Corrigendum to "Asymptotic behavior of Bayesian learners with misspecified models" [J. Econ. Theory 195 (2021) Article 105260]

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Esponda et al. (2021) (henceforth, EPY) consider a Bayesian learning problem by a misspecified agent, and characterize the asymptotic motion of the action frequency by using a stochastic approximation technique developed by Benaïm et al. (2005) (henceforth, BHS). Unfortunately, it turns out that some theorems of BHS involve minor errors, which affect the main characterization result of EPY, Theorem 2. In this note, we will explain what the problem is and how to fix the statement of Theorem 2. Many of the subsequent results in the paper relied on Theorem 2. With the corrected version of Theorem 2, all of the existing results are valid, except the results establishing lack of convergence to unstable equilibria (Propositions 3 and 9).

Consider a differential inclusion

$$\dot{x}(t) \in A(x(t)) \tag{1}$$

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where $A(b) \subset \boldsymbol{R}^m$ for all $b \in \boldsymbol{R}^m$. A function $x : \boldsymbol{R}^+ \to \boldsymbol{R}^m$ is a solution to this differential inclusion if it is absolutely continuous and (1) holds for almost all $t \in [0, \infty)$. Let $S_{x(0)}$ denote the set of solutions to this differential inclusion given an initial value $x(0) \in \boldsymbol{R}^m$, and let $S = \bigcup_{x(0)} S_{x(0)}$ denote the set of all solutions. Given $t \geq 0$ and $z : \boldsymbol{R}^+ \to \boldsymbol{R}^m$, let $f^t(z)$ denote a *translation flow* which is defined as

$$f^t(z)[s] = z(t+s) \quad \forall s \geq 0.$$

Theorem 4.1 of BHS claims that the following two statements are equivalent:

(i) A bounded continuous function $z : \boldsymbol{R}^+ \to \boldsymbol{R}^m$ is an *asymptotic pseudo trajectory* of the differential inclusion (1) in that given any T,

$$\lim_{t \to \infty} \inf_{x \in S_{z(t)}} \sup_{s \in [0,T]} \|z(t+s) - x(s)\| = 0 \tag{2}$$

(ii) A bounded function z is uniformly continuous and any limit point of the translation flows $\{f^t(z)\}_t$ solves the differential inclusion (1).

It turns out that this theorem is incorrect; (i) implies (ii) as stated, but the converse is not true. Indeed, BHS prove only that (ii) implies

$$\lim_{t \to \infty} \inf_{x \in S} \sup_{s \in [0,T]} \|z(t+s) - x(s)\| = 0, \tag{3}$$

which need not imply (2).

This influences their Theorem 4.2, which claims that if a bounded function z is a perturbed solution of the differential inclusion (1) in the sense of BHS, then it is an asymptotic pseudo trajectory. In the proof of this theorem, BHS show that any bounded perturbed solution z satisfies the property (ii) of Theorem 4.1. But as noted above, this implies only (3), rather than (2).

Now, we will explain how this error affects EPY's result. Consider a misspecified Bayesian learning problem studied by EPY. Following their notation, let X denote the set of actions, F denote the policy correspondence (i.e., when the agent's belief is μ, she chooses an action from the set $F(\mu) \subseteq X$), and $w : \boldsymbol{R}^+ \to \triangle X$ denote the continuous-time interpolation of the sequence of action frequencies $\{\sigma_t\}_{t=1}^{\infty}$. Theorem 2 claims that almost surely, w is an asymptotic pseudo trajectory of the differential inclusion

$$\dot{\sigma}(t) \in \triangle F(\triangle \Theta(\sigma(t))) - \sigma(t). \tag{4}$$

The proof shows that condition (ii) in Theorem 4.1 of BHS is satisfied. As noted above, this implies that expression (3), but not necessarily expression (2), is satisfied. Accordingly, the statement of the theorem should be modified as follows. With an abuse of notation, let $S_{\sigma(0)}$ denote the set of all solutions to the differential inclusion (4) for an initial value $\sigma(0)$, and let $S = \bigcup_{\sigma(0)} S_{\sigma(0)}$ denote the set of all solutions.

Theorem 1 (Correct version of Theorem 2 in EPY)**.** *Almost surely, given any $T > 0$, we have*

$$\lim_{t\to\infty} \inf_{\sigma\in S} \sup_{s\in[0,T]} \|w(t+s) - \sigma(s)\| = 0$$

The difference from the original statement is that now the infimum is taken over the set S, rather than $S_{w(t)}$. The theorem still implies that the asymptotic motion of the action frequency w is approximated by a solution σ to the differential inclusion, but now the initial value of the solution σ can be (slightly) different from the actual action frequency $w(t)$.

