Economics 201A
Problem Set 1
Due 31 August 2017

1. Let X be a finite set. A **random choice function** is a function $P : 2^X \setminus \emptyset \to [0,1]^X$ such that $P(A)(x) \geq 0$, $\sum_{a \in A} P(A)(a) = 1$, and $\sum_{b \notin A} P(A)(b) = 0$. For notational ease, let $P_A = P(A)$. That is, a random choice function takes a menu of options and outputs a probability distribution over the menu, where $P_A(x)$ denotes the probability that x is chosen from menu A.

   A random choice function admits a **Luce representation** if there exists a set of weights $\{w(x) \geq 0 : x \in X\}$ such that

$$P_A(x) = \begin{cases} \frac{w(x)}{\sum_{a \in A} w(a)} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

   Let $\mathcal{P}$ denote the set of all bijections from X to $\{1, 2, \ldots, |X|\}$. Note that $\mathcal{P}$ is equivalent to the set of all strict rankings of objects in X, where $f(x) = 1$ is interpreted as x being the most preferred object under ranking f. A random choice function admits a **Falmagne representation** if there exists a probability distribution $\mu \in [0,1]^{\mathcal{P}}$ (that is, $\mu(f) \geq 0$ and $\sum_{f \in \mathcal{P}} \mu(f) = 1$) such that

$$P_A(x) = \begin{cases} \sum \{\mu(f) : f \text{ such that } f(x) \leq f(a) \text{ for all } a \in A\} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

   A random choice function satisfies **Random Independence of Irrelevant Alternatives** if

$$\frac{P_A(x)}{P_A(y)} = \frac{P_B(x)}{P_B(y)}$$

   whenever $P_A(x), P_A(y), P_B(x), P_B(y) > 0$.

   (a) Interpret the Luce and the Falmagne representations.

   (b) Interpret the Random IIA condition.

   (c) Prove or provide a counterexample to the following statement: Let P admit a Luce representation. Define $\mathcal{C}(A) = \{x \in A : P_A(x) > 0\}$. Then $\mathcal{C}$ is rationalizable.

   (d) Prove or provide a counterexample to the following statement: If P admits a Luce representation, then it satisfies Random IIA.

   (e) Suppose $P_A(a) > 0$ whenever $a \in A$. Prove that if P satisfies Random IIA, then P admits a Luce representation. (Hint: Consider a candidate for w.)

   (f) Prove or provide a counterexample to the following statement: If P admits a Falmagne representation, then P satisfies Random IIA.

2. Consider the following classical condition:

   Definition 0.1. A choice rule $\mathcal{C}$ satisfies **path independence** if, for all $A, B \in 2^X \setminus \emptyset$,

$$\mathcal{C}(A \cup B) = \mathcal{C}\big(\mathcal{C}(A) \cup \mathcal{C}(B)\big)$$

   (a) Prove the following claim: If $\mathcal{C}$ is nonempty and rationalizable, then $\mathcal{C}$ satisfies path independence.

   (b) Provide an example of a nonempty choice rule that satisfies path independence, but is not rationalizable.