Conditional Retrospective Voting in Large Elections*

Ignacio Esponda (NYU Stern)

Demian Pouzo (UC Berkeley)

May 2, 2014

Abstract

We introduce a solution concept in the context of elections with private information by embedding a model of boundedly rational voters into an otherwise standard equilibrium setting. Voters evaluate alternatives based on past performance, but, since counterfactual outcomes remain unobserved, the sample from which they learn is potentially biased. A retrospective voting equilibrium formalizes the idea that voters learn from a biased sample and have systematically biased beliefs in large elections. This approach provides several novel insights regarding the preference and information aggregation properties of elections. When applied to a Downsian setting of two-party competition, we find that, in contrast to the standard Nash equilibrium case, parties have an incentive to exacerbate the degree to which their policy platforms differ. This incentive to polarize, however, increases the welfare of the median voter.

*We thank the Editor and several anonymous referees for extremely helpful feedback. We also thank Heski Bar-Isaac, Ernesto Dal Bó, Erik Eyster, Frederico Finan, Gregory Huber, Navin Kartik, Alessandro Lizzeri, Salvatore Nunnari, Santiago Oliveros, Wolfgang Pesendorfer, Andy Skrzypacz, and several seminar participants for their comments.

Esponda: Stern School of Business, New York University, 44 W Fourth Street KMC 7-76, New York, NY 10012, iesponda@stern.nyu.edu; Pouzo: Department of Economics, UC Berkeley, 530 Evans Hall #3880, Berkeley, CA 94704, dpouzo@econ.berkeley.edu.

1 Introduction

In the economics literature, voters are often portrayed as sophisticated individuals who have well-defined preferences, can solve complicated signal-extraction problems, and have correct expectations about the distribution of (counterfactual) payoffs.[1] The empirical evidence, on the other hand, often finds that voters are poorly informed and have little understanding of ideology and policy.[2] Consistent with the evidence, political scientists often view voters as boundedly rational individuals who vote "retrospectively" and reward or punish politicians and their parties based on their past performance. In the words of Fiorina (1981, p. 5), voters "need *not* know the precise economic or foreign policies of the incumbent administration in order to see or feel the *results* of those policies."

Many of the "rational" assumptions in the voting literature are implicitly embodied in the notion of Nash equilibrium. Sometimes these assumptions seem more driven by a methodological tradition than by a conviction in their empirical validity. On the other hand, equilibrium analysis and comparative statics lies at the heart of economics and has provided numerous insights in voting and several other contexts.

The main contribution of this paper is to combine the previous approaches by embedding a model of boundedly rational voters who learn from the previous performance of the policies or parties into an otherwise standard equilibrium setting. This approach provides several novel insights regarding the preference and information aggregation properties of elections. We apply our boundedly rational equilibrium framework to a Downsian model of two-party competition and find that, in contrast to the standard Nash equilibrium case, parties have an incentive to exacerbate the degree to which their policy platforms differ. This incentive to polarize, however, increases the welfare of the median voter.

Our model captures an important feature of elections that is often overlooked in the literature. To illustrate this feature, consider an election between a Republican and a Democratic candidate in the United States. Voters are likely to use information about the past performance of the parties to predict their future performance and determine which party to vote for. For example, voters who are currently unemployed may favor a Democratic candidate if they have experienced better results

[1] In the context of elections with private information that we consider in this paper, see, e.g., Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997).

[2] See, e.g., Delli Carpini and Keeter (1997) and Converse (2000).

from previously elected Democratic administrations, compared to Republican administrations, when also unemployed in the past. This tendency to learn from the past is not limited to political elections. When shareholders vote on takeover proposals, they benefit from learning the outcome of previous takeovers in the same or comparable firms. A similar phenomenon occurs with legislators choosing whether to vote along party lines, union members voting to accept or reject negotiated contracts, and residents voting whether to approve additional funding for school districts.

A key feature in these examples is that, when using past information to evaluate alternatives, voters only observe the performance of the *elected* alternatives, so that counterfactual outcomes are not observable. For example, we will never find out how Romney would have performed had he been elected President of the U.S. in 2012 instead of Obama. Similarly, shareholders will not learn the benefits of a takeover that is not approved. Consequently, the sample from which voters learn is potentially biased. The reason is that the selection of alternatives is not randomized: To the extent that voters have some private information, they will elect alternatives that are likely to perform better. It is reasonable to assume that voters will not be able to control for unobserved counterfactual outcomes, thus ending up with systematically biased beliefs (see Section 2 for the evidence).

In our setup, there is a continuum of voters and two alternatives. One of the alternatives wins the election if it receives a high enough proportion of votes; otherwise the other alternative wins. Voters have (possibly heterogeneous) payoffs that are increasing in the state of the world for one alternative and decreasing for the other. In addition, voters have some information about the state of the world. For example, in an election between two political parties, the state can represent the fundamentals of the economy. One of the parties might be best at governing during recessions and the other during booms (perhaps because of their different positions on monetary and fiscal policy).

We propose a new solution concept, *retrospective voting equilibrium* (RVE), to formalize the idea that voters learn from a biased sample and have systematically biased beliefs. An RVE consists of a strategy profile and an election cutoff that satisfy two conditions: (i) there is a tie at the cutoff, with one alternative being elected above and the other below the cutoff; (ii) the strategy profile must be *optimal* given the election cutoff. Optimality is defined in terms of retrospective voting: Voters' perceptions of the benefits of each alternative derive from the observed performance of each alternative, which depends on the states in which each alternative is elected, and, therefore, on the election cutoff. This parsimonious characterization of retrospective voting in large elections is a major advantage of the framework.[3]

Our model is inspired by the notion of retrospective voting advanced by Key (1966) and Fiorina (1981), among others. Our work, however, is conceptually very different to the formal literature in retrospective voting, beginning with Barro (1973) and Ferejohn (1986), which studies elections as incentive mechanisms that hold politicians accountable. Instead, our model follows Downs' (1957) view of retrospective voting as a way to predict how parties will perform in the future rather than as a way to simply punish or reward the party for past performance (Fiorina (1981), Chapter 1).

We compare RVE to the two standard solution concepts in the literature, sincere voting (SV) and Nash equilibrium (NE), and find that RVE constitutes a middle ground and exhibits the more realistic properties of these two solution concepts: Behavior is endogenous and depends not only on the characteristic of an individual voter but also on the aggregate characteristics of the electorate (as in NE, but unlike SV); outcomes are significantly affected by both the electoral rule and the precision of information (as in SV, but unlike NE); and there is often a non-negligible fraction of nonpartisans (as in SV, but unlike NE).

We also highlight a tension that often precludes information aggregation under RVE but is not present under NE or SV: If information were aggregated, then, in general, one of the alternatives would be observed to have the best performance. But then everyone would want to vote for that alternative irrespective of their private information, which precludes information aggregation in the first place. Moreover, the fact that mistakes are driven by selection bias implies that, contrary to folk wisdom, a more informed electorate will not necessarily make better decisions. The extent of mistakes, however, is limited by the fact that voters, while not fully sophisticated, still tend to penalize alternatives that yield bad outcomes. Finally, the bias tends to run in the direction of overestimating the benefits of the more risky alternative, thus justifying conservatism as a means to mitigate mistakes.

We then turn to the main application of the framework, which is to embed the voting model in a Downsian model of two-party competition, modified to allow for

[3] In a parallel with the definition of a competitive equilibrium, the role of prices is played here by the election cutoff. Voters take the cutoff as given when optimizing, and the cutoff is determined endogenously in equilibrium.

state-contingent payoffs. There are two political parties, $Left$ and *Right*, and each of these parties is committed to a "left" and "right" platform, respectively, but they can choose their degree of polarization. To illustrate, suppose that the state of the economy ranges from recession to boom. The $Left$ party is ideologically constrained to favor expansionary fiscal policy while the *Right* party is constrained to favor contractionary policy. Expansionary policy does best in a recession but hurts in a boom, while the opposite is true for a contractionary policy. There is also a neutral, hands-off policy that neither helps nor hurts the economy. The $Left$ and *Right* parties choose the degree of expansion or contraction in their policies, respectively, in order to maximize the chance of being elected. Voters take these policies as given and play an equilibrium of the voting game. We compare the cases where voters play Nash equilibrium (NE) and retrospective voting equilibrium (RVE).

When voters play NE, the policy platforms converge to the neutral policy, and the logic is similar to the standard convergence result (Downs, 1957). The idea is that polarization hurts the chances of a party not only in states that are in the opposite extreme of the policy but also in intermediate states. Thus, the parties end up converging to a common, middle policy.

When voters play RVE, they evaluate parties based on observed, not counterfactual, performance, and the standard logic no longer applies. A party has an incentive to choose relatively extreme policies that work well in those states in which it is elected into office, since those are the states that retrospective voters use to evaluate its performance. This incentive to polarize under RVE leads to a better match between states and policies under RVE compared to NE and, therefore, to higher welfare (of the median voter, under majority rule). In the previous example, under RVE voting, the $Left$ party chooses an expansionary policy and tends to be elected during recessions and the *Right* party chooses a contractionary policy and tends to be elected during booms. Under NE, in contrast, both parties choose a neutral policy that does not respond to a fluctuating economy.

These results highlight an important benefit of polarization that is analogous to the idea of specialization or division of labor: parties specialize in certain policies and they are elected into office when these policies tend to be best. The theory predicts these features of two-party competition to be present in a context where there is uncertainty about the best policy, the best policy depends on the state of the world, and parties are committed to different ideological platforms but can choose their degree of polarization (e.g., the Democratic and Republican parties in the U.S.; the Labour and Conservative parties in the U.K.). These predictions are also consistent with empirical evidence from the literature on the political business cycle. For example, Hibbs (1977) and Alesina and Roubini (1992) find that the economy tends to expand with left-wing governments and to contract with right-wing governments, which is consistent with left-wing governments having a relatively stronger concern for unemployment over inflation. In addition, Faust and Irons (1999) find that Democrats are more likely to be elected president in times of high unemployment and Republicans are more likely to be elected in times of high inflation. In general, the literature has a hard time establishing causality because it is difficult to assess to what extent the correlation between political and economic variables is due to politics affecting the economy or the other way around. By incorporating both private information and state-contingent payoffs, our analysis provides a novel mechanism through which the economy (i.e., the state of the world) influences politics that is consistent with the data and could potentially be incorporated into empirical models of the political business cycle.

Another implication of these results is that, in order to evaluate the functioning of an electoral system, it is misleading to focus exclusively on whether voters are sophisticated or well informed and to ignore the incentives of the parties. When policies are exogenous, it is not surprising that NE voting is more efficient compared to boundedly-rational RVE voting. But, when the parties' incentives to choose policies are taken into account, we actually find a reverse implication: a simple retrospective voting heuristic leads to higher welfare for the median voter than sophisticated voting.

This paper follows a recent literature that studies game-theoretic equilibrium concepts for boundedly rational players (e.g., Osborne and Rubinstein (1998), Jehiel (2005), Eyster and Rabin (2005), Jehiel and Samet (2007), Jehiel and Koessler (2008), and Esponda (2008))). Papers that study elections with non-Nash solution concepts include Osborne and Rubinstein (2003), Eyster and Rabin (2005), Costinot and Kartik (2007), and Martinelli (2011).[4]

Bendor et al. (2010, 2011) postulate a dynamic model of retrospective voting where voters follow a satisficing rule and vote for the incumbent if it has performed well given their endogenous aspiration level. Spiegler (2013) studies a dynamic model

[4]The original literature on the political business cycle also assumed boundedly rational voters (Nordhaus, 1975).

of reforms in which an infinite sequence of policy makers care about the public evaluation of their interventions. The public follows a simple attribution rule and (mistakenly) attributes changes in outcomes to the most recent intervention. Levy and Razin (2014) study a setting where voters have two correlated pieces of private information but naive voters fail to account for their correlation. They find that correlation neglect might lead to higher welfare under fixed policies and to less polarized policies when parties choose policies. These papers focus on other interesting aspects of bounded rationality and not on the type of sample selection problem that motivates our paper. [5]

Motivated by the empirical evidence on polarization (e.g., McCarty et al. (2006)), a large literature relaxes the assumptions of the Downsian framework to explain non-convergent policies.[6] While this literature restricts attention to a private values setting, McMurray (2013b) recently considers a pure common value setting and shows that Nash voting leads to convergent policies when parties can commit and are office-motivated, which is inefficient because policies do not match the state of the world.[7] In contrast, we allow for both private and common value elements in voter preferences and show that polarization obtains when voters are boundedly rational.

In Section 2, we discuss the behavioral assumptions underlying our solution concept and the related evidence. In Section 3, we introduce the framework and the solution concept. In Section 4, we study the degree of polarization under two-party competition and, in Section 5, we provide a game-theoretic foundation for our solution concept. We conclude in Section 6 by mentioning possible extensions.

[5] Callander (2011) studies a model of dynamic policy-making where "rational" voters learn the mapping between policies and outcomes.

[6] Some explanations include: policy motivated candidates with uncertainty about median voter preferences (Wittman (1977), Calvert (1985)), the threat of entry by a third party (Palfrey, 1984), the effect of executive-legislative compromise (Alesina and Rosenthal, 2000), lack of policy commitment (Osborne and Slivinski (1996), Besley and Coate (1997)), candidates with "valence" attributes, (Aragones and Palfrey (2002), Gul and Pesendorfer (2009), Kartik and McAfee (2007)), differentiation in the presence of multiple constituencies (Eyster and Kittsteiner, 2007), and convex voter preferences (Kamada and Kojima, 2012).

[7] Kartik et al. (2012) show that elections are also inefficient when candidates have socially valuable information about the state of the world.

time	signal	vote	election outcome	observed payoff	
1	**r**	R	R	**-1**	
2	**r**	L	R	**1**	
3	**r**	R	L	0	→ counterfactual not observed
4	**l**	L	R	-1	
5	**l**	L	L	0	
6	**r**	R	R	**1**	
7	**r**	R	L	0	→ counterfactual not observed
8	**l**	L	R	1	

Table 1: Illustration of retrospective voting rule

2 Behavioral assumptions and evidence

In the dynamic environment that motivates our (steady-state) solution concept, voters predict the future performance of an alternative based on its past observed performance. For every election, each voter first observes a private signal that is correlated with the performance of the alternatives. After observing their signals, voters simultaneously cast a vote. The election outcome is then determined and voters observe the performance of the elected alternative. The environment is stationary, in the sense that the signals and performances of the alternatives are drawn from the same distribution in every election.

Table 1 shows data for eight elections from the point of view of one particular voter. Suppose that there is an election in period 9 in which this voter observes signal r prior to voting. A *retrospective voter* behaves as follows. First, she uses past elections to form a belief about the performance of each alternative conditional on signal r. In this example, alternative L always delivers a payoff of zero, while alternative R delivers an average payoff of $(-1+1+1)/3 = 1/3$ when the signal is r (i.e., in periods 1, 2, and 6). Then, in period 9, the voter votes for R, which is the best alternative given the current evidence.

This retrospective voting rule is inspired by the idea of retrospective voting in the political science literature (Key (1966), Fiorina (1981)). This idea has received both empirical (e.g., Kramer (1971), Fiorina (1978), Lewis-Beck and Stegmaier (2000), Martorana and Mazza, 2012) and experimental support (Woon (2012), Huber et al. (forthcoming)). More generally, the evidence shows that the electorate is often poorly informed and has little understanding of ideology and policy (e.g., Delli Carpini and Keeter (1997) and Converse (2000)), and voter mistakes do not tend to cancel in the aggregate (e.g., Bartels, 1996).

One novelty with respect to the literature on retrospective voting is that we allow for private information—hence the use of the term *conditional* retrospective voting. An important insight is that the introduction of private information gives rise to sample selection problems. In Table 1, the voter also observes signal r in periods 3 and 7, but, since L is elected, the voter does not observe the performance of R in those periods. If L and R were randomly chosen each period, the fact that the performance of R is not observed in periods 3 and 7 should not affect beliefs in the long run. The problem, however, is that the election outcome depends on private information that is correlated with performance. In particular, it is likely that the reason why R was not elected in periods 3 and 7 is that voters obtained signals that were relatively unfavorable to R. So, if our voter had been somehow able to observe the counterfactual performance of R in periods 3 and 7, she would have probably observed a relatively bad performance. Thus, the fact that counterfactual performances are not observed likely leads to overestimating the value of electing R. Our model provides a tractable way to account for the systematic bias in beliefs that results from the selection problem.

As a starting point, we make the stark assumption that voters do not try to control for the selection problem, either because they do not understand the selection problem or because they do not know how to control for it. For example, even a sophisticated voter might have trouble thinking how Romney would have performed as president of the U.S. and including that counterfactual assessment in his overall evaluation of Republican candidates.[8]

The idea that voters do not try to correct for unobserved counterfactuals is consistent with the empirical findings of Achen and Bartels (2004), Leigh (2009), and Wolfers (2009), who find that voters punish politicians for events that are outside of their control. Healy and Malhotra (2010) find that punishment is related to the politician's response to these events. Our model allows voters to be fairly sophisticated and

[8] In another paper (Esponda and Pouzo, 2012), we discuss intermediate assumptions under which voters partially account for pivotality.

to condition their learning on private signals, such as campaign platforms, media reports, and economic indicators. This type of naiveté also underlies the winner's curse in common value auctions and has received robust support in experimental settings (e.g., Thaler (1988), Kagel and Levin (2002), and Charness and Levin (2009)).[9]

In this paper, we focus on the steady state of the previous dynamic voting environment when voters follow the retrospective voting rule described above. This steady state is formally captured by the notion of a (naive) behavioral equilibrium (Esponda, 2008)—we provide a formal proof of this statement in another paper (Esponda and Pouzo, 2012). This solution concept captures the failure of players to account for selection problems and differs from the standard notion of Nash equilibrium.[10] One of the contributions of the current paper is to characterize the (naive) behavioral equilibrium, which captures the retrospective behavior we want to analyze, as the number of voters goes to infinity. This exercise is analogous to that carried out by Feddersen and Pesendorfer (1997), who characterize *Nash equilibrium* as the number of voters goes to infinity. The formal results appear in Section 5. It turns out that this characterization takes a very convenient and intuitive form, analogous to the notion of a competitive equilibrium in market economies. Thus, we begin by directly postulating this convenient characterization as our definition of *retrospective voting equilibrium* in Section 3.

3 Voting framework

3.1 Setup

A continuum of voters participate in an election between two alternatives, R (right) and L (left). A state $\omega \in \Omega = [-1, 1]$ is first drawn according to a probability

[9] For voting experiments, Guarnaschelli et al. (2000) conclude that subjects' votes do not deviate much from Nash equilibrium play, although Eyster and Rabin (2005) find that these deviations can be systematically attributed to naiveté. Recently, Esponda and Vespa (2011) show that between 50 and 80 percent of their subjects are not sophisticated. These experiments are only indirect tests of our assumption because they inform subjects of the primitives of the game.

[10] In Esponda and Pouzo (2012), we also show that there is a rule that provides a foundation for Nash equilibrium. This *pivotal* rule is identical to the retrospective rule described above, with the exception that learning only takes place from those elections in which a voter was pivotal. We believe, however, that this rule is unrealistic in the context of this paper, where there is a large electorate, the probability of being pivotal is very small, and, hence, there would be essentially no learning opportunities from the past under this rule. Thus, we believe that the existence of a behavioral foundation for Nash equilibrium under a large electorate remains an open question.

distribution G and, conditional on the state, each player observes an independently-drawn private signal. Players then simultaneously submit a vote for either R or L. Votes are aggregated according to an electoral rule $\rho \in (0,1)$: Alternative R is elected if the proportion of votes in favor of R is greater or equal than ρ; otherwise, L is elected.

We model heterogeneity (in preferences and information) by assuming that each voter is of a particular type θ, where ϕ is the full-support probability distribution over the set of types $\Theta \subset \mathbb{R}$. Conditional on a state $W = \omega$, players of type θ independently draw a signal $S_\theta = s$ from a finite, nonempty set $\mathbb{S}_\theta \subset \mathbb{R}$ with probability $q_\theta(s \mid \omega)$; let s_θ^L and s_θ^R denote the lowest and highest signals in $\mathbb{S}_\theta$. The payoff of type θ is given by $u_\theta(o, \omega)$, where $o \in \{L, R\}$ is the winner of the election.

Let $\sigma_\theta : \mathbb{S}_\theta \to [0,1]$ denote the strategy of type θ, where $\sigma_\theta(s)$ is the probability that type θ votes for alternative R after observing signal s. A strategy σ_θ is *nondecreasing* if $\sigma_\theta(s') \geq \sigma_\theta(s)$ for all $s' > s$. A strategy profile $\sigma = (\sigma_\theta)_{\theta \in \Theta}$ is nondecreasing if σ_θ is nondecreasing for each θ.

We maintain the following assumptions throughout the paper, for all $\theta \in \Theta$:

A1. (i) $u_\theta(R, \cdot) : \Omega \to \mathbb{R}$ is nondecreasing and $u_\theta(L, \cdot) : \Omega \to \mathbb{R}$ is nonincreasing, and one of them is strictly monotone; (ii) $u_\theta(R, \cdot)$ and $u_\theta(L, \cdot)$ are both continuously differentiable, except possibly in a finite number of points, and $\sup_{\theta,o,\omega} |u_\theta(o, \omega)| \leq K < \infty$.

