17) therefore implies $\hat{v}(p^\alpha)\geq\hat{v}(q^\alpha)$ for $\alpha\in(0,1]$. By continuity, this inequality also holds for $\alpha=0$, and hence $\hat{v}(p)\geq\hat{v}(q)$, as claimed.

A.9 Proof of Proposition 4

To show that (1) implies (3), suppose that the individual is naive and take any lotteries p, q. Then

$$\begin{aligned}\big[u(p) > u(q) \text{ and } v(p) \geq v(q)\big] &\iff \mathcal{C}(\{p,q\}) = \{p\} \succ \{q\} \\ &\implies \{p,q\} \succ \{q\} \qquad \text{(by naivete)} \\ &\iff \big[u(p) > u(q) \text{ and } \hat{v}(p) \geq \hat{v}(q)\big].\end{aligned} \tag{18}$$

Lemma 1 cannot be applied in this case since the inequalities in Equation (18) are not all strict. However, regularity of $(\succsim, \mathcal{C})$ together with the continuity of expected-utility functions yields the following condition:[31]

$$\big[u(p) \geq u(q) \text{ and } v(p) \geq v(q)\big] \implies \hat{v}(p) \geq \hat{v}(q).$$

By Proposition 1 from De Meyer and Mongin (1995), this condition implies that $\hat{v} = au + bv + c$ for some $a, b \geq 0$ and $c \in \mathbb{R}$. Since $\hat{v}$ is assumed to be non-constant, we can further conclude that $a + b > 0$ and hence $\hat{v} \approx \alpha u + (1-\alpha)v$ for some $\alpha \in [0,1]$. If, in addition, the individual is sophisticated, we can likewise show

$$\big[u(p) > u(q) \text{ and } \hat{v}(p) \geq \hat{v}(q)\big] \implies v(p) \geq v(q),$$

and thus, since v is also assumed to be non-constant, we have $\hat{v} \approx v$.

To show that (3) implies (1), suppose that $\hat{v} \approx \alpha u + (1-\alpha)v$ for some $\alpha \in [0,1]$ and take any lotteries p, q. Then

$$\begin{aligned}\mathcal{C}(\{p,q\}) = \{p\} \succ \{q\} &\iff \big[u(p) > u(q) \text{ and } v(p) \geq v(q)\big] \\ &\implies \big[u(p) > u(q) \text{ and } \hat{v}(p) \geq \hat{v}(q)\big] \\ &\iff \{p,q\} \succ \{q\},\end{aligned}$$

and thus the individual is naive. If in addition $\hat{v} \approx v$, then we also have

$$\{p,q\} \succ \{q\} \implies \mathcal{C}(\{p,q\}) = \{p\} \succ \{q\},$$

and hence the individual is sophisticated.

The equivalence between (2) and (3) follows as in Ahn, Iijima, Le Yaouanq, and Sarver (2018). (While the definition of $\hat{v} \gg_u v$ in Ahn, Iijima, Le Yaouanq, and Sarver (2018) allows for the case of $v \approx -u$, this is ruled out by regularity).

[31] Formally, for the Strotz representation, regularity implies there exist lotteries p^* and q^* such that $u(p^*) > u(q^*)$ and $v(p^*) \geq v(q^*)$. Suppose $u(p) \geq u(q)$ and $v(p) \geq v(q)$, and let $p^\alpha \equiv \alpha p^* + (1-\alpha)p$ and $q^\alpha \equiv \alpha q^* + (1-\alpha)q$. Then $u(p^\alpha) > u(q^\alpha)$ and $v(p^\alpha) \geq v(q^\alpha)$ for any $\alpha \in (0,1]$ by the linearity of expected-utility functions. Equation (18) therefore implies $\hat{v}(p^\alpha) \geq \hat{v}(q^\alpha)$ for $\alpha \in (0,1]$. By continuity, this inequality also holds for $\alpha = 0$, and hence $\hat{v}(p) \geq \hat{v}(q)$, as claimed.

B Testing Naivete with Consumption-Savings Data

In Section 3.6 we considered an infinite-horizon consumption-savings problem with a CRRA felicity function. We found that a naive agent follows a stationary linear consumption policy, as in the standard time-consistent case. This might raise the question of whether in general an analyst is able to detect an agent's naivete by only observing choice data in consumption-savings problems. While this question is practically important, its full analysis is beyond the scope of the current paper. We do not know whether the "non-identification" finding in Section 3.6 generalizes to other felicity functions under infinite-horizon problems. Below we only provide a simple example using a finite horizon, which describes an instance of naive behavior that cannot be rationalized by sophisticated agents.