EPY used Theorem 2 to prove several convergence results. All results stating convergence to a point or more generally a set continue to be true under the correct version of Theorem 2, with minor changes to the proofs and the definition of a robustly attracting set.[1] The reason is that these results look at the case in which the limit point (or the limit set) is attracting, in the sense that any solution to the differential inclusion converges to the equilibrium, as long as the initial value is in a neighborhood of the equilibrium; this means that starting from this neighborhood, even if the initial value is slightly perturbed as stated in the correct theorem above, the solution to the differential inclusion still converges to the same point.

On the other hand, the proofs of results establishing lack of convergence to unstable equilibria (Propositions 3 and 9) relied importantly on the fact that the initial condition of the differential inclusion is *exactly* $w(t)$. This is because we used the fact that the solutions to the differential inclusion starting at a point $w(t)$

[1]The definition of a robustly attracting set should be modified as follows: A set A is *robustly attracting* if it is attracting and there is $\zeta > 0$ and $\varepsilon > 0$ such that for any initial value $\sigma(0) \in B_\zeta(A)$, any solution $\sigma \in S^{\infty,\varepsilon}_{\sigma(0)}$ to the perturbed differential inclusion never leaves the basin $\mathscr{U}_A$ and do not approach its boundary; formally, $B_\zeta(\sigma(t)) \in \mathscr{U}_A$ for all $t \geq 0$. The difference from the original definition is that now we need that the perturbed solution does not approach the boundary of the basin.

in the neighborhood of an unstable equilibrium move away from this equilibrium to conclude that the action frequency cannot converge to the unstable equilibrium. But with the correct version of Theorem 2, one should also consider solutions to the differential inclusion that start at points near the point $w(t)$, and, in particular, one such point is the unstable equilibrium itself, with a solution that forever remains at the unstable equilibrium. Therefore, the correct version of Theorem 2 alone cannot help establish that there is zero probability of convergence to unstable equilibria.

We have argued that the proof of Theorem 2 in EPY and of the results establishing lack of convergence to unstable equilibria are incorrect. We now use an example to show that the statement of Theorem 2 in EPY is indeed incorrect.

Example 1. Consider a bandit problem where there are two arms, a "safe arm" and a "risky arm." The safe arm always gives a payoff of zero. The risky arm gives a random payoff of either -1 (fail) or 1 (success) randomly. The success probability is $\theta^* = 0.8$. The agent does not know this success probability θ^*, and she believes that is is uniformly distributed on $\Theta = [0.3, 0.9]$.

In period one, the agent chooses the risky arm, because $E[\theta] = 0.6 > 0.5$. Suppose that the agent had a payoff of -1 (fail) in the first few periods, and her posterior mean of θ becomes lower than 0.5. She then starts to choose the safe arm and does so forever. In this case, the action frequency $\sigma(t)$, which represents the share of the risky arm up to time t, decreases over time and converges to zero. Let w denote the continuous-time interpolation of this particular path.

Obviously this path w satisfies the condition stated in Theorem 2 of EPY, so the theorem suggests that this path w be an asymptotic pseudo trajectory of the differential inclusion (4). However, this is not the case. Indeed, in this example, the differential inclusion (4) reduces to

$$\frac{d\sigma(t)}{dt} = \begin{cases} 1 - \sigma(t) > 0 & \text{if } \sigma(t) > 0 \\ [0,1] & \text{if } \sigma(t) = 0 \end{cases},$$

so any solution starting from an initial value $\sigma > 0$ converges to $\sigma = 1$ as time goes to infinity. This means that the path w above is *not* an asymptotic pseudo trajectory; since $\sigma(t) > 0$ for all t, any solution with an initial value $\sigma(t)$ converges to $\sigma = 1$, and it is very different from the actual path w which converges to zero. So this bandit problem is a counterexample to EPY's Theorem 2.

Note also that this bandit problem is a counterexample to Theorem 4.1 of BHS. Take a path w as stated. Then the limit point of the translation flows $\{f^t(w)\}_t$ is a constant function $w^*(t) = 1$ for all t, which obviously solves the differential inclusion (4). So this path w satisfies the property (ii) in Theorem 4.1 of BHS. However this w is not an asymptotic pseudo trajectory, as explained above.

On the other hand, the statement in the corrected version of Theorem 2 provided in this note is valid in this example. The point is that in a later period t, the share $\sigma(t)$ of the risky arm is very close to zero. So if we consider the differential inclusion with the "perturbed" initial value $\sigma = 0$, its solution can stay at $\sigma = 0$, which approximates the actual path w.

To conclude, it is an open question what conditions need to be satisfied for the action frequency to converge or not to unstable equilibria. This is a question that unfortunately cannot be answered with the tools developed by EPY.

References

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