A2. MLRP: For all $\omega' > \omega$, and $s' > s$:

$$\frac{q_\theta(s'|\omega')}{q_\theta(s'|\omega)} - \frac{q_\theta(s|\omega')}{q_\theta(s|\omega)} > 0.$$

A3. (i) G has a density function g, where $\inf_\Omega g(\omega) > 0$; (ii) there exists $d > 0$ such that $q_\theta(s|\omega) > d$ for all $\theta \in \Theta$, $s \in \mathbb{S}$ and $\omega \in \Omega$; (iii) $q_\theta(s \mid \cdot)$ is continuous for all $s \in \mathbb{S}$.

A4. $\Theta \subset \mathbb{R}$ is a compact interval (a singleton is a special case) and $\mathbb{S}_\theta = \mathbb{S}$; $u_\theta(R, \omega)$, $u_\theta(L, \omega)$, and $q_\theta(s \mid \omega)$ are jointly continuos in $\Theta \times \Omega$ for all $s \in \mathbb{S}$.

Assumptions A1-A2 provide an ordering between states, information, and players' preferences. Note that A2 is trivially satisfied for types with a unique signal (i.e., no private information). As mentioned in Section 5, we view the case without private information as the limit of environments with private information as information precision vanishes. Assumption A3 rules out “strong signals” in the sense of (Milgrom, 1979). Assumption A4 guarantees uniqueness of the equilibrium outcome and is made only for convenience. Thus, the voting environment essentially coincides with the standard setup in Feddersen and Pesendorfer (1997).[11]

Example 1. (*Homogenous informed voters*) The state is uniformly distributed in $[-1, 1]$ and there is a unique voter type with payoffs $u(R, \omega) = \omega - 1/2$, $u(L, \omega) = -\omega - 1/2$, so that the payoff from the Left [Right] policy is increasing [decreasing] in the state. In particular, $c^{FB} = 0$ is the first-best election cutoff, i.e., everyone prefers R in states $\omega > c^{FB}$ and L in states $\omega < c^{FB}$. In addition, each voter privately observes a binary signal from $\mathbb{S} = \{s^L, s^R\}$ with probability $q(s^R \mid \omega) = .5 + \iota\omega$, where $\iota \in (0, .5]$ is the precision of information.

For example, these primitives represent an election between two candidates with different proposals to lower unemployment. The Left candidate is committed to spending more resources in education and training while the Right candidate is committed to lowering corporate taxes to incentivize employment. The state of the world captures the true underlying cause for unemployment. In high states of the world, unemployment is mostly due to weak demand; in low states of the world, it is due to workers lacking the right skills. Voters observe information correlated with the true cause for unemployment, such as reasons for job loss, types of job listings, or the current premium for skilled labor. □

3.2 Retrospective voting equilibrium

Let

$$\kappa(\omega; \sigma) \equiv \int_{\Theta} \sum_{s \in \mathbb{S}} q_\theta(s \mid \omega)\sigma_\theta(s)\phi(d\theta)$$

denote the proportion of votes in favor of alternative R under state ω and strategy profile σ. Assumption A2 implies that $\kappa(\cdot; \sigma)$ is nondecreasing if σ is nondecreasing. In the case where the strategy depends on private information, so that σ is not flat, then $\kappa(\cdot; \sigma)$ is increasing and the outcome of the election can be characterized by a cutoff: R is elected if and only if $\kappa(\omega; \sigma) \geq \rho$, or, equivalently, for all sufficiently high

[11]One difference is that we require $u_\theta(L, \cdot)$ and $u_\theta(R, \cdot)$ to be separately monotone, rather than only their difference to be increasing.

states. This observation motivates the following definition.[12]

Definition 1. A state $\omega \in \Omega$ is an *election cutoff* given a strategy profile σ if $\kappa(\tilde{\omega}; \sigma) \geq \rho$ for all $\tilde{\omega} > \omega$ and $\kappa(\tilde{\omega}; \sigma) \leq \rho$ for all $\tilde{\omega} < \omega$.

When making her decision, each voter takes the cutoff as given. A cutoff determines the set of states for which each alternative is chosen, and, consequently, each voter's evaluation of the benefits of electing each alternative. For a given cutoff $\omega \in \Omega$, the difference in benefits from electing R over L that is perceived by a voter of type θ who observes signal s is

$$v_\theta\left(s;\omega\right) \equiv E\left(u_\theta(R,W) \mid W \geq \omega, S_\theta = s\right) - E\left(u_\theta(L,W) \mid W < \omega, S_\theta = s\right). \qquad (1)$$

To interpret the above expression, note that, for election cutoff ω, alternative R is elected whenever $W \geq \omega$, so that a voter's retrospective evaluation of R is given by the first term in the right hand side of (1). A similar interpretation holds for the second term. As a matter of comparison, the notion of Nash equilibrium differs by conditioning not on the events $\{W \geq \omega\}$ or $\{W < \omega\}$ but rather on the event that a voter is pivotal.

The following definition captures the idea that each voter votes for the alternative that she sincerely believes to have the highest perceived benefit.

Definition 2. A strategy profile σ is *optimal* given an election cutoff ω if, for all $\theta \in \Theta$ and $s \in \mathbb{S}$, $v_\theta(s;\omega) > 0$ implies $\sigma_\theta(s) = 1$ and $v_\theta(s;\omega) < 0$ implies $\sigma_\theta(s) = 0$.

By assumptions A1-A2, $v_\theta(\cdot;\omega)$ is increasing. Therefore, any strategy that is optimal given some cutoff must be nondecreasing.

Definition 3. A *retrospective voting equilibrium* (RVE) is a strategy profile σ^* and an election cutoff ω^* such that: (i) σ^* is optimal given ω^*, and (ii) ω^* is an election cutoff given σ^*.

[12] When σ, and, therefore, $\kappa(\cdot;\sigma)$ are constant, this definition is motivated by the limiting case where signals satisfy MLRP but become uninformative; see Section 5.

A retrospective voting equilibrium requires players to optimize given an election cutoff that is endogenously determined by players' strategies. In particular, unlike the standard notion of sincere voting, voting behavior now depends endogenously on the aggregate behavior of all voters. Moreover, the definition of equilibrium is reminiscent of the definition of a competitive equilibrium in market economies. In the voting context, the role of prices is played by the election cutoff. Voters take the election cutoff as given when they optimize, and their consequent behavior yields that election cutoff. In Section 5, we provide a foundation for Definition 3 by showing that a retrospective voting equilibrium characterizes the naive behavioral equilibrium (Esponda, 2008) of the voting game as the number of voters goes to infinity.

3.3 Characterization of RVE

We now characterize retrospective voting equilibrium. For each type θ and signal s, define the *personal cutoffs*

$$c_\theta(s) \equiv \arg\min_{\omega\in\Omega} |v_\theta(s;\omega)|, \tag{2}$$

which depend only on the primitives of the environment. Since Ω is compact and $v_\theta(s;\cdot)$ is continuous and increasing (by A1-A3), there exists a unique solution $c_\theta(s)$ that is nonincreasing in s. Moreover, A4 implies that $v_\theta(s;\omega)$ is jointly continuous in (θ,ω) and, by the Theorem of the Maximum, $c_\theta(s)$ is continuous in θ. Thus, we can define $\underline{c} \equiv \min_\theta c_\theta(s^R)$ and $\bar{c} = \max_\theta c_\theta(s^L)$ as the lowest and highest personal cutoffs across all types.

If we knew the equilibrium election cutoff ω^*, then it would be straightforward to characterize the equilibrium strategy: a type θ with signal s such that $c_\theta(s) < \omega^*$ must satisfy $v_\theta(s;\omega^*) > 0$ and, therefore, she will optimally vote for R; similarly, if $c_\theta(s) > \omega^*$, then she will optimally vote for L. For example, consider a voter with two signals and personal cutoffs $c(s^R) < c(s^L)$, as depicted in the left panel of Figure 1. If the equilibrium cutoff were lower than $c(s^R)$, this voter would always vote for L. Similarly, if the election cutoff were higher than $c(s^L)$, she would always vote for R. In the case where the election cutoff were between her personal cutoffs $c(s^R)$ and $c(s^L)$, this voter would vote her signal.

We now characterize the set of equilibrium cutoffs. For any election cutoff $\omega \in \Omega$,

$$\overline{\kappa}(\omega) \equiv \int_{\Theta} \sum_{\{s: c_\theta(s) < \omega\}} q_\theta\left(s \mid \omega\right) \phi(d\theta) \tag{3}$$

may be interpreted as the proportion of players that vote for R in state ω when the cutoff is also given by ω.[13]

Lemma 1. $\overline{\kappa} : \Omega \to [0,1]$ is left-continuous, increasing over the subdomain $(\underline{c}, \overline{c})$, and satisfies: $\overline{\kappa}(\omega) = 0$ if $\omega \leq \underline{c}$ and $\overline{\kappa}(\omega) = 1$ if $\omega > \overline{c}$.

Proof. See the Appendix. □

The next result says that there is a unique equilibrium cutoff and that it is essentially given by the state where the proportion of votes for R, as captured by the function $\overline{\kappa}$, coincides with the electoral rule ρ. By continuity of $\overline{\kappa}$, the proportion of votes for R is higher than ρ for states above this intersection and lower than ρ for states below this intersection.

Theorem 1. *For any electoral rule $\rho \in (0,1)$, there exists a unique retrospective voting equilibrium cutoff and it is given by $\overline{\kappa}^{-1}(\rho) \in [\underline{c}, \overline{c}]$.*[14]

Proof. See the Appendix. □

The following examples illustrate how to find a retrospective voting equilibrium.

Example 1, continued. An RVE can be found in four simple steps. First, we obtain the "belief functions"

$$\begin{aligned} v\left(s;\omega\right) &= E\left(W \mid W \geq \omega, s\right) - E\left(-W \mid W < \omega, s\right) \\ &= \frac{\frac{1}{4}(1-\omega^2) + I_s\frac{\iota}{3}(1-\omega^3)}{\frac{1}{2}(1-\omega) + I_s\frac{\iota}{2}(1-\omega^2)} + \frac{\frac{1}{4}(\omega^2-1) + I_s\frac{\iota}{3}(\omega^3+1)}{\frac{1}{2}(\omega+1) + I_s\frac{\iota}{2}(\omega^2-1)}, \end{aligned}$$

[13]The interpretation is exact except when there is a unique type (i.e., Θ is a singleton) and ω is one of its personal cutoffs.

[14]$\overline{\kappa}^{-1} : (0,1) \to [\underline{c}, \overline{c}]$ is defined as $\overline{\kappa}^{-1}(\rho) = \inf\{\omega \in \Omega : \overline{\kappa}(\omega) \geq \rho\}$.

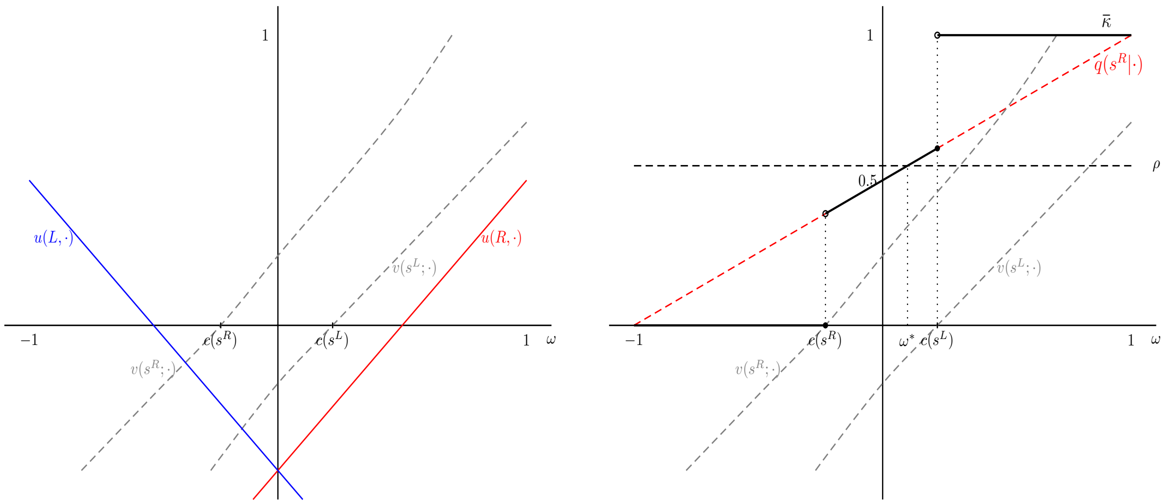

Figure 1: Example 1. Finding a retrospective voting equilibrium.
The left panel shows the personal cutoffs $c(s^R)$ and $c(s^L)$ that result from equating the perceived benefits from electing R over L, represented by $v(s^R;\cdot)$ and $v(s^L;\cdot)$, to zero. The right panel shows how to use the personal cutoffs to construct the vote share function $\bar{\kappa}$, and how to then find the equilibrium cutoff ω^* by intersecting the vote share function with the threshold rule ρ.

where $I_{s^R} = -I_{s^L} = 1$. Second, we compute the personal cutoffs $c(s)$, which solve $v(s; c(s)) = 0$. Since $v(s^R; 0) > 0 > v(s^L; 0)$, then $c(s^R) < c^{FB} < c(s^L)$. The belief functions and corresponding cutoffs are depicted in the left panel of Figure 1, together with the payoffs from the Left and Right policies.

Third, we compute the vote share for R,

$$\overline{\kappa}(\omega) = \begin{cases} 0 & \text{if}\, \omega \leq c(s^R) \\ .5 + \iota\omega & \text{if}\, c(s^R) < \omega \leq c(s^L) \\ 1 & \text{if}\, \omega > c(s^L) \end{cases}$$

Finally, we intersect the vote share for R with the electoral rule. The equilibrium cutoff as a function of the electoral rule ρ is then

$$\omega^* = \begin{cases} c(s^R) & \text{if}\, \rho \leq .5 - (-\iota c(s^R)) \\ \frac{1}{\iota}(\rho - .5) & \text{if}\, .5 - (-\iota c(s^R)) < \rho < .5 + \iota c(s^L) \\ c(s^L) & \text{if}\, \rho \geq .5 + \iota c(s^L) \end{cases}$$

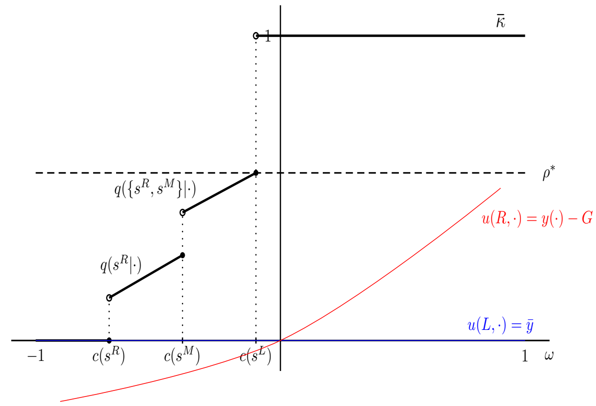

Figure 2: Example 2. RVE with safe and risky alternatives.
All personal cutoffs are negative and, therefore, the risky policy (Right) is excessively elected in equilibrium relative to the first best cutoff $c^{FB} = 0$. This welfare loss is mitigated by choosing any majority rule $\rho \geq \rho^*$, leading to an equilibrium cutoff $\omega^* = c(s^L)$.

Thus, the first-best outcome can be obtained with our boundedly rational voters if and only if the electoral rule is $\rho = 1/2$. In contrast, a rule that requires a supermajority to elect one of the alternatives will inefficiently elect the other alternative too often in equilibrium. The equilibrium is depicted in the right panel of Figure 1 for the case $\rho > 1/2$ and $\iota = 1/2$. In equilibrium, alternative R is elected in states higher than ω^* and L is elected in lower states. □

Example 2. *(Risky vs. safe alternatives)* A representative voter with uncertain gross income $y(\omega)$ that is increasing in the state chooses between two policies. Under a *full stabilization policy* (Left), taxes $t(\omega) = y(\omega) - \bar{y}$ are set to smooth recessions and booms and to obtain a constant disposable income $\bar{y}$. Under a budget balance policy (Right), taxes $t(\omega) = G$ are set to balance a fixed amount of expenditure G. We refer to these policies as the safe and risky policies, respectively. Figure 2 depicts the disposable income from each policy, where $\bar{y}$ is normalized to zero and higher states are associated with higher gross income. The first-best election cutoff is $c^{FB} = 0$. But, since $E(u(R, W) \mid W \geq 0, s) > 0 = u(L)$ for any signal s, it follows that all personal cutoffs are negative and, therefore, the risky policy (Right) will be excessively elected in equilibrium relative to the first-best outcome. The intuition is that, since voters tend to elect the risky alternative in those states in which it is best, they will overestimate its value and will be biased towards voting for the risky alternative. As shown by Figure 2, this bias can be partially mitigated by choosing a high enough threshold (above ρ^*) for electing the risky alternative, thus providing a new normative rationale for requiring supermajorities to adopt risky alternatives (see Buchanan and Tullock (1967), Caplin and Nalebuff (1988), Dal Bo (2006), and Holden (2009) for alternative justifications of conservatism).[15] $\square$

3.4 Comparison to other solution concepts

We illustrate how retrospective voting equilibrium (RVE) compares to the two standard solution concepts in the literature: sincere voting (SV) and Nash equilibrium (NE). The main finding is that RVE exhibits the more realistic features of the other solution concepts: behavior is endogenous (as in NE, but unlike SV), but outcomes depend on both the electoral rule and the precision of information, and individual voting behavior depends on private information for a significant fraction of the electorate (as in SV, but unlike NE).

As a reminder, under SV voters choose the alternative with the highest expected payoff conditional on their private signal. Under NE, voters choose the alternative with the highest expected payoff conditional on their private signal and conditional on the event that their vote is pivotal in equilibrium. Feddersen and Pesendorfer (1997) establish *full information equivalence*: the NE outcome in a sufficiently large election is essentially the outcome that would arise if voters had perfect information about the state of the world.[16]

Example 1, continued. Under SV, voters vote for R if they observe s^R and for L if they observe s^L. The election cutoff is given by the intersection of the electoral rule with the proportion of voters choosing R in each state, which is given by $q(s^R \mid \cdot)$. One can see from Figure 3 that SV is efficient (i.e., aggregates information) if and only if majority rule is used, $\rho = 1/2$. This result is an illustration of Condorcet's famous "jury theorem". Under NE, by full information equivalence, the NE outcome is efficient and R is elected for $\omega > 0$ and L is elected for $\omega < 0$. The striking aspect

[15] Recall that we restrict attention to non-unanimous rules. In this example, however, the pivotal event and the event that the risky alternative is chosen coincide under unanimity, so that NE and RVE behavior coincide and, as shown by Feddersen and Pesendorfer (1998), unanimity does poorly in aggregating information.

[16] We restrict attention to the symmetric Nash equilibrium that is characterized by Feddersen and Pesendorfer (1997).

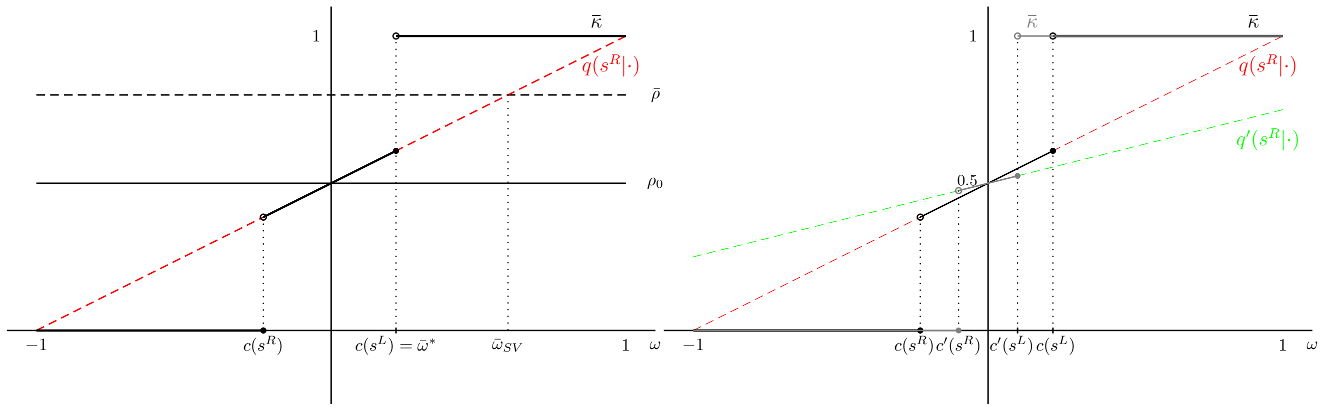

Figure 3: Example 1. Comparative statics and comparison to SV and NE.
The left panel shows that an increase in the electoral rule from ρ_0 to $\bar{\rho}$ leads to an increase in the election cutoff to $\bar{\omega}^*$ and $\bar{\omega}_{SV}$ under RVE and SV, respectively. The right panel shows that a decrease in information precision (i.e., from $q(s^R|\cdot)$ to $q'(s^R|\cdot)$) has two opposing effects under RVE: it leads to more extreme cutoffs along the flatter schedule $q'(s^R|\cdot)$ but the personal cutoffs also get closer to zero. The final effect on welfare is ambiguous. Under SV, only the first effect is present and lower information precision results in lower welfare.

of NE is that this is true for *any* (non-unanimous) election rule and for any precision of information $\iota > 0$, no matter how small. In contrast, changes in the election rule or information precision affect outcomes both under SV and RVE.