Consider a consumption-savings problem over a finite sequence of periods $t = 1, 2, \ldots, T$.[32] Let $R > 0$ denote the gross interest rate, and let m denote the current stock of savings. The analyst's data consists of the collection of choice functions $(c_t(m))_{m\in\mathbb{R}_+, t=1,\ldots,T}$, where $c_t(m) \in [0, m]$ for each $t = 1, \ldots, T$ and $m \geq 0$. For the naive quasi-hyperbolic model, these choices must satisfy

$$c_t(m) \in \operatorname*{argmax}_{c\in[0,m]} (1+\gamma)u(c) + (1+\gamma\beta)\delta \hat{W}_{t+1}(R(m-c))$$

for each $t = 1, \ldots, T$ and $m \geq 0$, where

$$\hat{W}_t(m) = \max_{c\in[0,m]} \Big((1+\hat{\gamma})u(c) + \delta(1+\hat{\gamma}\hat{\beta})\hat{W}_{t+1}(R(m-c))\Big) - \hat{\gamma} \max_{\hat{c}\in[0,m]} \Big(u(\hat{c}) + \hat{\beta}\delta\hat{W}_{t+1}(R(m-\hat{c}))\Big)$$

for each $t = 2, \ldots, T$, and $\hat{W}_{T+1}(m) = 0$ for all $m \geq 0$.

For $T = 3$, consider the following behavior for some $m > 0$:

$$c_1(m) = 0, \quad c_2(Rm) = Rm. \tag{19}$$

That is, the agent saves everything in period $t = 1$ and spends all of her wealth in period $t = 2$. We claim that Equation (19) is indicative of naivete and inconsistent with sophistication:

1. The consumption pattern in Equation (19) is inconsistent with sophistication:

   Take any u that is strictly increasing and strictly concave. For simplicity, suppose in addition that it is continuously differentiable and satisfies $u' > 0$. Also, take any (δ, β, γ). We show that the choice behavior in Equation (19) cannot result from such a sophisticated agent. Since $\hat{W}_3(\cdot) = u(\cdot)$, $c_2(Rm) = Rm$ implies

$$Rm \in \operatorname*{argmax}_{c\in[0,Rm]} (1+\gamma)u(c) + (1+\gamma\beta)\delta u(R(Rm-c)), \tag{20}$$

[32] Note that assuming a finite horizon for consumption-savings problems is standard in the revealed preference literature, and finite-horizon environments are also often used in applications including contracts and macroeconomics.

which implies the FOC

$$u'(Rm) \geq \frac{1+\gamma\beta}{1+\gamma}\delta R u'(0). \tag{21}$$

Note that this also implies

$$Rm \in \operatorname*{argmax}_{c\in[0,Rm]} u(c) + \beta\delta u(R(Rm-c)) \tag{22}$$

since $\frac{1+\gamma\beta}{1+\gamma}\delta \geq \beta\delta$. Therefore, the optimality of Rm in Equations (20) and (22) implies that $\hat{W}_2'(Rm) = u'(Rm)$ by the envelope theorem.

Then $c_1(m) = 0$ implies

$$0 \in \operatorname*{argmax}_{c\in[0,Rm]}(1+\gamma)u(c) + \delta(1+\beta\gamma)\hat{W}_2(R(m-c))$$

which implies the FOC

$$u'(0) \leq \frac{1+\gamma\beta}{1+\gamma}\delta R\hat{W}_2'(Rm) = \frac{1+\gamma\beta}{1+\gamma}\delta R u'(Rm). \tag{23}$$

But since $u' > 0$, Equations (21) and (23) together imply that $u'(Rm) \geq u'(0)$, which contradicts the strict concavity of u.

2. The consumption pattern in Equation 19 can arise under naivete:

   Consider a linear utility function $u(c) = c$ and parameters $\gamma > 0$, $\beta < 1$, and $\hat{\beta} = 1$ (complete naivete). (As will be clear, one can construct a similar example with strictly concave u and partial naivete.) Take an interest rate R that satisfies

   $$\frac{1+\gamma\beta}{1+\gamma}\delta R < 1 < \min\left\{\delta R, \frac{1+\gamma\beta}{1+\gamma}\delta^2 R^2\right\}. \tag{24}$$

   Take any $m > 0$. The consumption choice $c_2(Rm)$ in period 2 given wealth Rm is

   $$Rm = \operatorname*{argmax}_{c\in[0,Rm]}(1+\gamma)c + (1+\gamma\beta)\delta R(m-c),$$

   since $\frac{1+\gamma\beta}{1+\gamma}\delta R < 1$ by Equation (24). The continuation value at $t = 2$ as perceived at $t = 1$ is

   $$\hat{W}_2(m') = \max_{c\in[0,m']} c + \delta R(m'-c) = \delta R m'$$

   for all m' since $\hat{\beta} = 1$ and since $1 < \delta R$ by Equation (24). Based on this, the consumption choice $c_1(m)$ in period 1 given wealth m is

   $$0 = \operatorname*{argmax}_{c\in[0,m]}(1+\gamma)c + (1+\gamma\beta)\delta^2 R^2(m-c),$$

   since $1 < \frac{1+\gamma\beta}{1+\gamma}\delta^2 R^2$ by Equation (24).

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