*Changes in election rule*s. Suppose that the electoral rule increases from ρ_0 to $\bar{\rho}$, as shown in the left panel of Figure 3. Under SV, behavior is exogenous and voters continue to vote in the same way, so that the election cutoff increases from zero to ω_{SV}. In contrast, the RVE cutoff increases from zero to $\bar{\omega}^* = c(s^L) < \bar{\omega}_{SV}$, thus mitigating welfare losses from naive voting. Intuitively, the out of equilibrium dynamics implied by the dynamic retrospective rule described in Section 2 is as follows. As the rule increases to $\bar{\rho}$, voters initially do not react to this change but the election outcome of course changes: R is now elected for states higher than $\bar{\omega}_{SV}$ and L is elected for all lower states. As a result, R's observed performance improves and L's observed performance worsens, so that voters start voting for R even if they get s^L signals. But this change in voting behavior implies that the election cutoff will decrease and that R will begin to be chosen in states lower than $\bar{\omega}_{SV}$. This change in cutoff in turn makes L more desirable, and voters once again begin to vote for L when their signal is s^L. This process stops at the new RVE cutoff $\bar{\omega}^* = c(s^L)$, where voters who receive signal s^L vote L and voters who receive signal s^R are indifferent and randomize.

In particular, under RVE, voter behavior is disciplined by the performance of the parties, and dismal performances (e.g., an extreme election cutoff) produces changes in behavior that in turn affects the cutoff and mitigates welfare losses from changes in the primitives.

Changes in information precision. Under SV, a decrease in information precision flattens $q(s^R \mid \cdot)$, thus leading to more extreme election cutoffs and lower welfare. The situation is more subtle under RVE. On the one hand, a flatter $q(s^R \mid \cdot)$ leads to a flatter $\bar{\kappa}(\cdot)$ over some range. This effect leads to more extreme equilibrium cutoffs. On the other hand, the personal cutoffs $c(s^R)$ and $c(s^L)$ get closer to zero as information decreases, therefore bringing the equilibrium cutoff closer to the first-best cutoff of zero. Thus, as can be seen from the right panel of Figure 3, information has an ambiguous welfare effect. This result makes sense because voters learn from a biased sample and have systematically biased beliefs, so there is no reason why better information should mitigate this bias. □

Example 3. (*Heterogeneous voters*) It is straightforward to incorporate heterogeneity among voters. Consider an environment identical to Example 1, with the exception that preferences are heterogeneous: Payoffs of type θ are $u_\theta(R,\omega) = \omega - 1/2 + \theta$ and $u_\theta(L,\omega) = -\omega - 1/2$ for all $\omega \in \Omega$, and types are distributed uniformly on the interval $[-1, 1]$. In particular, higher types get higher payoffs under R. For concreteness, suppose majority rule, $\rho = .5$, and information precision $\iota = .5$.

Figure 4 illustrates equilibrium with heterogeneous voters for all solution concepts. Under RVE, the vote share function is given by

$$\bar{\kappa}(\omega) = q(s^R \mid \omega) \Pr\left(c_\theta(s^R) < \omega\right) + q(s^L \mid \omega) \Pr\left(c_\theta(s^L) < \omega\right),$$

where $c_\theta(s)$ is the personal cutoff of type θ for signal s. The equilibrium cutoff is $\omega^* = 0$ and all types lower than $-.22$ always vote L, all types higher than $.22$ always vote R, but all types between $-.22$ and $.22$ vote their signal. In particular, there is a significant fraction of "independent types" that vote according to their signal.

Under SV, a voter votes R whenever she either observes s^R and has type $\theta > -2/3$ or she observes s^L and has type $\theta > 2/3$. Thus, the proportion of R-votes in state ω is

$$\bar{\kappa}^{SV}(\omega) = q(s^R \mid \omega) \Pr(\theta > -2/3) + q(s^L \mid \omega) \Pr(\theta > 2/3) = .5 + \omega/3$$

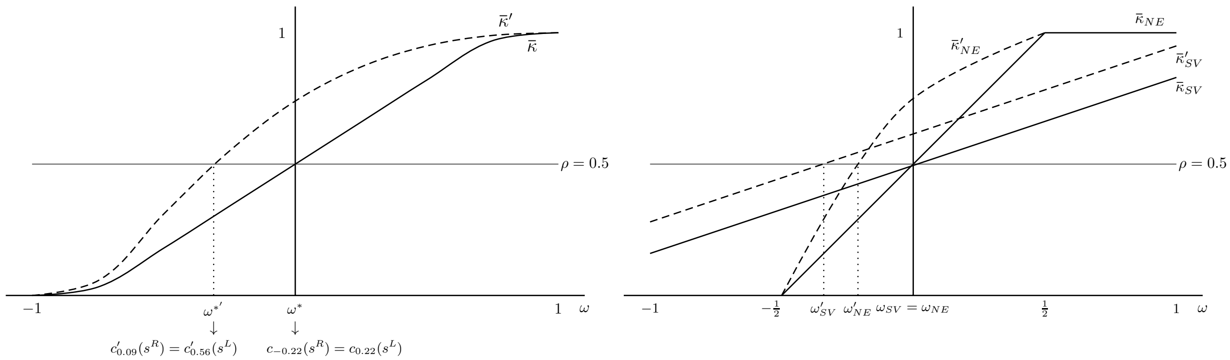

Figure 4: Example 3: Heterogeneous voters.

The vote share functions are smoothed out with heterogenous voters. The figure shows the effect of an increase (in first order stochastic sense) in the distribution of voters who prefer R (left panel shows RVE and right panel shows SV and NE).

and the election cutoff is $\omega_{SV} = 0$. Types below $-2/3$ always vote L, types above $2/3$ always vote R, and types in between $-2/3$ and $2/3$ vote their signal.

Under NE, it suffices to characterize the outcome under full information. Since type θ prefers R whenever $u_\theta(R,\omega) > u_\theta(L,\omega)$, or, equivalently, $\omega > -.5\theta$, then the proportion of votes for R is $\bar{\kappa}_{NE}(\omega) = .5 + \omega$ for $\omega \in [-.5, .5]$. Thus, the NE election cutoff is also $\omega_{NE} = 0$. Moreover, Feddersen and Pesendorfer (1997) show that NE voting behavior is as follows. For a given $\varepsilon > 0$, there is a sufficiently large number of voters such that, for all larger elections, voters that have a type in an ε-neighborhood of $\theta = 0$ vote their signal, but everyone else votes always R or always L.

Shift in the distribution of preferences. Suppose that there is an increase (in the first order stochastic dominance sense) in the distribution of voters who prefer R. For concreteness, let the new distribution of types be $\phi'(\theta) = .5(1+\theta)$ for $\theta \in [-1, 1]$. Figure 4 shows the new functions $\bar{\kappa}'_{SV}$, $\bar{\kappa}'_{NE}$, and $\bar{\kappa}'$, under SV, NE, and RVE. As expected, in all cases there is an upward shift in the proportion of votes for R, and the cutoff moves to the left, so that R is chosen in more states of the world. But the effect on voting behavior is different in each case.

Under SV, voting behavior is exogenous and so every type behaves exactly as before. The increase in the proportion of votes for R is driven by the fact that there are more high types and these types vote for R. In contrast, behavior is endogenously affected by the change in equilibrium cutoff under both NE and RVE. Under NE, the new "marginal type" is $\theta = .42$: lower types always vote L and higher types always vote R (while a vanishing fraction of types around .42 vote their signal). Under RVE, types lower than .09 always vote L, types higher than .56 always vote R, and types in between .09 and .56 vote their signal. In particular, more popular alternatives are supported by voters with more extreme preferences under NE and RVE, which helps mitigate the preference shift in favor of the more popular alternative.[17] $\square$

Example 3 also illustrates the more general point that voting behavior under both NE and RVE is influenced by the composition of the electorate.[18] The presence of composition effects has been documented in empirical work (e.g., Gelman et al. (2008), Leigh (2005)). One novel implication is that individual voting behavior might differ in local vs. national elections, even if the underlying alternatives are similar, because the composition of the electorate is different (see Fiorina (1992), Chari et al. (1997), and Franck and Tavares (2008) for evidence and alternative explanations.)

4 Endogenous policies and the degree of polarization

In this section, we apply the framework to study the degree of polarization in a two-party system. In the first stage, candidates commit to certain policies. In the second stage, voters play a voting equilibrium—we study both Nash (NE) and retrospective voting equilibrium (RVE).

We assume that there are two parties, that these two parties are ideologically constrained to choose policies from different platforms, but that they are free to choose the degree of polarization within their platform (e.g., a Republican candidate chooses whether to be closer to the center or far to the right). These assumptions seem consistent with constraints faced by candidates in the real world (e.g., because of their affiliation to different national parties), and these constrains are arguably unrelated to the type of equilibrium (Nash or RVE) played by voters.[19] We also assume that

[17]In the case of NE, the reason for more extreme supporters of R is that the pivotal voter believes that the state is given by the cutoff state, since this is where the proportions voting for R and L are equal. When preferences shift and the cutoff decreases, then the pivotal voter believes the state is lower. In order to be indifferent between voting for L or R, then its type must be higher. In the case of RVE, the reason for more extreme supporters of R is that, if R is elected more often, then it must be elected in worse states and its observed performance must be lower. Therefore, types that were marginally willing to vote for R will no longer desire to vote for R.

[18]For example, McMurray (2013a) shows that, under NE, the relative quality of information in the electorate influences individual voting behavior.

[19]The two-party assumption is realistic and often attributed to the fact that there is only one

policies are not state contingent, which is a simple way to capture the fact that policies are not fully responsive to states due to both political and informational constraints. Finally, we assume that there is no private information (to be interpreted as the limiting case of a negligible amount of information). This assumption is made for tractability and because it constitutes a relatively small departure from the standard setting of two-party competition (Downs, 1957).[20] The main insight that arises is that parties have incentives to polarize under RVE, but not NE, and that this polarization actually increases welfare.

The environment is described by $\{I, \Omega, g, \Theta, \phi, \mathbb{X}, u_\theta\}$, where: $I = \{Left, Right\}$ is the set of players; $\Omega = [-1, 1]$ is the state space; g is the density function over states satisfying $\inf_\Omega g(\omega) > 0$; $\Theta \subset \mathbb{R}$ is a compact interval (a singleton is a special case) representing the set of preference types; ϕ is the probability distribution over types; $\mathbb{X} = [-1, 1]$ is the set of policies, with the interpretation that $x = L < 0$ represents a Left policy, $x = R > 0$ represents a Right policy, and $x = 0$ is a Neutral policy; $u_\theta : \mathbb{X} \times \Omega \to \mathbb{R}$ is the payoff function of type θ, which is assumed to be bounded, continuously differentiable in Ω (except possibly in a finite number of points), and jointly continuous in $\mathbb{X} \times \Omega$. We assume (for simplicity) that the election is decided by majority rule and denote the type of the median voter by θ_M. We make the following assumptions.[21]

B1. For all $\theta \in \Theta$: $u_\theta(L, \cdot)$ is decreasing for all $L < 0$, $u_\theta(R, \cdot)$ is increasing for all $R > 0$, and $u_\theta(0, \omega) = 0$ for all $\omega \in \Omega$.

B2. For all $L < 0$ and $R > 0$, $u_\theta(L, \omega)$ is decreasing in θ and $u_\theta(R, \omega)$ is increasing in θ for all $\omega \in \Omega$.

B3. (i) $u_{\theta_M}(x, 0) < 0$ for all policies $x \neq 0$; (ii) There exist policies $\bar{L} < 0$ and $\bar{R} > 0$ such that $E\left(u_{\theta_M}(\bar{L}, W) | W \leq 0\right) > 0$ and $E\left(u_{\theta_M}(\bar{R}, W) | W \geq 0\right) > 0$.

Assumption B1 says that, the higher the state, then the higher the payoff from

winner of the election (e.g., Duverger, 1954). The constraint on policies breaks the symmetry of the model while guaranteeing that the monotonicity assumptions of Section 3 are satisfied (the literature often breaks the symmetry by assuming that parties have policy preferences, though in some cases the restriction is placed directly on the strategy space, e.g., Gul and Pesendorfer, 2009). From an empirical perspective, these constraints on policies are often attributed to the weight of the national parties' distinctive ideologies (e.g., Ansolabehere et al. (2001)) and the influence of closed primaries (e.g., Gerber and Morton (1998)).

[20] In fact, with no private information (not even a negligible amount), our setting corresponds to a Downsian setting with random payoffs.

[21] Formally, the median voter is $\theta_M = \min\{\theta' : \phi(\{\theta : \theta \leq \theta'\}) \geq 1/2\}$.

Right policies and the lower the payoff from Left policies. In addition, there is a Neutral policy $x = 0$ with a constant payoff (normalized to zero) that captures the potential for policy convergence among the parties. Assumption B2 says that types are also ordered: higher types have higher payoffs from Right policies and lower payoffs from Left policies. Finally, assumption B3 says that polarized policies are bad in "neutral" states but can be beneficial in "extreme" states from the point of view of the median voter. If the state is $\omega = 0$, then the Neutral policy is best and any polarized policies $L < 0$ and $R > 0$ result in lower payoffs. There exist, however, polarized policies $\bar{L} < 0$ and $\bar{R} > 0$ that are on average better than the Neutral policy when evaluated in the Left ($\omega < 0$) and Right ($\omega > 0$) states, respectively.

We fix the previous environment and consider a policy game between two players or parties, $Left$ and *Right*. In the first stage, the parties simultaneously choose and commit to policies (L, R) with the objective of maximizing their probability of winning the election. The $Left$ party is restricted to choose a Left or Neutral policy, $L \leq 0$, and the *Right* party is restricted to choose a Right or Neutral policy, $R \geq 0$. In the second stage, voters play an equilibrium of the voting game, where the policies in the first stage determine voters' payoffs. We consider two notions of voting equilibrium for the second stage: Nash (NE) and retrospective voting equilibrium (RVE). We assume that parties are sophisticated and know which notion of voting equilibrium is played in the second stage, and we focus on the Nash equilibrium policies of the policy game, which we refer to as *policy equilibrium*.

Voting behavior under NE and RVE is characterized by Feddersen and Pesendorfer (1997) and the results in Section 3, respectively, for all policies $(L, R) \neq (0, 0)$.[22] For the case $(L, R) = (0, 0)$, we make the natural assumption that, under both NE and RVE, the $Left$ party is elected in states $\omega < 0$ and the *Right* party is elected in states $\omega > 0$.[23]

Finally, to compare outcomes we will use the welfare of the median voter, which, for a fixed policy profile (L, R), is

$$\boldsymbol{W}_{\theta_M}(L, R) \equiv \Pr(W \geq \omega^*) \, E\left(u_{\theta_M}(R, W) \mid W \geq \omega^*\right) + \Pr(W < \omega^*) \, E\left(u_{\theta_M}(L, W) \mid W < \omega^*\right),$$

[22]For simplicity, we equate Nash equilibrium with its limit, full information equivalence. The Online Appendix provides the formal justification for this approach.

[23]This assumption can be rationalized by adding a small constant $\epsilon > 0$ to the payoff of electing party $Left$ whenever $\omega < 0$ and party *Right* whenever $\omega > 0$, and then taking the limit of equilibria as ϵ goes to 0.

where ω^* is the equilibrium election cutoff that results either under NE or RVE.

The following examples satisfy the previous assumptions and illustrate the range of environments to which our results apply.

Example 1, continued. Suppose that, in addition to a Left policy that focuses on training and a Right policy that focuses on corporate subsidies, there is also a Neutral policy that mitigates the costs of unemployment by a magnitude that does not depend on whether unemployment is due to weak demand or poor skills. A typical example is welfare policy intended to bring people out of poverty. In particular, the Neutral policy does not depend on the state of the world, and we normalize its payoff to zero. Then $x = L \leq 0$ represents the weight given to education policies relative to welfare policies, with $L = -1$ representing all the weight on education: $u(L, \omega) = L(\omega + 1/2)$. Similarly, $x = R \geq 0$ represents the weight given to corporate subsidies relative to welfare policies, with $R = 1$ representing all the weight on subsidies: $u(R, \omega) = R(\omega - 1/2)$. □

Example 4. Consider an election between two district attorneys in a county plagued by drug-related crime. There is a choice between a "left" intervention that targets the supply for drugs and a "right" intervention that targets demand. There is also a neutral policy under which prosecution efforts are not increased for drug-related crimes. The $Left$ and *Right* candidates choose the level of resources devoted to a left and right intervention, respectively. Drug-related crime is constant across states, but crime is mostly driven by the demand side in high states and by the supply side in low states.□

A Neutral policy also naturally arises in cases where voters compare the performance of the policies to an observable benchmark. As the next examples illustrate, the benchmark can be the payoff in a control group or the payoff before the policy is enacted (for empirical evidence of such comparisons, see Healy and Malhotra (2010)).

Example 5. Two candidates compete in a local union election. The $Left$ and *Right* candidates adhere to a tough and soft bargaining platform, respectively. All workers/voters have the same quadratic utility

$$\Pi(x, \omega) = -(x - \omega)^2.$$

The *Left* candidate commits to a relatively tough demand $x = L \leq 0$ and the *Right* candidate commits to a relatively soft demand $x = R \geq 0$. The interpretation is that, the higher the state of the world, the higher the firm's bargaining power and, therefore, the softer is the optimal demand by the union. Workers also observe, as a benchmark, the payoffs $\Pi(0, \omega)$ of a non-unionized sector that is equivalent to implementing a Neutral policy $x = 0$. Workers evaluate their union representative against this benchmark,:

$$u(x, \omega) = \Pi(x, \omega) - \Pi(0, \omega) = -x^2 + 2x\omega.$$

In particular, higher states make it more desirable to adopt softer demands. □

Example 6.[24] The natural rate of unemployment is given by a function $\bar{U}(\omega)$ that is decreasing in the state. The actual unemployment rate U_θ of a voter of type $\theta > 0$ is

$$U_\theta(x, \omega) = \bar{U}(\omega) + x/\theta > 0,$$

where $x \in [-1, 1]$ is the policy. A policy $x = L < 0$ is a fiscal stimulus and decreases unemployment; a policy $x = R > 0$ is contractionary (e.g., expenditure reduction) and increases unemployment. The Neutral policy $x = 0$ results in the natural unemployment rate. Voters dislike both unemployment and increases in government expenditure, and their utility is given by

$$\Pi_\theta(x, \omega) = -\left(U_\theta(x, \omega)\right)^2 + x.$$

At the beginning of a period, the Neutral policy of $x = 0$ is in place and voters observe the effects of this benchmark policy. Then, the party in power implements its chosen policy and voters observe the effects of this policy. Voters' then assess the extent to which the policy implemented by the party was beneficial; thus

$$u_\theta(x, \omega) = \Pi_\theta(x, \omega) - \Pi_\theta(0, \omega) = -x^2/\theta^2 - 2x\bar{U}(\omega)/\theta + x.$$

We assume that the median voter prefers the Neutral policy if she believes the state to be $\omega = 0$, i.e., $\theta_M = 2\bar{U}(0)$.

In particular, higher states represent better economic fundamentals and make

[24] This example is based on a model by Persson and Tabellini (2000, p. 426).

$x = R > 0$ policies more desirable; similarly, $x = L < 0$ policies are more desirable when fundamentals are bad. Higher types are less affected by economic policy and prefer less stimulus and more expenditure reduction. Finally, the median voter prefers the Neutral policy in state $\omega = 0$ but prefers some $x = R > 0$ policy if the economy is better than average and some $x = L < 0$ if it is worse than average. $\square$

The next result says that, because of the ordering over types, equilibrium is determined by the preferences of the median voter.

Lemma 2. *For any policy profile $(L, R) \neq (0, 0)$, the unique NE election cutoff is given by*

$$\omega_{NE}(L, R) = \arg\min_{\omega \in [-1,1]} |u_{\theta_M}(R, \omega) - u_{\theta_M}(L, \omega)|$$

and the unique RVE election cutoff is given by the personal cutoff of the median voter,

$$\omega_{RVE}(L, R) = \arg\min_{\omega \in [-1,1]} |v_{\theta_M}(\omega)|.$$

Moreover, welfare of the median voter under RVE is given by

$$\boldsymbol{W}_{\theta_M}(L, R) = E\left(u_{\theta_M}(R, W)|W \geq \omega^*\right) \geq E\left(u_{\theta_M}(L, W)|W \leq \omega^*\right)$$

if $\omega^* < 1$ and

$$\boldsymbol{W}_{\theta_M}(L, R) = E\left(u_{\theta_M}(L, W)|W \leq \omega^*\right) \geq E\left(u_{\theta_M}(R, W)|W \geq \omega^*\right)$$

if $\omega^* > -1$, where $\omega^* = \omega_{RVE}(L, R)$.

Proof. See the Appendix. $\square$

The last part of Lemma 2 says that observed equilibrium performance must be equalized, from the perspective of the median voter, when voters follow an RVE with an interior election cutoff. The next result compares equilibrium policies and welfare under NE and RVE.

Proposition 1. *The Neutral policy profile* $(0,0)$ *is the unique policy equilibrium when voters play NE, but it is not an equilibrium when voters play RVE. Moreover, any policy equilibrium under RVE yields strictly higher welfare to the median voter than the unique policy equilibrium under NE.*

Proof. Suppose voters play NE. By Lemma 2, the election cutoff for a profile (L,R) is given by the intersection of $u_{\theta_M}(R,\cdot)$ and $u_{\theta_M}(L,\cdot)$. By continuity of the payoff functions in the state, monotonicity of the payoff functions in policies (B1), and the assumption that $u(x,0)<0$ for all $x\neq 0$ (B1), it follows that, for all $R\geq 0$, $\arg\max_{L\geq 0}\omega_{NE}(L,R)=0$. Therefore, the Neutral policy is a strictly dominant strategy for party $Left$. A similar argument shows that the Neutral policy is a strictly dominant strategy for party *Right*.

Now suppose voters play RVE and that $(L,R)=(0,0)$, so that $\omega_{RVE}(0,0)=0$. By B3, there exists $\bar{R}$ such that $E\left(u_{\theta_M}(\bar{R},W)|W\geq 0\right)>0=u(0,\cdot)$. Thus, $\bar{R}$ is a profitable deviation for party *Right* because, by B1 and Lemma 2, $\omega_{RVE}(\bar{R},0)<\omega_{RVE}(0,0)$.

Finally, consider any policy equilibrium under RVE, (L,R), with election cutoff $\omega^*\equiv\omega_{RVE}(L,R)\leq 0$. By Lemma 2, the equilibrium welfare of the median voter is $\boldsymbol{W}_{\theta_M}(L,R)=E\left(u_{\theta_M}(R,W)|W\geq\omega^*\right)$. Suppose, in order to obtain a contradiction, that $\boldsymbol{W}_{\theta_M}(L,R)\leq 0$, where 0 is the payoff in the policy equilibrium under NE. Then, by B3, there exists $\bar{L}$ such that

$$\begin{aligned}E\left(u_{\theta_M}(R,W)|W\geq\omega^*\right)\leq 0&<E\left(u_{\theta_M}(\bar{L},W)|W\leq 0\right)\\&\leq E\left(u_{\theta_M}(\bar{L},W)|W\leq\omega^*\right),\end{aligned}$$

where the last inequality follows by B1. Then $\bar{L}$ is a profitable deviation for the $Left$ party because, by B1, $\omega_{RVE}(\bar{L},R)>\omega_{RVE}(L,R)$, thus contradicting that (L,R) is an equilibrium under RVE. The case where $\omega_{RVE}(L,R)>0$ is similar and, therefore, omitted. □

The first result in Proposition 1 says that, when voters play NE, both parties choose the same, Neutral policy. This result extends the standard Downsian logic of the median voter theorem to a setting where there is uncertainty about the best alternative.[25] The idea is that polarization hurts the chances of a party not only

[25]McMurray (2013b) recently shows this convergence result in a pure common value setting.

in states that are in the opposite extreme but also in intermediate states. Thus, parties end up converging to a common, middle platform. This simple logic does not apply under RVE. The reason is that retrospective voters evaluate parties based on *observed*, not counterfactual, performance. A party which chooses a polarized policy wins in those extreme states in which the policy is best, and, therefore, voters will assess the party to have a relatively high performance. In equilibrium, this advantage over the other party cannot persist and will have to be mitigated by electing the party even in intermediate states in which its policy is not superior. Thus, the party with a polarized policy ends up elected in both extreme and intermediate states of the world.

Proposition 1 also says that the welfare of the median voter is strictly higher under RVE compared to NE. The reason is that, under RVE, the parties choose different policies and there is a better match between policies and the state of the world. Thus, while Nash voting is efficient in aggregating information for fixed policies, it does poorly if policies are endogenous when compared to retrospective voting.

Example 5 (continued). Figures 5 and 6 illustrate how to find a policy equilibrium for this example under the assumption that the state is uniformly distributed. First, consider the case where voters follow NE. Suppose that $L = -1/2$ and $R = 1/2$, so that the election cutoff is at $\omega_{NE}(L, R) = 0$, which is where the two quadratic utility functions intersect. Then, as shown in the left panel of Figure 5, the *Right* party can deviate to $R' = 0$, move the election cutoff to $\omega_{NE}(L, R') = -1/4$, and, therefore, increase its chances of being elected. The unique Nash equilibrium, $(0, 0)$, is shown in the right panel of Figure 5. Any deviation (such as $R' > 0$ in the figure) makes a party worse off.

The left panel of Figure 6 shows that $(0, 0)$ is not an equilibrium under RVE because the *Right* party can decrease the election cutoff to $\omega^*(0, R') < 0$ by deviating to policy R'. This is clear from the picture because the integral of $\Pi(R', \omega)$ is higher than $\Pi(0, \omega)$ over $\omega > 0$, which implies that $v(0) > 0$ and, therefore, the election cutoff decreases.[26] The right panel of Figure 6 depicts the unique equilibrium under RVE: $(L, R) = (-1/2, 1/2)$ with election cutoff $\omega^*(-1/2, 1/2) = 0$. Any deviation, such as $R' > 0$ shown in the figure, results in a lower integral and, therefore, a decrease in the $v(\cdot)$ function and an increase in the election cutoff. Inspection of the right panels

[26]It suffices to look at Π, rather than u, because the same benchmark term $E\left(u(L, W) \mid W \leq 0\right) = E\left(u(R, W) \mid W \geq 0\right)$ is subtracted from payoffs whenever $L = -R$ and the election cutoff is zero.

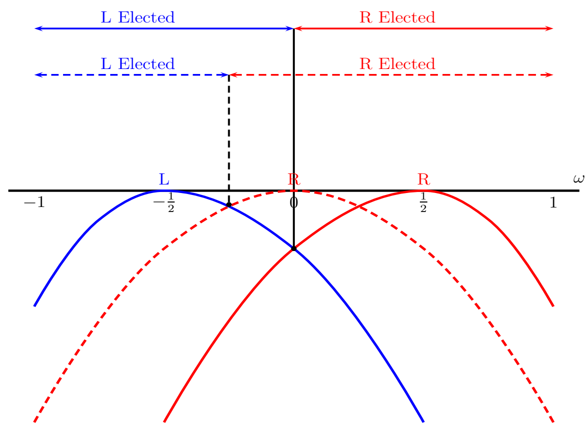

Figure 5: Example 5: Policy equilibrium under NE voting.

Left panel: $(-1/2, 1/2)$ is not an equilibrium because, for example, *Right* has a profitable deviation to $R = 0$. Right panel: $(0, 0)$ is the unique policy equilibrium. The figure illustrates how a deviation by *Right* to $R' > 0$ increases the NE cutoff and makes *Right* worse off.

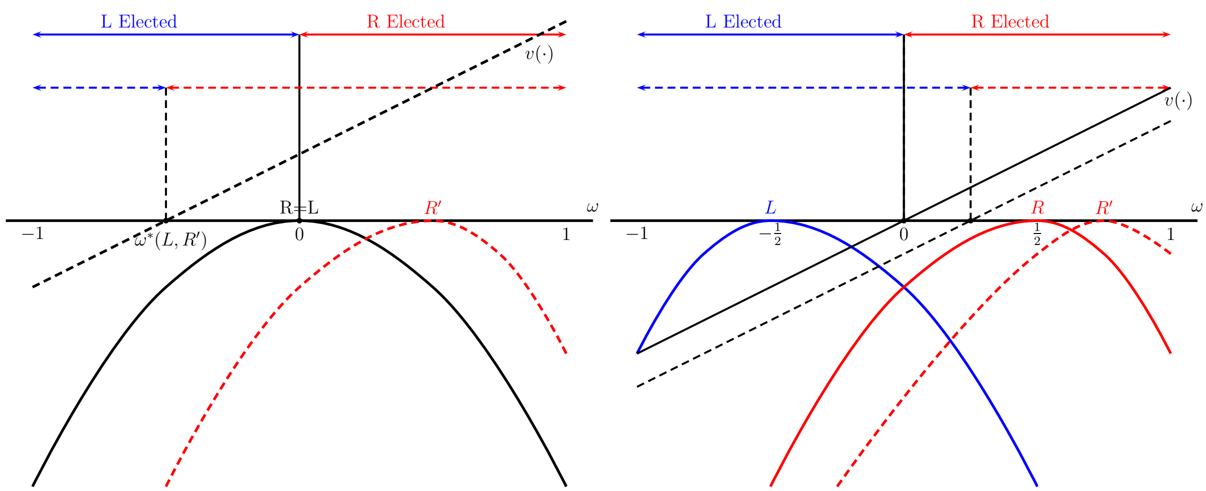

Figure 6: Example 5. Policy equilibrium under RVE voting.

Left panel: $(0, 0)$ is not an equilibrium because, for example, *Right* can deviate to $R' > 0$ and decrease the RVE cutoff from zero to $\omega^*(L, R')$. Right panel: $(-1/2, 1/2)$ is the unique policy equilibrium. The figure illustrates how a deviation by *Right* to $R' > 1/2$ increases the RVE cutoff and makes *Right* worse off; a similar figure applies for a deviation to $R' < 1/2$.

of Figures 5 and 6 also reveals that welfare is strictly higher under RVE compared to NE. In both cases, $Left$ is elected for $\omega < 0$ and $Right$ is elected for $\omega > 0$. The difference is that the average performance of policy $-1/2$ is strictly higher than the performance of policy 0 in states $\omega < 0$, and similarly for policies 1/2 and 0 in states $\omega > 0$.[27] $\square$

In the previous example, welfare is not only higher under RVE compared to NE, but it is also efficient, in the sense that a planner who has to commit to two policies and who wants to maximize voter welfare would choose policies $L = -1/2$ and $R = 1/2$ and would then decide to implement L for $\omega < 0$ and R for $\omega > 0$. Efficiency does not hold in general, but we now show that a policy equilibrium (L, R) under RVE is *constrained-efficient* in the sense that it maximizes the welfare of the median voter when the election is decided by retrospective voting.

Proposition 2. *Suppose that voters play RVE.*

(i) If (L, R) maximizes $\mathbf{W}_{\theta_M}$, then (L, R) is a policy equilibrium.

(ii) If (L, R) is a policy equilibrium with interior election cutoff, then (L, R) maximizes $\mathbf{W}_{\theta_M}$.

Proof. (i) Suppose that (L, R) with election cutoff $\omega^* \equiv \omega_{RVE}(L, R)$ maximizes $\mathbf{W}_{\theta_M}$. Suppose that $Right$ has a profitable deviation to R', so that $\omega' \equiv \omega_{RVE}(L, R') < \omega^* \leq 1$. Then, by Lemma 2,

$$\begin{aligned}\mathbf{W}_{\theta_M}(L, R') &= E\left(u_{\theta_M}(R', W) \mid W \geq \omega'\right) \geq E\left(u_{\theta_M}(L, W) \mid W \leq \omega'\right)\\ &> E\left(u_{\theta_M}(L, W) \mid W \leq \omega^*\right) = \mathbf{W}_{\theta_M}(L, R),\end{aligned}$$

where the strict inequality follows by B1. But the above expression contradicts the assumption that (L, R) maximizes $\mathbf{W}_{\theta_M}$. Therefore, $Right$ has no profitable deviation. A similar proof establishes that $Left$ has no profitable deviation. Therefore, (L, R) is a policy equilibrium.

(ii) Suppose not, so that (L', R') with election cutoff $\omega' \equiv \omega_{RVE}(L', R')$ gives strictly higher welfare to type θ_M than (L, R) with election cutoff $\omega^* \equiv \omega_{RVE}(L, R) \in$

[27] In a RVE, the median voter is indifferent between $Right$ and $Left$ and, if faced with the option, would prefer either of these policies to the Neutral policy. With heterogenous preferences, half of the electorate would favor $Left$ and the other half $Right$.

$(-1, 1)$, where we assume that $\omega' \leq \omega^*$ (the case $\omega' > \omega^*$ is similar and, therefore, omitted). Then

$$\begin{aligned}
\mathbf{W}_{\theta_M}(L, R) &= E\left(u_{\theta_M}(L, W) \mid W \leq \omega^*\right) = E\left(u_{\theta_M}(R, W) \mid W \geq \omega^*\right) \\
&< E\left(u_{\theta_M}(R', W) \mid W \geq \omega^{'}\right) = \mathbf{W}_{\theta_M}(L', R') \\
&\leq E\left(u_{\theta_M}(R', W) \mid W \geq \omega^*\right),
\end{aligned}$$

where the first two lines follow by Lemma 2 and the assumption that $\mathbf{W}_{\theta_M}(L', R') > \mathbf{W}_{\theta_M}(L, R)$ and the last line follows by B1 and the fact that $\omega' \leq \omega^*$. In particular, $E\left(u_{\theta_M}(L, W) \mid W \leq \omega^*\right) < E\left(u_{\theta_M}(R', W) \mid W \geq \omega^*\right)$ and, therefore, $\omega_{RVE}(L, R') < \omega^*$ and, therefore, (L, R) is not an equilibrium because player *Right* can increase its chances of being elected by deviating to R'. □

In Lemma 3 in the Appendix, we show that there always exists a policy profile (L, R) that maximizes $\boldsymbol{W}_{\theta_M}$ under RVE. Thus, a corollary of Proposition 2 is that a policy equilibrium under RVE always exists. We conclude by illustrating how Proposition 2 can also be useful for finding a policy equilibrium.

Example 6 (continued). Suppose that the state is uniformly distributed and that the natural rate of unemployment is given by $\bar{U}(\omega) = 1 - \omega/2$ and the median voter is $\theta_M = 2\bar{U}(0) = 2$. First, we argue that there is no equilibrium policy profile with a corner election cutoff. Suppose, for example, that R is always elected in equilibrium. Then $E\left(u_{\theta_M}(R, W)\right) = -R^2/4 < 0$ and, therefore, player *Left* could deviate to $L = 0$ and win the election with probability one. Thus, from now on we analyze cases where the election cutoff is interior. For every policy profile (L, R),

$$\begin{aligned}
v_{\theta_M}(\omega) &= E\left(u_{\theta_M}(R, W) \mid W \geq \omega\right) - E\left(u_{\theta_M}(L, W) \mid W \leq \omega\right) \\
&= \frac{1}{4}\left(L^2 - R^2 + R + L + \omega(R - L)\right).
\end{aligned}$$

Then, for every $(L, R) \neq (0, 0)$ with interior cutoff, the equilibrium cutoff is given by

$$c_{RVE}(L, R) = L + R - \frac{R + L}{R - L}.$$

Moreover, by Lemma 2, $\mathbf{W}_{\theta_M}(L, R) = E\left(u_{\theta_M}(R, W) \mid W \geq c_{RVE}(L, R)\right)$. It is straightforward to check that $(L, R) = (-1/2, 1/2)$ is the unique policy profile that maximizes $\mathbf{W}_{\theta_M}(L, R)$. Then, by Proposition 2, $(-1/2, 1/2)$ is the unique policy equilibrium under RVE. Thus, the $Left$ party chooses to stimulate the economy and is elected in states where unemployment is high, while the *Right* party chooses a contractionary policy and is elected in states where unemployment is low. On the other hand, the unique equilibrium under NE voting is $(0, 0)$, so that both parties choose the same Neutral policy, which yields the natural rate of unemployment. Welfare is therefore higher for the median voter under RVE compared both to NE voting and to a single-party system which implements the ex-ante optimal policy for the median voter, $x = 0$. $\square$

By incorporating state-contingent payoffs and (even a negligible amount of) private information, our analysis provides a novel mechanism through which the economy (i.e., the state of the world) interacts with the political environment: the parties specialize in different policies and the electorate tends to elect them in states in which these policies are best. Moreover, the evidence appears to be consistent with this view of party polarization: the economy tends to expand with left-wing governments and contract with right-wing government (Hibbs (1977) and Alesina and Roubini (1992)) and left-wing government are more likely to be elected during periods of high unemployment and right-wing government during periods of high inflation (Faust and Irons (1999)). Finally, it is interesting to note that this better match between policies and states arises as a result of boundedly rational voters who use a simple retrospective heuristic and evaluate parties based on their observed performance. A more sophisticated electorate, capable of complicated counterfactual and pivotal computations, would actually decrease welfare.

5 Foundation for voting equilibrium

We provide a game-theoretic foundation for RVE by showing that it corresponds to the limit of naive behavioral equilibrium (Esponda, 2008) as the number of players in our voting environment goes to infinity. All proofs appear in the Online Appendix.

5.1 Voting game

In this section, we describe the voting game with n voters and provide the definition of naive behavioral equilibrium. The rules of the game are as described in Section 3. The difference is that there are now a finite number of players, indexed by $i = 1, ..., n$, with types $(\theta_1, ..., \theta_n)$, where we now assume that Θ is a finite set rather than a compact interval (see the end of the section for a discussion). Player i's payoff when the election outcome is $o \in \{R, L\}$ is now

$$u_{\theta_i}(o, \omega) + 1\{o = L\}\nu,$$

where $\nu \in \mathbb{R}$ is a privately-observed payoff perturbation drawn independently for each player from a probability distribution F_{θ_i}. Recall that K is the uniform bound on payoffs postulated in assumption A1. In addition to A1-A3, we maintain the following assumptions for all $\theta \in \Theta$:

A5. F_θ is absolutely continuous and satisfies $F_\theta(-2K) > 0$ and $F_\theta(2K) < 1$; its density f satisfies $\inf_{x \in [-2K, 2K]} f_\theta(x) > 0$.

A6. $\mathbb{S}$ has at least two elements and there exists $z > 0$ such that for all $\omega' > \omega$ and $s' > s$,

$$\frac{q_\theta(s'|\omega')}{q_\theta(s'|\omega)} - \frac{q_\theta(s|\omega')}{q_\theta(s|\omega)} \geq z(\omega' - \omega).$$

Assumption A5 guarantees that each alternative is voted with positive probability. This property implies that the probability that players are pivotal (i.e., that their vote decides the election) becomes negligible as $n \to \infty$.[28] Assumption A6 is a strengthening of MLRP that establishes a uniform bound on the rate at which the likelihood ratio changes.

Following Harsanyi (1973), for each player there is a threshold perturbation above which the player will vote for L and below which she will vote for R. Thus, integrating over such perturbations and noting that F_θ is absolutely continuous, we

[28] A5 also yields a refinement, which is standard in the literature, that rules out equilibria where everyone votes for the same alternative because a unilateral deviation cannot change the outcome. Esponda and Pouzo (2012) show that the perturbations are also important for providing a learning foundation for naive equilibrium.

obtain a (mixed) strategy for each player i, $\alpha_i : \mathbb{S} \to [0,1]$, where $\alpha_i(s)$ is the probability of voting for R after observing signal s. In addition, each strategy profile $\alpha = (\alpha_1, ..., \alpha_n)$, together with the primitives of the game, induces a distribution $P^n(\alpha)$ over the outcomes of the game $\{R, L\} \times \mathbb{S}^n \times \Omega$.

To gain intuition for the notion of a naive equilibrium, suppose that player i repeatedly faces a sequence of stage games where players use strategies α every period. Then, under the assumption that the payoff to alternative R is observed only whenever R is chosen, player i will come to observe that, conditional on observing signal s, alternative R yields in expectation $E_{P^n(\alpha)}\left(u_{\theta_i}(R, W) \mid o = R, S_i = s\right)$.[29] A similar expression holds for alternative L.

A naive player who observes ν and s believes that expected utility is maximized by voting for R whenever $\Delta_i(P^n(\alpha), s) - \nu > 0$ and voting for L otherwise, where

$$\Delta_i(P^n(\alpha), s) \equiv E_{P^n(\alpha)}\left(u_{\theta_i}(R, W) \mid o = R, S_i = s\right) - E_{P^n(\alpha)}\left(u_{\theta_i}(L, W) \mid o = L, S_i = s\right) \tag{4}$$

is well-defined because of the payoff perturbations.

Definition 4. A strategy profile $\alpha = (\alpha_1, ..., \alpha_n)$ is a *(naive) equilibrium* of the voting game if for every player $i = 1, ..., n$ and for every $s \in \mathbb{S}$,

$$\alpha_i(s) = F_{\theta_i}\left(\Delta_i(P^n(\alpha), s)\right).$$

In equilibrium, each player best responds to a belief that depends endogenously on everyone's strategy and that is consistent with observed equilibrium outcomes. Naive players, however, do not account for the correlation between others' votes and the state of the world (conditional on their own private information).[30]

It is important to note that the definition of naive equilibrium does not rely on the monotonicity assumptions on payoff functions and the information structure. We do make use, however, of these monotonicity assumptions when characterizing naive equilibrium with a large number of players. A characterization of equilibrium without these monotonicity assumptions is outside the scope of this paper and is unlikely to

[29] Whenever an expectation E_P has a subscript P, this means that the probabilities are taken with respect to the distribution P.

[30] See Esponda and Pouzo (2012) for the proof that equilibrium exists and additional discussion.

yield the tractable framework introduced in Section 3.[31]

5.2 Large number of players

Our technical contribution is to analyze (naive) equilibrium as the number of voters goes to infinity. We do so by studying sequences of voting games. We build such sequences by independently drawing infinite sequences of types $\xi = (\theta_1, \theta_2, ..., \theta_n, ...) \in \Xi$ according to the probability distribution $\phi \in \Delta(\Theta)$; we denote the distribution over Ξ by Φ and we let $\theta_i(\xi)$ denote the type of player i, i.e., the ith component of ξ. We interpret each sequence of types as describing an infinite number of n-player games by letting the first n elements of ξ represent the types of the n players.

Let $\boldsymbol{\alpha}$ denote a *strategy mapping* from sequences of types Ξ to sequences of strategy profiles–i.e., for all $\xi \in \Xi$, let $\boldsymbol{\alpha}(\xi) = (\alpha^1(\xi), ..., \alpha^n(\xi), ...)$, where

$$\alpha^n(\xi) = (\alpha_1^n(\xi), ..., \alpha_n^n(\xi))$$

is the strategy profile that is played in the n-player game with types $\theta_1, ..., \theta_n$. Let $P^n(\boldsymbol{\alpha}(\xi))$ be the probability distribution over $\{R, L\} \times \mathbb{S}^n \times \Omega$ induced by the strategy profile $\alpha^n(\xi)$ in the n-player game. We define two properties of strategy mappings.[32]

Definition 5. A strategy mapping $\boldsymbol{\alpha}$ is an *ε-equilibrium mapping* if for a.e. $\xi \in \Xi$ there exists $n_{\varepsilon,\xi}$ such that for all $n \geq n_{\varepsilon,\xi}$

$$\left\| \alpha_i^n(\xi) - F_{\theta_i(\xi)} \left(\Delta_i(P^n(\boldsymbol{\alpha}(\xi)), \cdot) \right) \right\| \leq \varepsilon \tag{5}$$

for all $i = 1, ..., n$. A strategy mapping $\boldsymbol{\alpha}$ is *asymptotically interior* if, for a.e. $\xi \in \Xi$,

$$\liminf_{n \to \infty} P^n(\boldsymbol{\alpha}(\xi)) (o = R) > 0 \quad \text{and} \quad \limsup_{n \to \infty} P^n(\boldsymbol{\alpha}(\xi)) (o = R) < 1. \tag{6}$$

[31] Recently, there has been some progress in relaxing monotonicity assumptions under Nash equilibrium (Bhattacharya, 2008).

[32] The a.e. in "for a.e. $\xi \in \Xi$" stands for "almost every" and means that there is a set Ξ' with $\Phi(\Xi') = 1$ such that a condition is true for all $\xi \in \Xi'$. The results continue to hold if we only require $\Phi(\Xi') > 0$.

The first property in Definition 5 requires that, for large enough n, players play strategies that constitute an ε equilibrium. Our notion of limit equilibrium will require this property to hold for all $\varepsilon > 0$; while being slightly weaker than requiring strategies to constitute an equilibrium, this condition yields a full characterization of limit equilibrium.[33] The second property requires that the probabilities of choosing R and L remain bounded away from zero as the number of players increases. The reason for this restriction is that we can always obtain extreme equilibria where everyone votes for the same alternative, no information is obtained about the other alternative, and, therefore, beliefs about the other alternative can be arbitrary. The restriction to asymptotically interior strategies allows us to focus on equilibria where beliefs are not arbitrary.

In addition to characterizing the equilibrium cutoff, we characterize the profile of equilibrium strategies. Given a strategy mapping $\boldsymbol{\alpha}$ and a sequence of types $\xi \in \Xi$, let $\sigma^n(\xi; \boldsymbol{\alpha}) : \Theta \to [0,1]^{\mathbb{S}}$ represent the average strategy of each type in the n-player game. Formally, for all $\theta \in \Theta$ and $s \in \mathbb{S}$,

$$\sigma^n_\theta(\xi; \boldsymbol{\alpha})(s) = \frac{\sum_{i=1}^n 1\{\theta_i(\xi) = \theta\}\, \alpha_i^n(\xi)(s)}{\sum_{i=1}^n 1\{\theta_i(\xi) = \theta\}} \tag{7}$$

whenever $\sum_{i=1}^n 1\{\theta_i(\xi) = \theta\} > 0$, and arbitrary otherwise. We call any element $\sigma : \Theta \to [0,1]^{\mathbb{S}}$ an *average strategy profile* and say that σ is *increasing* if $s' > s$ implies $\sigma_\theta(s') > \sigma_\theta(s)$ for every type $\theta \in \Theta$.

Definition 6. An average strategy profile $\sigma^* : \Theta \to [0,1]^{\mathbb{S}}$ is a *limit ε-equilibrium* if there exists an asymptotically interior ε-equilibrium mapping $\boldsymbol{\alpha}$ such that

$\lim_{n\to\infty} \|\sigma^n(\xi; \boldsymbol{\alpha}) - \sigma^*\| = 0$ for a.e. $\xi \in \Xi$. An average strategy profile σ^* is a *limit equilibrium* if it is a limit ε-equilibrium for all $\varepsilon > 0$.

The following result characterizes limit equilibria.

Theorem 2. *σ^* is a limit equilibrium if and only if there exists a cutoff $\omega^* \in (-1, 1)$ such that $\kappa(\omega^*; \sigma^*) = \rho$ and $\sigma^*_\theta(s) = F(v_\theta(s; \omega^*))$ for all $\theta \in \Theta$ and $s \in \mathbb{S}$.*

[33] Our result that a limit equilibrium is a fixed point of a particular correspondence remains true under the stronger requirement that strategies constitute an equilibrium. But the converse result, that any fixed point is also a limit equilibrium, relies on the notion of ε equilibrium.

The intuition of Theorem 2 is as follows. Suppose that there is a sequence of average strategy profiles σ^n that converges to an increasing profile σ^*.[34] Then the probability that a randomly chosen player votes for R in state of the world ω converges to $\kappa(\omega;\sigma^*)$. By standard asymptotic arguments, the proportion of votes for R in state ω becomes concentrated around $\kappa(\omega;\sigma^*)$. So, for states where $\kappa(\omega;\sigma^*) > \rho$, the probability that R is elected converges to 1. Similarly, for states where $\kappa(\omega;\sigma^*) < \rho$, the probability that R is elected converges to 0. Since σ is increasing, then there is at most one (measure zero) state ω^* such that $\kappa(\omega^*;\sigma^*) = \rho$, so that the election outcome is characterized by an election cutoff ω^*. Moreover, the fact that the election outcome is characterized by a cutoff means that the beliefs of player i, defined by Δ_i in equation (4), can be approximated by the belief function v_θ defined in equation (1), where θ is the type of player i. Thus, the optimal strategy of a player of type θ who observes signal s is $\sigma^*_\theta(s) = F\left(v_\theta\left(s;\omega^*\right)\right)$.

5.3 Vanishing perturbations

We now consider sequences of equilibria where the perturbations vanish. We index games by a parameter η that indexes the cdf F^η_θ from which perturbations are drawn.

Definition 7. A family of perturbations $\{\mathbf{F}^\eta\}_{\eta\in\mathbb{N}}$, where $\mathbf{F}^\eta = \{F^\eta_\theta\}_{\theta\in\Theta}$, is *vanishing* if for all $\theta \in \Theta$ and η: assumption A5 is satisfied and

$$\lim_{\eta\to 0} F^\eta_\theta(\nu) = \begin{cases} 0 & \text{if } \nu < 0 \\ 1 & \text{if } \nu > 0 \end{cases}$$

Under a vanishing family of perturbations, the payoff perturbations converge to zero and we recover the original, unperturbed game. The next two results provide a foundation for the notion of RVE introduced in Section 3.

Theorem 3. *(i) Suppose that there exists a vanishing family of perturbations* $\{\mathbf{F}^\eta\}_\eta$ *and a sequence* $(\sigma^\eta,\omega^\eta)_\eta$ *such that* $\lim_{\eta\to 0}(\sigma^\eta, c^\eta) = (\sigma^*,\omega^*)$ *and where* σ^η *is a limit*

[34]We show in the Online Appendix that optimal strategies are increasing when the number of players is sufficiently large.

equilibrium and ω^η its corresponding cutoff for all η. Then (σ^*, ω^*) is a retrospective voting equilibrium.

(ii) Suppose that (σ^*, ω^*) is a retrospective voting equilibrium with $\omega^* \in (-1, 1)$. Then there exists a vanishing family of perturbations $\{\mathbf{F}^\eta\}_\eta$ and a sequence $(\sigma^\eta, \omega^\eta)_\eta$ such that $\lim_{\eta \to 0}(\sigma^\eta, c^\eta) = (\sigma^*, \omega^*)$ and where σ^η is a limit equilibrium and ω^η its corresponding cutoff for all η.

The first part of Theorem 3 follows by standard continuity arguments and the second part by construction.[35]

We conclude by making three observations. First, situations where one alternative is never chosen are easily justified: if an alternative is never chosen, then beliefs about its performance can be arbitrary. Our solution concept in Section 3 considers, say, $\omega^* = 1$ (i.e., *Right* is never chosen) to be an equilibrium cutoff only if equilibrium beliefs are such that *Right* yields the payoff at state $\omega = 1$ and $Left$ yields the unconditional payoff. The formal justification is that, if players follow symmetric increasing strategies such that the probability of *Right* being elected converges to zero, then the probability of state $\omega = 1$ conditional on *Right* being elected converges to 1. Second, our game-theoretic foundation uses assumption A6, which is stronger than A2 in Section 3. In particular, A2 allows for the case where voters have no private information. We can provide a foundation for such a case by considering a sequence of voting games indexed by $r \in \mathbb{N}$, where $z^r > 0$ denotes the constant defined in assumption A6, and where $\lim_{r \to \infty} z^r = 0$. Therefore, the case of no private information must be viewed as the limiting case of an information structure that satisfies A6 but where informativeness vanishes. Finally, in Section 3 we assumed that Θ was a compact interval, rather than a finite set, in order to obtain uniqueness of equilibrium and facilitate the application of the framework. However, we can view the case where Θ is a compact interval as the limiting case of a sequence of environments where the finite number of elements in Θ goes to infinity.

[35] As shown in the proof, the argument holds for *any* family of perturbations if ω^* is the unique equilibrium cutoff and $\phi\{(\theta : c_\theta(s) = \omega^*,\ s \in \mathbb{S}_\theta)\} = 0$.

6 Conclusion

We provided a framework that formalizes a previously ignored feature of many elections: voters learn to make decisions by observing past outcomes, but they cannot observe the consequences of outcomes that are never chosen. Voters hold systematically biased beliefs, but these beliefs arise endogenously from the biased sample that derives from the aggregate behavior of all voters. The framework is easy to apply and yields several new insights about large elections. When embedded into a setting of two-party competition, the model predicts that parties with differentiated platforms will tend to exacerbate their differences. This polarization, however, increases the welfare of the median voter.

The model can be generalized in several nontrivial directions: considering ways in which voters could account for selection (for example, by conditioning on vote shares); allowing for more than two alternatives and letting voters be strategic, as in Myatt, 2007; considering nonstationary environments; and relaxing the monotonicity assumptions on preferences and information that drive our characterization results.

References

Achen, C.H. and L.M. Bartels, "Blind retrospection: Electoral responses to drought, flu, and shark attacks," 2004.

Alesina, A. and H. Rosenthal, "Polarized platforms and moderate policies with checks and balances," *Journal of Public Economics*, 2000, *75* (1), 1–20.

Alesina, Alberto and Nouriel Roubini, "Political cycles in OECD economies," *The Review of Economic Studies*, 1992, *59* (4), 663–688.

Ansolabehere, Stephen, James M Snyder Jr, and Charles Stewart III, "Candidate positioning in US House elections," *American Journal of Political Science*, 2001, pp. 136–159.

Aragones, E. and T. Palfrey, "An Economic Model of Representative Democracy," *Journal of Economic Theory*, 2002, *103*, 131–161.

Austen-Smith, D. and J.S. Banks, "Information aggregation, rationality, and the Condorcet jury theorem," *American Political Science Review*, 1996, pp. 34–45.

Barro, R., "The Control of Politicians: An Economic Model," *Public Choice*, 1973, *14*, 19–42.

Bartels, Larry M, "Uninformed votes: Information effects in presidential elections," *American Journal of Political Science*, 1996, pp. 194–230.

Bendor, J., D. Diermeier, Siegel D.A., and M.M. Ting, *A behavioral theory of elections*, Princeton Univ Pr, 2011.

_ , S. Kumar, and D.A. Siegel, "Adaptively Rational Retrospective Voting," *Journal of Theoretical Politics*, 2010, *22* (1), 26.

Besley, T. and S. Coate, "An Economic Model of Representative Democracy*," *Quarterly Journal of Economics*, 1997, *112* (1), 85–114.

Bhattacharya, S., "Preference Monotonicity and Information Aggregation in Elections," 2008.

Buchanan, J.M. and G. Tullock, *The calculus of consent: Logical foundations of constitutional democracy*, Vol. 100, University of Michigan Press, 1967.

Callander, Steven, "Searching for good policies," *American Political Science Review*, 2011, *105* (4), 643–662.

Calvert, R. L., "Robustness of the multidimensional voting model: Candidate motivations, uncertainty, and convergence," *American Journal of Political Science*, 1985, pp. 69–95.

Caplin, A. and B. Nalebuff, "On 64%-majority rule," *Econometrica*, 1988, *56* (4), 787–814.

Chari, V.V., L.E. Jones, and R. Marimon, "The economics of split-ticket voting in representative democracies," *The American Economic Review*, 1997, pp. 957–976.

Charness, G. and D. Levin, "The origin of the winner's curse: a laboratory study," *American Economic Journal: Microeconomics*, 2009, *1* (1), 207–236.

Converse, P.E., "Assessing the capacity of mass electorates," *Annual Review of Political Science*, 2000, *3* (1), 331–353.

Costinot, A. and N. Kartik, "On Optimal Voting Rules under Homogeneous Preferences," 2007.

Dal Bo, E., "Committees with supermajority voting yield commitment with flexibility," *Journal of Public Economics*, 2006, *14*, 573–599.

Delli Carpini, M.X. and S. Keeter, *What Americans know about politics and why it matters*, Yale Univ Pr, 1997.

Downs, A., *An economic theory of democracy*, New York: Harper and Row, 1957.

Duverger, M., *Political parties: their organization and activity in the modern state.*, New York: Wiley, 1954.

Esponda, I., "Behavioral equilibrium in economies with adverse selection," *The American Economic Review*, 2008, *98* (4), 1269–1291.

_ **and D. Pouzo**, "Learning Foundation for Equilibrium in Voting Environments with Private Information," *working paper*, 2012.

_ **and E.I. Vespa**, "Hypothetical Thinking and Information Extraction: Pivotal Voting in the Laboratory," 2011.

Eyster, E. and M. Rabin, "Cursed equilibrium," *Econometrica*, 2005, *73* (5), 1623–1672.

_ **and T. Kittsteiner**, "Party platforms in electoral competition with heterogeneous constituencies," *Theoretical Economics*, 2007, (2), 41–70.

Faust, Jon and John S Irons, "Money, politics and the post-war business cycle," *Journal of Monetary Economics*, 1999, *43* (1), 61–89.

Feddersen, T. and W. Pesendorfer, "Voting behavior and information aggregation in elections with private information," *Econometrica*, 1997, pp. 1029–1058.

_ **and** _ , "Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting," *American Political Science Review*, 1998, pp. 23–35.

Ferejohn, J., "Incumbent performance and electoral control," *Public choice*, 1986, *50* (1), 5–25.

Fiorina, M., *Divided government*, New York: Macmillan, 1992.

Fiorina, M.P., "Economic retrospective voting in American national elections: A micro-analysis," *American Journal of Political Science*, 1978, pp. 426–443.

_ , *Retrospective voting in American national elections*, Yale University Press, 1981.

Franck, R. and S. Tavares, "Income and vote switching between local and national elections: evidence from New York State," *Economics Bulletin*, 2008, *4* (10), 1–10.

Gelman, A., D. Park, B. Shor, J. Bafumi, and J. Cortina, *Red State, Blue State, Rich State, Poor State: Why Americans Vote the Way They Do*, 1st edition ed., Princeton University Press, 2008.

Gerber, Elisabeth R and Rebecca B Morton, "Primary election systems and representation," *Journal of Law Economics and Organization*, 1998, *14* (2), 304–324.

Guarnaschelli, S., R.D. McKelvey, and T.R. Palfrey, "An experimental study of jury decision rules," *American Political Science Review*, 2000, pp. 407–423.

Gul, F. and W. Pesendorfer, "Partisan politics and election failure with ignorant voters," *Journal of Economic Theory*, 2009, *144* (1), 146–174.

Harsanyi, J.C., "Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points," *International Journal of Game Theory*, 1973, *2* (1), 1–23.

Healy, A. and N. Malhotra, "Random Events, Economic Losses, and Retrospective Voting: Implications for Democratic Competence," *Quarterly Journal of Political Science*, 2010, *5* (2), 193–208.

Hibbs, Douglas A, "Political parties and macroeconomic policy," *American political science review*, 1977, *71* (04), 1467–1487.

Holden, R., "Supermajority voting rules," 2009.

Huber, A., S.J. Hill, and G.S. Lenz, "Sources of Bias in Retrospective Decision-Making: Experimental Evidence on Voters' Limitations in Controlling Incumbents," *American Political Science Review*, forthcoming.

Jehiel, P., "Analogy-based expectation equilibrium," *Journal of Economic theory*, 2005, *123* (2), 81–104.

_ **and D. Samet**, "Valuation equilibrium," *Theoretical Economics*, 2007, *2* (2), 163–185.

_ **and F. Koessler**, "Revisiting games of incomplete information with analogy-based expectations," *Games and Economic Behavior*, 2008, *62* (2), 533–557.

Kagel, J.H. and D. Levin, *Common value auctions and the winner's curse*, Princeton Univ Pr, 2002.

Kamada, Y. and F. Kojima, "Voter preferences, polarization, and electoral policies," Technical Report, Working Paper, May 2012.

Kartik, N. and R. P. McAfee, "Signaling character in electoral competition," *The American economic review*, 2007, pp. 852–870.

_ **, F. Squintani, and K. Tinn**, "Information Revelation and (Anti-) Pandering in Elections," Technical Report, Working paper 2012.

Key, V.O., *The responsible electorate*, Vintage Books New York, 1966.

Kramer, G.H., "Short-term fluctuations in US voting behavior, 1896-1964," *The American Political Science Review*, 1971, pp. 131–143.

Leigh, A., "Economic Voting and Electoral Behavior: How Do Individual, Local, and National Factors Affect the Partisan Choice?," *Economics & Politics*, 2005, *17* (2), 265–296.

_ , "Does the World Economy Swing National Elections?," *Oxford Bulletin of Economics and Statistics*, 2009, *71* (2), 163–181.

Levy, G. and R. Razin, "Correlation neglect, voting behaviour and polarization," *Working Paper*, 2014.

Lewis-Beck, M.S. and M. Stegmaier, "Economic determinants of electoral outcomes," *Annual Review of Political Science*, 2000, *3* (1), 183–219.

Martinelli, C., "Ignorance and Naivete in Large Elections," 2011.

Martorana, M.F. and I. Mazza, "Adaptive voting: an empirical analysis of participation and choice," 2012.

McCarty, N.M., K.T. Poole, and H. Rosenthal, *Polarized America: The dance of ideology and unequal riches*, MIT Press Cambridge, MA, 2006.

McMurray, J. C., "Aggregating information by voting: The wisdom of the experts versus the wisdom of the masses," *The Review of Economic Studies*, 2013, *80* (1), 277–312.

_ , "Ideology as Opinion: A Spatial Model of Common-value Elections," 2013.

Milgrom, P.R., "A convergence theorem for competitive bidding with differential information," *Econometrica*, 1979, pp. 679–688.

_ , "Good news and bad news: Representation theorems and applications," *The Bell Journal of Economics*, 1981, pp. 380–391.

Myatt, D.P., "On the theory of strategic voting," *The Review of Economic Studies*, 2007, *74* (1), 255.

Nordhaus, W.D., "The political business cycle," *The Review of Economic Studies*, 1975, *42* (2), 169–190.

Osborne, M. J. and A. Slivinski, "A model of political competition with citizen-candidates," *The Quarterly Journal of Economics*, 1996, *111* (1), 65–96.

Osborne, M.J. and A. Rubinstein, "Games with procedurally rational players," *American Economic Review*, 1998, *88*, 834–849.

_ **and** _ , "Sampling equilibrium, with an application to strategic voting," *Games and Economic Behavior*, 2003, *45*, 434–441.

Palfrey, T. R., "Spatial equilibrium with entry," *The Review of Economic Studies*, 1984, *51* (1), 139–156.

Spiegler, R., "Placebo reforms," *The American Economic Review*, 2013, *103* (4), 1490–1506.

T., Persson and G. Tabellini, *Political economics - Explaining economic policy*, MIT Press: Cambridge, 2000.

Thaler, R.H., "Anomalies: The winner's curse," *The Journal of Economic Perspectives*, 1988, *2* (1), 191–202.

Wittman, D., "Candidates with policy preferences: A dynamic model," *Journal of Economic Theory*, 1977, *14* (1), 180–189.

Wolfers, J., "Are Voters Rational? Evidence from Gubernatorial Elections," 2009.

Woon, J., "Democratic Accountability and Retrospective Voting: A Laboratory Experiment," *American Journal of Political Science*, 2012.

7 Appendix

Proof of Lemma 1. $\bar{\kappa}(\cdot)$ is left-continuous: By the Dominated Convergence Theorem, it suffices to show left-continuity of $q_\theta\left(c_\theta(s) < \omega \mid \omega\right) \equiv \sum_{\{s:c_\theta(s)<\omega\}} q_\theta\left(s \mid \omega\right)$ for all $\theta \in \Theta$. Fix any $\theta \in \Theta$. Since there are a finite number of personal cutoffs for type θ (defined by equation (2)), then for each $c \in (-1, 1)$ there exists $\omega'_\theta < c$ such that all personal cutoffs of θ are outside the interval $[\omega'_\theta, c)$. Then, for all $\hat{\omega}$, $q_\theta\left(c_\theta(s) < \omega \mid \hat{\omega}\right) = q_\theta\left(c_\theta(s) < c \mid \hat{\omega}\right)$ for all $\omega \in [\omega'_\theta, c]$. In addition, $q_\theta(s \mid \cdot)$ is continuous by A3(iii). Therefore, $\lim_{\omega \uparrow c} q_\theta\left(c_\theta(s) < \omega \mid \omega\right) = q_\theta\left(c_\theta(S_\theta) < c \mid c\right)$.

$\bar{\kappa}(\cdot)$ is increasing over $(\underline{c}, \overline{c})$: Let $\underline{c} < \omega < \omega' < \overline{c}$. Then

$$\begin{aligned}\int_\Theta \sum_{\{s:c_\theta(s)<\omega'\}} q_\theta\left(s \mid \omega'\right)\phi(d\theta) &\geq \int_\Theta \sum_{\{s:c_\theta(s)<\omega\}} q_\theta\left(s \mid \omega'\right)\phi(d\theta) \\ &\geq \int_\Theta \sum_{\{s:c_\theta(s)<\omega\}} q_\theta\left(s \mid \omega\right)\phi(d\theta), \qquad (8)\end{aligned}$$

where the last inequality follows because, since $c_\theta(\cdot)$ is nondecreasing, the event $\{c_\theta(s) < \omega\}$ is equivalent to $\{s \leq s_\theta(\omega)\}$ for some threshold $s_\theta(\omega)$, and, therefore, MLRP implies that $\sum_{\{s:c_\theta(s)<\omega\}} q_\theta\left(s \mid \omega'\right) \geq \sum_{\{s:c_\theta(s)<\omega\}} q_\theta\left(s \mid \omega\right)$ (see (Milgrom, 1981)). Next, we show that the inequality in (8) holds strictly. This is trivially true if there exists a positive ϕ-measure of types with personal cutoffs in $[\omega, \omega')$, so suppose that is not the case. Since, by A4, $c_\theta(s^L)$ is continuous in θ and Θ is a compact interval, the union of $c_\theta(s^L)$ over all $\theta \in \Theta$ is a compact interval. Given that there is no positive measure of types with personal cutoffs in $[\omega, \omega')$, then, the facts that ϕ has full support and $\omega' < \bar{c}$ implies that, for all $\theta \in \Theta$, $c_\theta(s^L) \geq \omega' > \omega$ and, therefore, $\{c_\theta(s) < \omega\} \neq \mathbb{S}$. Then, because MLRP holds strictly (by A2), the second inequality in (8) is strict.

Finally: If $\omega \leq \underline{c}$, then $\{c_\theta(s) < \omega\} = \emptyset$ for all θ, so that $\bar{\kappa}(\omega) = 0$. Similarly, if $\omega > \bar{c}$, then $\{c_\theta(S_\theta) < \omega\} = \mathbb{S}$ for all θ, so that $\bar{\kappa}(\omega) = 1$. $\square$

Proof of Theorem 1. The proof relies on the following claim.

Claim 1.1 Suppose that σ is optimal given election cutoff ω^*. Then

$$\kappa(\omega;\sigma) = \int_\Theta \left(\sum_{\{s: c_\theta(s) < \omega^*\}} q_\theta(s \mid \omega) + \sum_{\{s: c_\theta(s) = \omega^*\}} q_\theta(s \mid \omega)\sigma_\theta(s) \right) \phi(d\theta) \qquad (9)$$

for all $\omega \in \Omega$. In addition, $\bar{\kappa}(\omega) \geq \kappa(\omega;\sigma)$ for $\omega > \omega^*$ and $\bar{\kappa}(\omega) \leq \kappa(\omega;\sigma)$ for $\omega < \omega^*$.

Proof. Since σ is optimal given ω^*, then

$$\sigma_\theta(s) = \begin{cases} 0 & \text{if } c_\theta(s) > \omega^* \\ 1 & \text{if } c_\theta(s) < \omega^* \end{cases} \qquad (10)$$

and equation (9) follows. In addition, for all $\omega > \omega^*$,

$$\begin{aligned} \kappa(\omega;\sigma) &\leq \int_\Theta \sum_{\{s: c_\theta(s) \leq \omega^*\}} q_\theta(s \mid \omega)\phi(d\theta) \\ &\leq \int_\Theta \sum_{\{s: c_\theta(s) < \omega\}} q_\theta(s \mid \omega)\phi(d\theta) = \bar{\kappa}(\omega). \end{aligned}$$

Similarly, for all $\omega < \omega^*$, $\kappa(\omega;\sigma) \geq \bar{\kappa}(\omega)$. $\square$

We now prove Theorem 1. Fix $\rho \in (0,1)$ and let

$$\omega^* \equiv \kappa^{-1}(\rho) = \inf\{\omega : \bar{\kappa}(\omega) \geq \rho\}. \qquad (11)$$

Note that, by Lemma 1, $\omega^* \in [\underline{c}, \bar{c}]$. We begin by showing that there exists σ^* such that (σ^*, ω^*) is a voting equilibrium. Let σ^* satisfy (10). It remains to specify $\sigma^*_\theta(s)$ for (θ, s) such that $c_\theta(s) = \omega^*$. First, suppose that $\omega^* \notin \{-1, 1\}$. If ω^* is the election cutoff, then (θ, s) such that $c_\theta(s) = \omega^*$ is indifferent between R and L, and, therefore, $\sigma^*_\theta(s) = \alpha$ is optimal for any $\alpha \in [0, 1]$. Let σ^*_α denote the strategy profile constructed above. We now pick α such that ω^* is an election cutoff given σ^*_α. Let $\hat{\kappa}(\alpha) \equiv \kappa(\omega^*; \sigma_\alpha)$. By Claim 1.1,

$$\hat{\kappa}(\alpha) = \int_\Theta \left(\sum_{\{s: c_\theta(s) < \omega^*\}} q_\theta(s \mid \omega^*) + \sum_{\{s: c_\theta(s) = \omega^*\}} q_\theta(s \mid \omega^*)\alpha \right) \phi(d\theta),$$

which is continuous in α. First, we establish that $\hat{\kappa}(0) \leq \rho$. Suppose not, so that $\hat{\kappa}(0) = \bar{\kappa}(\omega^*) > \rho$. Since $\bar{\kappa}$ is left-continuous (Lemma 1), then there exists $\omega' < \omega^*$ such that $\bar{\kappa}(\omega') > \rho$. But then (11) is contradicted. Second, we establish that $\hat{\kappa}(1) \geq \rho$. Suppose not, so that $\hat{\kappa}(1) = \lim_{\omega \downarrow \omega^*} \bar{\kappa}(\omega) < \rho$. Then, there exists $\omega'' > \omega^*$ such that $\bar{\kappa}(\omega'') < \rho$. But, since $\bar{\kappa}(\cdot)$ is increasing (Lemma 1), then (11) is contradicted. Since $\hat{\kappa}(0) \leq \rho$ and $\hat{\kappa}(1) \geq \rho$, by continuity of $\hat{\kappa}$ there exists α^* such that $\hat{\kappa}(\alpha^*) = \kappa(\omega^*; \sigma^*_{\alpha^*}) = \rho$. Since $\kappa(\cdot; \sigma^*_{\alpha^*})$ is nondecreasing (because $\sigma^*_{\alpha^*}$ is nondecreasing), then ω^* is an election cutoff given $\sigma^*_{\alpha^*}$. Hence, $(\sigma^*_{\alpha^*}, \omega^*)$ is a voting equilibrium. Next, suppose that $\omega^* = -1$ (the case $\omega^* = 1$ is similar and, therefore, omitted). Now let $\alpha^* = 1$; in particular, $\sigma^*_{\alpha^*}$ is optimal given ω^* (note it would not necessarily be optimal for a different value of α^*). In addition, we just established above that $\hat{\kappa}(1) = \kappa(\omega^*; \sigma^*_{\alpha^*}) \geq \rho$. Since $\kappa(\cdot; \sigma^*_{\alpha^*})$ is nondecreasing, it follows that $\kappa(\omega; \sigma^*_{\alpha^*}) \geq \rho$ for all ω, implying that $\omega^* = -1$ is a cutoff given $\sigma^*_{\alpha^*}$.

Finally, we show that, for all $\omega \neq \omega^*$, there exists no σ such that (σ, ω) is a voting equilibrium. Suppose, in order to obtain a contradiction, that (σ, ω) is a voting equilibrium, where $\omega < \omega^*$ (the case $\omega > \omega^*$ is similar and, therefore, omitted) . Let $\omega' \in (\omega, \omega^*)$. Then $\bar{\kappa}(\omega') \geq \kappa(\omega'; \sigma) \geq \rho$, where the first inequality follows from Claim 1.1 and the second from the fact that ω is an election cutoff given σ. But then (11) is contradicted. $\square$

Proof of Lemma 2. The equilibrium cutoff under $j = NE, RVE$ is unique and given by $\omega_j(L, R) = \inf\{\omega \in \Omega : \bar{\kappa}_j(\omega) \geq 1/2\}$, where $\bar{\kappa}_j(\omega) = \phi(\{\theta : c_{\theta,j} < \omega\})$, and where $c_{\theta,NE} = \arg\min_{\omega \in [-1,1]} |u_\theta(R, \omega) - u_\theta(L, \omega)|$ and $c_{\theta,CRV}$ is the personal cutoff defined in (2). By Theorem 1, the above statement is correct for $j = RVE$. For the case of Nash equilibrium, $j = NE$, the statement follows from full information equivalence (Feddersen and Pesendorfer, 1997): at each state, the proportion of people voting for an alternative is given by the proportion of people that prefer that alternative at the given state. By B1-B2, $v_\theta(\omega)$ and $u_\theta(R,\omega) - u_\theta(L,\omega)$ are continuous in ω and increasing in θ and ω; thus $c_{\theta,j}$ is continuous, nonincreasing, and it is decreasing for all θ such that $c_{\theta,j} \in (-1,1)$. Therefore, because θ_M is the median voter, it follows that $\omega_j(L,R) = c_{\theta_M,j} = \inf\{\omega \in \Omega : \phi(\{\theta : c_{\theta,j} < \omega\}) \geq 1/2\}$. The last statement in the lemma follows because v_{θ_M} is increasing. For example, if $\omega^* \equiv \omega_{RVE}(L,R) \in (-1,1)$, then, since v_{θ_M} is increasing, it follows that $v_{\theta_M}(\omega^*) = 0$. Thus, $\boldsymbol{W}_{\theta_M}(L,R) = E\left(u_{\theta_M}(R,W)|W \geq \omega^*\right) = E\left(u_{\theta_M}(L,W)|W \leq \omega^*\right)$. $\square$

Lemma 3. *There exists a policy profile (L,R) that maximizes $\boldsymbol{W}_{\theta_M}$ under RVE.*

Proof. By B1, v_θ is continuous and, therefore, the Theorem of the Maximum implies that $\omega_{RVE}(L,R)$ is continuous for all $(L,R) \neq (0,0)$. Then, B1, Lemma 2, and the Dominated convergence theorem imply that $\boldsymbol{W}_{\theta_M}$ is continuous for all $(L,R) \neq (0,0)$. In addition, $\mathbf{W}_{\theta_M}(0,0) = 0$ and, for all R,

$$\limsup_{L\to 0} E\left(u_{\theta_M}(L,W) \mid W \leq \omega_{RVE}(L,R)\right) \leq \lim_{L\to 0} u_{\theta_M}(L,-1) = 0$$

where the first inequality follows because $u_{\theta_M}(L,\cdot)$ is decreasing (assumption B1) and the last equality because $u_{\theta_M}(\cdot,-1)$ is continuous. A similar result holds for $E\left(u_{\theta_M}(R,W) \mid W \geq \omega_{RVE}(L,R)\right)$. Thus

$$\limsup_{(L,R)\to(0,0)} \mathbf{W}_{\theta_M}(L,R) \leq 0.$$

Therefore, $\mathbf{W}_{\theta_M}$ is upper semi-continuous. Since $[-1,0]\times[0,1]$ is compact, then the maximum of $\mathbf{W}_{\theta_M}$ is attained. $\square$

8 Online Appendix

Online appendix "Conditional Retrospective Voting in Large Elections, " by Ignacio Esponda and Demian Pouzo.

1.1 Full information equivalence

Throughout the paper, we have used Feddersen and Pesendorfer's (1997) result that the symmetric Nash equilibrium as the number of players goes to infinity is characterized by full information equivalence. In this section, we formally state this result and show how to provide the appropriate limiting counterparts for the setting with endogenous policies.

Feddersen and Pesendorfer (1997, Theorem 3) and Lemma 4 in this paper imply the following result: Let ω^* denote the election cutoff under Nash equilibrium (or RVE), defined in Lemma 2. For all $\varepsilon > 0$, there is an n_ε such that for all $n' > n_\varepsilon$, the following holds: if $\omega < \omega^* - \varepsilon$, then L is elected with probability greater than $1 - \varepsilon$; if $\omega < \omega^* + \varepsilon$, then R is elected with probability greater than $1 - \varepsilon$.

The following result provides a limiting foundation for Proposition 1. For simplicity, we assume that parties can choose from a finite number of policies that includes the neutral policy and at least one polarized right $R > 0$ and one polarized left $L < 0$ policies.

Proposition 3. *There exists n such that for all $n' > n$, the Neutral policy $(0, 0)$ is the unique policy equilibrium when voters play NE, but it is not an equilibrium when voters play naive behavioral equilibrium.*

Proof. Consider first the case of NE and let $\omega_{NE}(L, R)$ denote the election cutoff. Define $m_L \equiv \min_{L<0,R\geq 0} G\left(\omega_{NE}(0, R)\right) - G\left(\omega_{NE}(L, R)\right)$ and $m_R \equiv \min_{L\leq 0,R>0} G\left(\omega_{NE}(L, R)\right) - G\left(\omega_{NE}(L, 0)\right)$. By B1, the finiteness of the policy space, the continuity of payoff functions in the state, and the fact that $\inf_\Omega g(\omega) > 0$, it follows that $m_L > 0$ and $m_H > 0$. Let $m \equiv \min\{m_L, m_R\} > 0$. First, fix R and compare the probability that party $Left$ gets elected with a policy $L < 0$ vs. $L = 0$. Let $\varepsilon < m/(1 + m)$ and consider an election with $n' > n_\varepsilon$ voters, where n_ε is defined directly above the proposition. By full information equivalence, the highest probability of electing $Left$ under (L, R) is $G\left(\omega_{NE}(L, R)\right) + \varepsilon\left(1 - G\left(\omega_{NE}(L, R)\right)\right)$ and the lowest probability of electing $Left$

under $(0, R)$ is $(1-\varepsilon)G(\omega_{NE}(0, R))$. The latter probability is strictly higher than the former if $G(\omega_{NE}(0, R)) - G(\omega_{NE}(L, R)) > \varepsilon/(1-\varepsilon)$, which is the case here by our choice of ε.[36] Thus, it is a strictly dominant strategy for party $Left$ to choose the Neutral policy. A similar proof establishes that it is a strictly dominant strategy for party *Right* to choose the Neutral policy.

Next, consider the case of a naive behavioral equilibrium. Suppose that $(L, R) = (0, 0)$ and consider a deviation by party $Left$ to policy $\bar{L}$ defined in assumption B3. By assumption, $E\left(u(\bar{L}, W) \mid W < 0\right) > 0$ and, therefore, the election cutoff $\omega_{RVE}(\bar{L}, 0) > 0$. Let $k \equiv G\left(\omega_{RVE}(\bar{L}, 0)\right) - G(0)$ and note that $k > 0$ by absolute continuity of G. Let $\varepsilon < k/(1+k)$ and consider an election with $n' > n_\varepsilon$ voters, where n_ε is defined directly above the proposition. By the statement directly above the proposition, the highest probability of electing $Left$ under $(0, 0)$ is $G(0)+\varepsilon(1-G(0))$ and the lowest probability of electing $Left$ under $(\bar{L}, 0)$ is $(1-\varepsilon)G\left(\omega_{RVE}(\bar{L}, 0)\right)$. The latter probability is strictly higher than the former if $G\left(\omega_{RVE}(\bar{L}, 0)\right) - G(0) > \varepsilon/(1-\varepsilon)$, which is the case here by our choice of k. Thus, $Left$ has strict incentives to deviate from the Neutral policy. $\square$

1.2 Foundation for voting equilibrium: Proofs

Here we prove the statements in Section 5.

1.2.1 Preliminary lemma

The proof of Theorem 2 relies on the following lemma.

Lemma 4. *Let $\boldsymbol{\alpha}$ be such that $\lim_{n\to\infty} \|\sigma^n(\xi; \boldsymbol{\alpha}) - \sigma^*\| = 0$ for a.e. Ξ, where σ^* is increasing. Then there exists $\omega^* \in \arg\min_{\omega\in\Omega} |\kappa(\omega; \sigma^*) - \rho|$ such that for all $\epsilon > 0$ and a.e. $\xi \in \Xi$,*

$$\lim_{n\to\infty} \inf_{\omega\in\Omega:\omega\geq\omega^*+\epsilon} P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega) = 1 \tag{12}$$

and

$$\lim_{n\to\infty} \sup_{\omega\in\Omega:\omega\leq\omega^*-\epsilon} P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega) = 0. \tag{13}$$

[36]Note that we use the finite number of policies to get uniformity of n_ε in the policy profile (L, R). The proof with a continuous policy space is more tedious, but it can be shown that uniformity holds under continuity assumptions on the payoff functions. In that case, however, the result is that the equilibrium must be in a neighborhood of $(0, 0)$, and this neighborhood decreases in size as the number of players increases.

Moreover, if $\omega^* \in (-1,1)$, then for a.e. $\xi \in \Xi$ and for all $\varepsilon > 0$ there exists $n_{\xi,\varepsilon}$ such that for all $n \geq n_{\xi,\varepsilon}$,

$$\left\|\Delta_i(P^n(\boldsymbol{\alpha}(\xi)),\cdot) - v_{\theta_i(\xi)}(\cdot;\omega^*)\right\| \leq \varepsilon \tag{14}$$

for all $i = 1,...,n$.

Proof. We use the following notation. Let $x_i \in \{R, L\}$ denote the vote of player i, let $\kappa_i^n(\omega;\xi) \equiv P^n(x_i = R \mid \omega)$ be the probability that player $i = 1,...,n$ votes for R conditional on the state being ω, and let $\kappa^n(\omega;\xi) \equiv \frac{1}{n}\sum_{i=1}^n \kappa_i^n(\omega;\xi)$ be the average over all players.

First, note that, for a.e. $\xi \in \Xi$, for all $\omega \in \Omega$,

$$\begin{aligned}
\lim_{n\to\infty} \kappa^n(\omega;\xi) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \sum_{\theta\in\Theta}\sum_{s\in\mathbb{S}} q_\theta(s|\omega) 1\{\theta_i(\xi)=\theta\}\alpha_i^n(\xi)(s) \\
&= \lim_{n\to\infty} \sum_{\theta\in\Theta}\sum_{s\in\mathbb{S}} q_\theta(s|\omega)\left\{\frac{1}{n}\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}\alpha_i^n(\xi)(s)\right\} \\
&= \sum_{\theta\in\Theta}\sum_{s\in\mathbb{S}} q_\theta(s|\omega)\left\{\lim_{n\to\infty}\sigma_\theta^n(\xi;\boldsymbol{\alpha})(s) \times \left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}\right)\right\} \\
&= \sum_{\theta\in\Theta}\sum_{s\in\mathbb{S}} q_\theta(s|\omega)\sigma_\theta^*(s)\phi(\theta) = \kappa(\omega;\sigma^*),
\end{aligned} \tag{15}$$

where we have used the assumption that $\lim_{n\to\infty}\|\sigma^n(\xi;\boldsymbol{\alpha}) - \sigma^*\| = 0$ a.s.-Ξ and the strong law of large numbers applied to $\frac{1}{n}\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}$. Note also that, for all $\omega, \omega' \in \Omega$,

$$\begin{aligned}
|\kappa^n(\omega;\xi) - \kappa^n(\omega';\xi)| &\leq \sum_{\theta\in\Theta}\sum_{s\in\mathbb{S}} |q_\theta(s|\omega) - q_\theta(s|\omega')| \left\{\sigma_\theta^n(\xi;\boldsymbol{\alpha})(s) \times \left(\frac{1}{n}\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}\right)\right\} \\
&\leq \max_{\theta\in\Theta}\max_{s\in\mathbb{S}} |q_\theta(s|\omega) - q_\theta(s|\omega')|
\end{aligned}$$

and since $|\Theta| < \infty$ and $|\mathbb{S}| < \infty$, this display and A3(iii) imply that the family $\{\kappa^n(\cdot;\xi) : \Omega \to [0,1] \colon n = 1,2,...\}$ is equicontinuous for all $\xi \in \Xi$. This result, the one in (15) and the fact that Ω is compact, implies that

$$\lim_{n\to\infty}\sup_{\omega\in\Omega} |\kappa^n(\omega;\xi) - \kappa(\omega;\xi)| = 0 \tag{16}$$

a.s.-Ξ.

Second, let $Y^n(\omega;\xi) \equiv n^{-1/2}\sum_{i=1}^n (1\{x_i^n = R\} - \kappa_i^n(\omega;\xi))$. It follows that for all $\delta > 0$ and for a.e. ξ, there exists $n'(\delta,\xi)$ such that, for all $n \geq n'(\delta,\xi)$,

$$\begin{aligned} P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega) &= P^n(\boldsymbol{\alpha}(\xi))\left(Y^n(\omega;\xi) \geq \sqrt{n}(\rho - \kappa^n(\omega;\xi)) \mid \omega\right) \\ &\leq P^n(\boldsymbol{\alpha}(\xi))\left(Y^n(\omega;\xi) \geq \sqrt{n}\delta \mid \omega\right) \\ &\leq (n^2\delta^2)^{-1}\sum_{i=1}^n E\left[(1\{x_i^n = R\} - \kappa_i^n(\omega;\xi))^2 \mid \omega\right] \\ &\leq 4(n\delta^2)^{-1}, \end{aligned}$$

for all $\omega \in \{\omega \in \Omega : \kappa(\omega;\sigma) \leq \rho - \delta\}$, where the second line follows from (16) and the third from the Markov inequality.

Third, the facts that $\kappa(\cdot;\sigma^*)$ is increasing (because σ^* is increasing) and continuous (by A3(iii)) imply that there exists $\omega^* \in [-1,1]$ such that $\omega^* \in \arg\min_{\omega\in\Omega}|\kappa(\omega;\sigma^*) - \rho|$ and that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\kappa(\omega;\sigma^*) \leq \rho - \delta$ for all $\omega \leq \omega^* - \epsilon$. Hence, $\{\omega \in \Omega : \omega \leq \omega^* - \epsilon\} \subseteq \{\omega \in \Omega : \kappa(\omega;\sigma^*) \leq \rho - \delta\}$, and the previous argument implies that

$$\lim_{n\to\infty}\sup_{\omega\in\Omega:\omega\leq\omega^*-\epsilon} P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega) = 0. \tag{17}$$

By employing a similar argument, it follows that $\lim_{n\to\infty}\inf_{\omega\in\Omega:\omega\geq\omega^*+\epsilon} P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega) = 1$ a.e. $\xi \in \Xi$.

We now establish the second part of the lemma. Suppose that $\omega^* \in (-1,1)$. First note that the previous part of the proof implies that, for any $\omega \in \Omega$

$$\lim_{n\to\infty} P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega) = 1\{\omega > \omega^*\}. \tag{18}$$

Second, note that, for all n and all $\omega \in \Omega$,

$$P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega) = \sum_{s\in\mathbb{S}} P^n(\xi)(o = R \mid \omega, S_i = s)q_{\theta_i(\xi)}(s|\omega) \tag{19}$$

for all $i \leq n$. By (18), (19), and A3(ii), for a.e. $\xi \in \Xi$ and all $s \in \mathbb{S}$,

$$\lim_{n\to\infty} P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega, S_i = s) = 0 \ (= 1) \tag{20}$$

for $\omega < \omega^*$ ($\omega > \omega^*$), where convergence is uniform in $i \leq n$.[37] Therefore, for a.e. $\xi \in \Xi$ and all $s \in \mathbb{S}$, $\lim_{n\to\infty} E_{P^n(\boldsymbol{\alpha}(\xi))}\left(u_{\theta_i(\xi)}(R,W) \mid o = R, S_i = s\right) =$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\int_\Omega P^n(\boldsymbol{\alpha}(\xi))\,(o = R \mid W, S_i = s)\, q_{\theta_i(\xi)}(s \mid W) u_{\theta_i(\xi)}(R,W) G(dW)}{\int_\Omega P^n(\boldsymbol{\alpha}(\xi))\,(o = R \mid W, S_i = s)\, q_{\theta_i(\xi)}(s \mid W) G(dW)} \\
&= \frac{\int_\Omega \lim_{n\to\infty} P^n(\boldsymbol{\alpha}(\xi))\,(o = R \mid W, S_i = s)\, q_{\theta_i(\xi)}(s \mid W) u_{\theta_i(\xi)}(R,W) G(dW)}{\int_\Omega \lim_{n\to\infty} P^n(\boldsymbol{\alpha}(\xi))\,(o = R \mid W, S_i = s)\, q_{\theta_i(\xi)}(s \mid W) G(dW)} \\
&= \frac{\int_\Omega 1\{W \geq \omega^*\} q_{\theta_i(\xi)}(s|W) u_{\theta_i(\xi)}(R,W) G(dW)}{\int_\Omega 1\{W \geq \omega^*\} q_{\theta_i(\xi)}(s|W) G(dW)} \\
&= E\left(u_{\theta_i(\xi)}(R,W) \mid W \geq \omega^*, S_i = s\right), \qquad (21)
\end{aligned}$$

where convergence is uniform in $i \leq n$. The first and fourth lines in (21) follow by definition, the second line follows from the dominated convergence theorem and the fact that u_{θ_i} is bounded (and the denominator being greater than zero, as established next), and the third line follows from (20) and the fact that G is absolutely continuous, so we can ignore the case $\{W = \omega^*\}$ (also, note the importance of $\omega^* < 1$ for the denominator to be well-defined). A similar argument holds for $E_{P^n(\boldsymbol{\alpha}(\xi))}\left(u_{\theta_i(\xi)}(L,W) \mid o = L, S_i = s\right)$, thus establishing the lemma. □

1.2.2 Proof of Theorem 2

Proof. Only if: Let σ^* be a limit equilibrium, so that σ^* is a limit ε-equilibrium for all $\varepsilon > 0$. Lemma OA in the Online Appendix, Section 1.2.4, shows that σ^* must be increasing. Fix any $\varepsilon > 0$ and let $\boldsymbol{\alpha}$ be the corresponding ε-equilibrium mapping that is asymptotically interior. Because $\boldsymbol{\alpha}$ is asymptotically interior, then $\omega^* \in (-1,1)$ and, therefore, (14) holds by Lemma 4. Then, for all $\theta \in \Theta$, there exists $\xi \in \Xi$ and

[37] Formally, suppose that $\omega < \omega^*$. Then for all $\varepsilon > 0$ there exists $n_{\xi,\omega,\varepsilon}$ such that, for all $n \geq n_{\xi,\omega,\varepsilon}$, $P^n(\boldsymbol{\alpha}(\xi))(o = R \mid \omega, S_i = s) q_{\theta_i(\xi)}(s|\omega) \leq \varepsilon$ for all $i \leq n$ and $s \in \mathbb{S}$.

n' such that for all $n \geq n'$,

$$\begin{aligned}\|\sigma_\theta^* - F_\theta(v_\theta(\cdot;\omega^*))\| \leq & \|\sigma_\theta^* - \sigma_\theta^n(\xi;\boldsymbol{\alpha})\| \\ & + \left\| \frac{\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}\,\alpha_i^n(\xi)(s)}{\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}} - \frac{\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}\,F_\theta\left(\Delta_i(P^n(\boldsymbol{\alpha}(\xi)),s)\right)}{\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}} \right\| \\ & + \left\| \frac{\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}\,F_\theta\left(\Delta_i(P^n(\boldsymbol{\alpha}(\xi)),s)\right)}{\sum_{i=1}^n 1\{\theta_i(\xi)=\theta\}} - F_\theta\left(v_\theta\left(\cdot;\omega^*\right)\right) \right\| \\ & \leq \varepsilon + \varepsilon + \varepsilon,\end{aligned}$$

where the last inequality follows because: (i) σ^* being a limit equilibrium implies that $\lim_{n\to\infty}\|\sigma^n(\xi;\boldsymbol{\alpha}) - \sigma^*\| = 0$ for a.e. $\xi \in \Xi$; (ii) $\boldsymbol{\alpha}$ is an ε-equilibrium mapping; and (iii) equation (14) and continuity of F_θ (A5). Since the above relationship holds for every $\varepsilon > 0$, then $\|\sigma_\theta^* - F_\theta(v_\theta(\cdot;\omega^*))\| = 0$ for all θ.

If: Consider the strategy mapping $\boldsymbol{\alpha}$ defined by letting players of type θ always play σ_θ^*–i.e., for all ξ, s, n, and $i \leq n$, $\alpha_i^n(\xi)(s) = \sigma_{\theta_i(\xi)}^*(s)$. First, note that $\sigma^n = \sigma^*$ converges trivially to σ^*, and σ^* is increasing because F_θ and $v_\theta(\cdot;\omega^*)$ are increasing (by A1-A3 and A5). Moreover, $\omega^* \in (-1,1)$ by assumption. Then, equations (12) and (13) in Lemma 4 and the dominated convergence theorem imply that $\boldsymbol{\alpha}$ is asymptotically interior. In addition, for a.e. $\xi \in \Xi$ and for every $\varepsilon > 0$, there exists $n_{\xi,\varepsilon}$ such that for all $n \geq n_{\xi,\varepsilon}$,

$$\begin{aligned}\left\|\alpha_i^n(\xi) - F_{\theta_i(\xi)}\left(\Delta_i(P^n(\boldsymbol{\alpha}(\xi)),\cdot)\right)\right\| &= \left\|\sigma_{\theta_i(\xi)}^* - F_{\theta_i(\xi)}\left(\Delta_i(P^n(\boldsymbol{\alpha}(\xi)),\cdot)\right)\right\| \\ &= \left\|F_{\theta_i(\xi)}\left(v_{\theta_i(\xi)}\left(\cdot;\omega^*\right)\right) - F_{\theta_i(\xi)}\left(\Delta_i(P^n(\boldsymbol{\alpha}(\xi)),\cdot)\right)\right\| \leq \varepsilon\end{aligned}$$

for all $i = 1, ..., n$, where the first line follows by construction of the strategy and the second line follows by (14) and continuity of F_θ (A5). Thus, (σ^*, ω^*) is a limit equilibrium. □

1.2.3 Proof of Theorem 3

Proof. Part (i): Theorem 2 implies that $\sigma_\theta^*(s) = \lim_{\eta\to 0}\sigma_\theta^\eta(s) = \lim_{\eta\to 0}F_\theta^\eta\left(v_\theta(s;\omega^*)\right)$ for all $\theta \in \Theta$ and $s \in \mathbb{S}$. Since $\mathbf{F}^\eta$ is vanishing, then $\sigma_\theta(s) = 1$ if $v_\theta(s;\omega^*) > 0$ and $\sigma_\theta(s) = 0$ if $v_\theta(s;\omega^*) < 0$. Therefore, σ^* is optimal given ω^*. Next, fix any $\omega' < \omega^*$. Since $\omega^\eta \to \omega^*$, there exists $\bar{\eta}$ such that, for all $\eta < \bar{\eta}$, $\omega' < \omega^\eta$, and, by Theorem 2, $\kappa(\omega';\sigma^\eta) \leq \rho$. Since $\sigma^\eta \to \sigma^*$, continuity of $\kappa(\omega';\cdot)$ implies that $\kappa(\omega';\sigma^*) \leq \rho$.

Similarly, $\kappa(\omega''; \sigma^*) \geq \rho$ for all $\omega'' > \omega^*$. Therefore, ω^* is an election cutoff given σ^*.

Part (ii): For any family of vanishing perturbations $\{\mathbf{F}^\eta\}_\eta$, define

$$\bar{\kappa}^\eta(\omega) \equiv \sum_{\theta \in \Theta} \phi(\theta) \sum_{s \in \mathbb{S}} q_\theta(s \mid \omega) F_\theta^\eta \left(v_\theta(s; \omega)\right).$$

Let (σ^*, ω^*) be a voting equilibrium with $\omega^* \in (-1, 1)$. Because $\omega^* \in (-1, 1)$ is an election cutoff given σ^* and $\kappa(\cdot; \sigma^*)$ is continuous, then $\kappa(\omega^*; \sigma^*) = \rho$. We split the proof into two cases: Either it is the case that all players vote for the same alternative (which may be different for each player) irrespective of their private information–so that $\kappa(\cdot; \sigma^*)$ is a constant function–or not–so that $\kappa(\cdot; \sigma^*)$ is increasing.

Case 1 ($\kappa(\cdot; \sigma^*)$ is increasing): Rewrite $\overline{\kappa}^\eta$ as

$$\begin{aligned}\bar{\kappa}^\eta(\omega) =& \sum_{\theta \in \Theta} \phi(\theta) \Bigg\{ \sum_{s: c_\theta(s) < \omega^*} q_\theta(s \mid \omega) F_\theta^\eta \left(v_\theta(s; \omega)\right) + \sum_{s: c_\theta(s) = \omega^*} q_\theta(s \mid \omega) F_\theta^\eta \left(v_\theta(s; \omega)\right) \\ &+ \sum_{s: c_\theta(s) > \omega^*} q_\theta(s \mid \omega) F_\theta^\eta \left(v_\theta(s; \omega)\right) \Bigg\} \equiv T_1^\eta(\omega) + T_2^\eta(\omega) + T_3^\eta(\omega).\end{aligned}$$

Since $v_\theta(s; \cdot)$ is increasing and $\omega^* \in (-1, 1)$, then: for all (θ, s) such that $c_\theta(s) \geq \omega^*$, $v_\theta(s; \omega) < 0$ for all $\omega < \omega^*$ and, for all (θ, s) such that $c_\theta(s) \leq \omega^*$, $v_\theta(s; \omega) > 0$ for all $\omega > \omega^*$. Therefore, since $\{\mathbf{F}^\eta\}_\eta$ is vanishing, $\lim_{\eta \to 0} T_2^\eta(\omega) + T_3^\eta(\omega) = 0$ for all $\omega < \omega^*$ and $\lim_{\eta \to 0} T_1^\eta(\omega) + T_2^\eta(\omega) = \sum_{\theta \in \Theta} \phi(\theta) q_\theta(c_\theta(S_\theta) \leq \omega^* \mid \omega) \geq \kappa(\omega; \sigma^*)$ for all $\omega > \omega^*$. In addition, $T_1^\eta(\omega) \leq \kappa(\omega; \sigma^*)$ and $T_3^\eta(\omega) \geq 0$ for all ω. Therefore, $\lim_{\eta \to 0} \bar{\kappa}^\eta(\omega) \leq \kappa(\omega; \sigma^*) < \kappa(\omega^*; \sigma^*) = \rho$ for all $\omega < \omega^*$ and $\lim_{\eta \to 0} \bar{\kappa}^\eta(\omega) \geq \kappa(\omega; \sigma^*) > \kappa(\omega^*; \sigma^*) = \rho$ for all $\omega > \omega^*$. Consequently, by continuity of $\kappa^\eta(\cdot)$, there exists $(\omega^\eta)_\eta$ such that $\omega^\eta \to \omega^* \in (-1, 1)$ and $\bar{\kappa}^\eta(\omega^\eta) = \rho$ for all sufficiently small η. By letting $\sigma_\theta^\eta(s) = F_\theta^\eta(v_\theta(s; \omega^\eta))$ for all θ, s, it follows that $\kappa(\omega^\eta; \sigma^\eta) = \bar{\kappa}^\eta(\omega^\eta) = \rho$ for all sufficiently small η and, by Theorem 2, that σ^η is a limit equilibrium and ω^η its corresponding cutoff for all sufficiently small η. Finally, it remains to establish that $\sigma^\eta \to \sigma^*$. Consider a type and signal such that $c_\theta(s) < \omega^*$, so that $v_\theta(s; \omega^*) > 0$. By continuity of $v_\theta(s; \cdot)$ and the fact that $\omega^\eta \to \omega^*$, it follows that $v_\theta(s; \omega^\eta) > 0$ for all sufficiently small η and, therefore, because $\{\boldsymbol{F}^\eta\}_\eta$ is vanishing, it also follows that $\lim_{\eta \to 0} \sigma_\theta^\eta(s) = 1 = \sigma_\theta^*(s)$, where the last equality follows since σ^* is optimal given ω^*–see equation (10). A similar argument establishes that $\lim_{\eta \to 0} \sigma_\theta^\eta(s) = 0 = \sigma_\theta^*(s)$

for types and signals such that $c_\theta(s) > \omega^*$. Therefore, if $\{s : c_\theta(s) = \omega^*\} = \emptyset$ for all θ, we have shown that, for *any* family of vanishing perturbations, there exists a sequence of limit equilibria that converge to a voting equilibria. In the case where $\{s : c_\theta(s) = \omega^*\} \neq \emptyset$ for some θ, we construct a specific family of perturbations $\left\{\hat{\mathbf{F}}^\eta\right\}_\eta$ with the property that $\lim_{\eta\to 0} \hat{F}^\eta_\theta(v_\theta(s;\omega^\eta)) = \sigma_\theta(s)$ for all (θ, s) such that $c_\theta(s) = \omega^*$. The details that show existence of such a family are tedious but straightforward and are as follows.First, observe that $\omega^\eta \to \omega^*$ and thus, by continuity of $v_{\theta'}(s_{\theta'};\cdot)$, it follows that $v_{\theta'}(s_{\theta'};\omega^\eta) \to 0$. Since there are a finite number of such $(\theta', s_{\theta'})$, there exists a sequence $(r^\eta)_\eta$ such that $r^\eta \to 0$ and $|v_{\theta'}(s_{\theta'};\omega^\eta)| \leq r^\eta$ uniformly over $(\theta', s_{\theta'})$. Second, for each $\theta' \in \Theta$, let $\hat{F}^\eta_{\theta'}(0) = \sigma_{\theta'}(s_{\theta'})$ and for all $t \in [-r^\eta, r^\eta]$, $\hat{F}^\eta_{\theta'}(t) = \sigma_{\theta'}(s_{\theta'}) + t$ if $\sigma_{\theta'}(s_{\theta'}) + t \in (0, 1)$ and either 0 or 1 if $\sigma_{\theta'}(s_{\theta'}) + t \leq 0$ or $\sigma_{\theta'}(s_{\theta'}) + t \geq 1$, respectively. For any $t \notin [-r^\eta, r^\eta]$ $\hat{F}^\eta_{\theta'}(t)$ can be chosen arbitrarily, provided it conforms with the properties of a cdf and $\hat{F}^\eta_{\theta'}(t) \to 0$ and $\hat{F}^\eta_{\theta'}(t) \to 1$ as $\eta \to 0$ for $t < -r^n$ and $t > r^n$, respectively. Note that since $r^\eta \to 0$, for each fixed $t \neq 0$, $\lim_{\eta\to 0} \hat{F}^\eta_{\theta'}(t) = 1\{t > 0\}$, and thus $\left\{\hat{\mathbf{F}}^\eta\right\}_\eta$ is vanishing.

Also note that, if $\sigma_{\theta'}(s_{\theta'}) \in (0, 1)$, $\hat{F}^\eta_{\theta'}(v_{\theta'}(s;\omega^\eta)) = \sigma_{\theta'}(s_{\theta'}) + v_{\theta'}(s;\omega^\eta)$ for sufficiently small η and thus converges to $\sigma_{\theta'}(s_{\theta'})$. If $\sigma_{\theta'}(s_{\theta'}) = 1$, then $1 \geq \hat{F}^\eta_{\theta'}(v_{\theta'}(s;\omega^\eta)) \geq 1 + v_{\theta'}(s;\omega^\eta)$ and it also converges to $\sigma_{\theta'}(s_{\theta'}) = 1$. If $\sigma_{\theta'}(s_{\theta'}) = 0$ a similar result applies.

Case 2 ($\kappa(\omega;\sigma^*) = \rho$ for all ω): Without loss of generality, suppose that $\mathbb{S}_\theta \subset (0,\infty)$ for all θ. Let $\mathcal{T}_L = \{(\theta, s) : v_\theta(s;\omega^*) < 0 \,\text{or}\, (v_\theta(s;\omega^*) = 0 \,\&\, \sigma^*_\theta(s) = 0)\}$, $\mathcal{T}_R = \{(\theta, s) : v_\theta(s;\omega^*) > 0 \,\text{or}\, (v_\theta(s;\omega^*) = 0 \,\&\, \sigma^*_\theta(s) = 1)\}$, and $\mathcal{T}_0 = \{(\theta, s) : v_\theta(s;\omega^*) = 0 \,\&\, \sigma^*_\theta(s) \in (0, 1)\}$. Note that, since (σ^*, ω^*) is a voting equilibrium, then $\sigma^*_\theta(s) = 0$ if $(\theta, s) \in \mathcal{T}_L$ and $\sigma^*_\theta(s) = 1$ if $(\theta, s) \in \mathcal{T}_R$. Define $X_L \equiv \sum_{(\theta,s)\in\mathcal{T}_L} \phi(\theta)q(s \mid \omega^*)s \geq 0$, $X_R \equiv \sum_{(\theta,s)\in\mathcal{T}_R} \phi(\theta)q(s \mid \omega^*)\frac{1}{s} \geq 0$, and $X_0 \equiv \sum_{(\theta,s)\in\mathcal{T}_0} \phi(\theta)q(s \mid \omega^*) \geq 0$. The proof constructs a specific family of perturbations. For all η and all $\theta \in \Theta$ and $s \in \mathbb{S}_\theta$ and for any $(\zeta_L, \zeta_0, \zeta_R)$ let

$$F^\eta_\theta\left(v_\theta\left(s;\omega^*\right)\right) = \begin{cases} \zeta_L s\eta & \text{if}\, v_\theta(s;\omega^*) < 0 \,\text{or}\, (v_\theta(s;\omega^*) = 0 \,\&\, \sigma^*_\theta(s) = 0) \\ \sigma^*_\theta(s) + \zeta_0\eta & \text{if}\, \{v_\theta(s;\omega^*) = 0 \,\&\, \sigma^*_\theta(s) \in (0, 1)\} \\ 1 - \frac{\zeta_R}{s}\eta & \text{if}\, v_\theta(s;\omega^*) > 0 \,\text{or}\, (v_\theta(s;\omega^*) = 0 \,\&\, \sigma^*_\theta(s) = 1) \end{cases}$$

By construction, for all $\zeta_j \in (0,\infty)$,$j = R, L$ and $\zeta_0 \in [0,\infty)$, and for all η sufficiently low, there exists a vanishing family $\{\mathbf{F}^\eta\}_\eta$ that satisfies the above restrictions; note that, by MLRP, for each θ there is at most one signal that satisfies $v_\theta(s;\omega^*) = 0$. Then, since $\omega^* \in (-1,1)$,

$$\begin{aligned}\bar{\kappa}^\eta(\omega^*) - \rho = \bar{\kappa}^\eta(\omega^*) - \kappa(\omega^*;\sigma^*) &= \sum_{(\theta,s)} \phi(\theta) q(s \mid \omega^*) \left(F_\theta^\eta\left(v_\theta\left(s;\omega^*\right)\right) - \sigma_\theta^*(s)\right) \\ &= \eta\left(-\zeta_R X_R + \zeta_L X_L + \zeta_0 X_0\right).\end{aligned}$$

It is straightforward to check that we can always pick $\zeta_R, \zeta_L, \zeta_0$ such that $-\zeta_R X_R + \zeta_L X_L + \zeta_0 X_0 = 0$ and, therefore, $\bar{\kappa}^\eta(\omega^*) = \rho$ for all η sufficiently small. As in Case 1, by letting $\sigma_\theta^\eta(s) = F_\theta^\eta(v_\theta(s;\omega^\eta))$ for all (θ,s), it follows that σ^η is a limit equilibrium and ω^* its corresponding cutoff for all sufficiently small η. The proof is completed by noting that, by construction, $\lim_{\eta\to 0} \sigma^\eta = \sigma^*$. $\square$

1.2.4 Supplementary lemma: increasing strategies

Lemma OA. There exists $\bar{\varepsilon}$ such that for all $\varepsilon < \bar{\varepsilon}$: If σ is a limit ε-equilibrium, then it is increasing.

Proof: We use the following notation. Let $x_i \in \{R, L\}$ denote the vote of player i, let $\kappa_i^n(\omega;\xi) \equiv P^n\left(x_i = R \mid \omega\right)$ be the probability that player $i = 1, ..., n$ votes for R conditional on the state being ω, and let $\kappa^n(\omega;\xi) \equiv \frac{1}{n}\sum_{i=1}^n \kappa_i^n(\omega;\xi)$ be the average over all players.

Throughout the proof let Ξ' be the set in Definition 6 and fix $\xi \in \Xi'$ and a strategy mapping $\overline{\boldsymbol{\alpha}}$ such that 1.-3. in Definition 6 are satisfied. We drop ξ and $\overline{\alpha}$ from the notation, let $P^n \equiv P^n\left(\overline{\boldsymbol{\alpha}}(\xi)\right)$ and, for each strategy α_i^n, let $P_{\alpha_i}^n \equiv P^n\left(\alpha_i^n, \overline{\alpha}_{-i}^n(\xi)\right)$. The proof relies on the following claims; the proofs of the first three claims appear at the end of this section.

Claim OA.1: For all $\delta > 0$ and $\omega \in \Omega$, there exits $n_{\delta,\omega}$ such that for all $n \geq n_{\delta,\omega}$,

$\left|P_{\alpha_i}^n\left(o = R \mid \omega, s_i\right) - P_{\alpha_i'}^n\left(o = R \mid \omega, s_i'\right)\right| < \delta$ uniformly over $i, s_i, s_i', \alpha_i^n, \alpha_i'^n$.

Claim OA.2: For all $\delta > 0$ there exist n_δ such that for all $n \geq n_\delta$, $|\Delta_i(P^n, s_i) - \Delta_i(P_{\alpha_i}^n, s_i)| < \delta$ uniformly over i, s_i, α_i^n.

Claim OA.3: There exists $c > 0$ and n_c such that for all $n \geq n_c$, $\Delta_i\left(P_{\alpha_i}^n, s_i'\right) - \Delta_i\left(P_{\alpha_i}^n, s_i\right) \geq c$ for all i and $s_i' > s_i$ such that $\alpha_i^n(s_i') = \alpha_i^n(s_i)$.

Claim OA: There exists $c' > 0$ and $n_{c'}$ such that for all $n \geq n_{c'}$, $\Delta_i(P^n, s_i') - \Delta_i(P^n, s_i) \geq c'$ for all i and $s_i' > s_i$.

Proof of Claim OA. Fix any α_i^n such that $\alpha_i^n(s_i') = \alpha_i^n(s_i)$. By Claims OA.2 and OA.3, for all $n \geq \max\{n_c, n_\delta\}$

$$\begin{aligned}\Delta_i(P^n, s_i') - \Delta_i(P^n, s_i) &\geq \left(\Delta_i\left(P^n_{\alpha_i}, s_i'\right) - \delta\right) - \left(\Delta_i\left(P^n_{\alpha_i}, s_i\right) + \delta\right) \\ &\geq c - 2\delta.\end{aligned}$$

The claim follows by setting $\delta = c/4$ and $c' = c/2 > 0$. □

Proof of Lemma OA. The definition of ε-equilibrium implies that for all i, $s_i' > s_i$, $n \geq n_\varepsilon$,

$$\begin{aligned}\overline{\alpha}_i^n(s_i') - \overline{\alpha}_i^n(s_i) \geq\ & F_{\theta_i}(\Delta_i(P^n, s_i')) - F_{\theta_i}(\Delta_i(P^n, s_i)) - 2\varepsilon. \\ & + F_{\theta_i}(\Delta_i(P^n, s_i) + c') - F_{\theta_i}(\Delta_i(P^n, s_i) + c'),\end{aligned} \tag{22}$$

where we have added and subtracted the same term to the RHS. Let $c' > 0$ be as defined in Claim OA. Since F_{θ_i} is absolutely continuous, then

$$F_{\theta_i}(\Delta_i(P^n, s_i) + c') - F_{\theta_i}(\Delta_i(P^n, s_i)) = \int_{\Delta_i(P^n, s_i)}^{\Delta_i(P^n, s_i) + c'} f_{\theta_i}(t)\, dt \geq c'' > 0,$$

where the inequality follows from A5 and the fact that $c' > 0$. Hence, the sum of the second and fourth terms in the RHS of (22) is at least $c'' > 0$. By Claim OA, the sum of the first and last terms in the RHS of (22) is positive. Therefore, for all $i, s_i' > s_i$, $n \geq n_\varepsilon$,

$$\overline{\alpha}_i^n(s_i') - \overline{\alpha}_i^n(s_i) \geq c'' - 2\varepsilon > 0.$$

Since $\sigma_\theta^n(\xi, \alpha)$ are averages of the strategies, then for all $\theta, s' > s$, and $n \geq n_\varepsilon$, it follows that $\sigma_\theta^n(s') - \sigma_\theta^n(s) \geq c'' - 2\varepsilon$. Since $\lim_{n\to\infty} \|\sigma^n - \sigma\| = 0$, then it follows that $\sigma_\theta(s') - \sigma_\theta(s) \geq c'' - 2\varepsilon > 0$, thus establishing that limit ε-equilibrium are increasing as long as $0 < \varepsilon < \overline{\varepsilon} \equiv c''/2 > 0$. □

Proof of Claim OA.1. The proof is divided into 3 steps.

Step 1. We first show that the probability of being pivotal goes to zero; i.e., for all $\omega \in \Omega$, for all i, $\lim_{n\to\infty} Piv^n_{\omega,i} = 0$, where

$$Piv^n_{\omega,i} \equiv P^n_1\left(o = R \mid \omega\right) - P^n_0\left(o = R \mid \omega\right),$$

where the "1" and "0" are understood as vectors of the same dimension as α_i. The sub-index "i" indicates that agent i is the one being pivotal.

By simple algebra,

$$Piv^n_{\omega,i} = P^n\left(\sqrt{n}K^n_\omega + \frac{\kappa^n_{i\omega} - 1}{V^n_\omega\sqrt{n}} + \frac{Z^n_{i\omega}}{\sqrt{n}} \leq \frac{\sum_{j=1}^n Z^n_{j\omega}}{\sqrt{n}} < \sqrt{n}K^n_\omega + \frac{\kappa^n_{i\omega}}{V^n_\omega\sqrt{n}} + \frac{Z^n_{i\omega}}{\sqrt{n}} \mid \omega\right),$$

where $Z^n_{j\omega} \equiv \frac{\{1\{x^n_j = R\} - \kappa^n_{j\omega}\}}{V^n_\omega}$, $V^n_\omega \equiv \sqrt{\frac{1}{n}\sum_{j=1}^n \kappa^n_{j,\omega}\left(1 - \kappa^n_{j,\omega}\right)}$, and $K^n_\omega \equiv \frac{\rho - \kappa^n_\omega}{V^n_\omega}$. Note that, for a given n, $\{Z^n_{j\omega}\}_j$ are independent, they have zero mean and unit variance. Moreover, by Step 3 below, $\liminf_{n\to\infty} V^n_\omega > 0$, so that

$$\sum_{j=1}^n E\left[\left|\frac{Z^n_{j\omega}}{\sqrt{n}}\right|^3\right] \leq \frac{2}{\sqrt{n}\left(V^n_\omega\right)^3} \to 0 \; as \; n \to \infty,$$

Hence by Lindeberg-Feller CLT, it follows that, given ω, $\sum_{j=1}^n \frac{Z^n_{j\omega}}{\sqrt{n}} \Rightarrow N(0,1)$ as $n \to \infty$.

Note also that, $\frac{Z^n_{i\omega}}{\sqrt{n}} \to 0$ a.s. as $n \to \infty$ and this limit is uniform on i.

We divide the remainder of the proof in 3 cases: (a) $\sqrt{n}K^n_\omega \to -\infty$, (b) $\sqrt{n}K^n_\omega \to K \in (-\infty, \infty)$ or (c) $\sqrt{n}K^n_\omega \to \infty$ (if necessary, we take a subsequence that converges, which exists since $(V^n_\omega(\xi))_n$ and $(\kappa^n_\omega(\xi))_n$ are uniformly bounded).

We first explore case (a) (case (c) is symmetrical). Note that, since $\liminf_{n\to\infty} V^n_\omega > 0$, then $\frac{\kappa^n_{i\omega}}{V^n_\omega\sqrt{n}} \to 0$. Therefore, $\sqrt{n}K^n_\omega + \frac{\kappa^n_{i\omega}}{V^n_\omega\sqrt{n}} + \frac{Z^n_{i\omega}}{\sqrt{n}} \to -\infty$, (and this limit holds uniformly for $i = 1, ..., n$) so that we can take $n \geq n_{M,\epsilon}$ such that $\sqrt{n}K^n_\omega + \frac{\kappa^n_{i\omega}}{V^n_\omega\sqrt{n}} + \frac{Z^n_{i\omega}}{\sqrt{n}} \leq -M$, where $\mathcal{L}_N(-M) < 0.5\epsilon$ (where $\mathcal{L}_N$ is the standard Gaussian cdf) for any ϵ. Therefore, for all $\epsilon > 0$ there exists $n_{\epsilon,\omega}$ such that for all $n \geq \max\{n_{\epsilon,\omega}, n_{M,\epsilon}\}$:

$$Piv^n_{\omega,i} \leq P^n\left(\frac{\sum_{j=1}^n Z^n_{j,\omega}}{\sqrt{n}} < -M \mid \omega\right) \leq 0.5\epsilon + \mathcal{L}_N(-M) < \epsilon$$

uniformly over $i = 1, ..., n$, where the first inequality follows from the fact that $n \geq n_{M,\epsilon}$ and the second follows from CLT and our choice of M.

For case (b) (i.e., K finite). Let $\delta > 0$ be such that $\mathcal{L}_N(K+\delta) - \mathcal{L}_N(K-\delta) < 0.5\epsilon$. Note that since $\lim_{n\to\infty}(V^n_\omega \sqrt{n})^{-1} = 0$, there exists a $n_{\delta,\omega}$ such that $(V^n_\omega \sqrt{n})^{-1} < 0.5\delta$ for all $n \geq n_{\delta,\omega}$; also since $\frac{Z^n_{i\omega}}{\sqrt{n}} \to 0$ a.s. as $n \to \infty$, we can take $n_{\delta,\omega}$ such that $\left|\frac{Z^n_{i\omega}}{\sqrt{n}}\right| < 0.5\delta$ (note that $n_{\delta,\omega}$ does not depend on i since convergence is uniform on i). Then, it follows for all $\epsilon > 0$, there exists $n_{\epsilon,\omega}$ such that for all $n \geq \max\{n_{\epsilon,\omega}, n_{\delta,\epsilon}\}$:

$$\begin{aligned} Piv^n_{\omega,i} \leq & P^n\left(\sqrt{n}K^n_\omega - \frac{1}{V^n_\omega\sqrt{n}} + \frac{Z^n_{i\omega}}{\sqrt{n}} \leq \frac{\sum_{j=1}^n Z^n_{j\omega}}{\sqrt{n}} < \sqrt{n}K^n_\omega + \frac{1}{V^n_\omega\sqrt{n}} + \frac{Z^n_{i\omega}}{\sqrt{n}} \mid \omega\right) \\ \leq & P^n\left(K - \delta < \frac{\sum_{j=1}^n Z^n_{j\omega}}{\sqrt{n}} \leq K + \delta \mid \omega\right) \\ \leq & 0.5\epsilon + \mathcal{L}_N(K+\delta) - \mathcal{L}_N(K-\delta) < \epsilon, \end{aligned}$$

where the third inequality follows from the CLT. We showed that for any convergent subsequence $(K^n_\omega)_n$, the associated subsequences of probabilities converge to zero, thus this result must hold for the whole sequence.

Step 2. Note that:

$$\begin{aligned} P^n_{\alpha_i}\left(o = R \mid \omega, s_i\right) = & \alpha^n_i(s_i) P^n_1\left(o = R \mid \omega\right) + (1 - \alpha^n_i(s_i)) P^n_0\left(o = R \mid \omega\right) \\ = & P^n_0\left(o = R \mid \omega\right) \\ & + \alpha^n_i(s_i)\left(P^n_1\left(o = R \mid \omega\right) - P^n_0\left(o = R \mid \omega\right)\right) \\ \equiv & P^n\left(o = R \mid \omega\right) + \alpha^n_i(s_i) Piv^n_{\omega,i} \end{aligned}$$

Therefore

$$|P^n_{\alpha_i}\left(o = R \mid \omega, s_i\right) - P^n_{\alpha'_i}\left(o = R \mid \omega, s'_i\right)| \leq |\alpha^n_i(s_i) - \alpha'^n_i(s_i)| \cdot |Piv^n_{\omega,i}|.$$

By step 1, it follows that for all $n \geq n_{\delta,\omega}$: $|Piv^n_{\omega,i}| \leq \delta$. Since $|\alpha^n_i(s_i) - \alpha'^n_i(s_i)| \leq 1$ the desired result follows.

Step 3. We now show that for all $\omega \in \Omega$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \kappa^n_{j\omega}\left(1 - \kappa^n_{j\omega}\right) > 0. \tag{23}$$

Fix any n and $j \leq n$. By assumption, $\alpha_j^n(s_j) \in [F_j(-2K), F_j(2K)] \subset (0,1)$ for all s_j. Therefore, $0 < \kappa_{j\omega}^n < 1$ for all ω, thus implying equation (23). $\square$

Proof of Claim OA.2. We prove that

$$\lim_{n\to\infty}\Big(E_{P^n}\left(u_{\theta_i}(R,W) \mid o=R, S=s_i\right) - E_{P^n_{\alpha_i}}\left(u_{\theta_i}(R,W) \mid o=R, S=s_i\right)\Big) = 0;$$

the proof for $o = L$ is similar and therefore omitted. We first show that, for all i, s_i, α_i,

$$E_{P^n_{\alpha_i}}\left(u_{\theta_i}(R,W) \mid o=R, S=s_i\right) = \frac{\int_\Omega P^n_{\alpha_i}\left(o=R \mid W, s_i\right) q_{\theta_i}(s_i \mid W) u_{\theta_i}(R,W) G(dW)}{\int_\Omega P^n_{\alpha_i}\left(o=R \mid W, s_i\right) q_{\theta_i}(s_i \mid W) G(dW)}$$

is well-defined for sufficiently large n. Fix any i. A3(ii) and the fact that $\overline{\boldsymbol{\alpha}}$ is asymptotically interior imply that there exists $\overline{n}$ such that for all $n \geq \overline{n}$, there exists s_i^* such that

$$P^n(o=R, s_i^*) = \int_\Omega P^n(o=R \mid W, s_i^*) q_{\theta_i}\left(s_i^* \mid W\right) G(dW) \geq c > 0,$$

which implies that $\int_\Omega P^n(o=R \mid W, s_i^*) G(dW) \geq c > 0$. By Claim OA.1, for each s_i, α_i^n, $P^n\left(o=R \mid \omega, s_i^*\right) - P^n_{\alpha_i}\left(o=R \mid \omega, s_i\right)$ converges to zero as $n \to \infty$. Since both probabilities are bounded by one, then the dominated convergence theorem implies that $\int_\Omega \big(P^n\left(o=R \mid W, s_i^*\right) - P^n_{\alpha_i}\left(o=R \mid W, s_i\right)\big) G(dW) \to 0$ as $n \to \infty$, uniformly over α_i. Therefore, there exists $n_{.5c}$ such that $\sup_{\alpha_i} \left| \int_\Omega [P^n\left(o=R \mid W, s_i^*\right) - P^n_{\alpha_i}\left(o=R \mid W, s_i\right)] G(dW)\right| < .5c$ for all $n \geq n_{.5c}$. So for all $n \geq \max \bar{n}, n_{.5c} \equiv \overline{n}_c$,

$$\int_\Omega P^n_{\alpha_i}\left(o=R \mid W, s_i\right) q_{\theta_i}\left(s_i \mid W\right) G(dW) \geq d \int_\Omega P^n_{\alpha_i}\left(o=R \mid W, s_i\right) G(dW) > .5dc > 0.$$

Hence, $E_{P^n_{\alpha_i}}\left(u_{\theta_i}(R,W) \mid o=R, S=s_i\right)$ is well defined.

By simple algebra, and letting $\Delta P^n_{\alpha_i}(R,\omega,s_i) \equiv P^n\left(o=R \mid \omega, s_i\right) - P^n_{\alpha_i}\left(o=R \mid \omega, s_i\right)$,

$$\begin{aligned}
&\left| E_{P^n}\left(u_{\theta_i}(R,W) \mid o=R, S=s_i\right) - E_{P^n_{\alpha_i}}\left(u_{\theta_i}(R,W) \mid o=R, S=s_i\right)\right| \\
&\leq \frac{\left|\int_\Omega \Delta P^n_{\alpha_i}(R,W,s_i) q_{\theta_i}(s_i \mid W) u_{\theta_i}(R,W)\, G(dW)\right|}{\int_\Omega P^n_{\alpha_i}\left(o=R \mid W\right) q_{\theta_i}(s_i \mid W) G(dW)} \\
&+ \frac{\left|\int_\Omega \Delta P^n_{\alpha_i}(R,W,s_i) q_{\theta_i}(s_i \mid W) G(dW)\right| \int_\Omega P^n\left(o=R \mid W\right) q_{\theta_i}(s_i \mid W) u_{\theta_i}(R,W)\, G(dW)}{\int_\Omega P^n\left(o=R \mid W\right) q_{\theta_i}(s_i \mid W) G(dW) \int_\Omega P^n_{\alpha_i}\left(o=R \mid W\right) q_{\theta_i}(s_i \mid W) G(dW)}
\end{aligned}$$

To establish the desired result, it is sufficient to show that each of the two absolute value terms in the numerator of the second and third line converge to zero as $n \to \infty$. However, this result follows by the dominated convergence theorem since $|u_{\theta_i}(R,\omega)| < K$, $q_{\theta_i}(s|\omega) \leq 1$, and pointwise convergence (for each ω) is obtained by Claim OA.1. □

Proof of Claim OA.3. For each $O \in \{R, L\}$: Let $G^n_{\alpha_i}(\omega \mid O, s_i) \equiv P^n_{\alpha_i}(\{W \leq \omega\} \mid o = O, s_i)$ denote the cdf of ω conditional on $o = O$ and s_i, and let $g^n_{\alpha_i}(\omega \mid O, s_i) \equiv P^n_{\alpha_i}(d\omega \mid o = O, s_i)$ denote the density. Let $\Delta g^n_{\alpha_i}(\omega \mid O, s'_i, s_i) \equiv g^n_{\alpha_i}(\omega \mid O, s'_i) - g^n_{\alpha_i}(\omega \mid O, s_i)$ and $\Delta G^n_{\alpha_i}(\omega \mid O, s'_i, s_i) \equiv G^n_{\alpha_i}(\omega \mid O, s'_i) - G^n_{\alpha_i}(\omega \mid O, s_i)$.

Then

$$\begin{aligned}
\Delta_i\left(P^n_{\alpha_i}, s'_i\right) - \Delta_i\left(P^n_{\alpha_i}, s_i\right) &= \int_\Omega \left(u_{\theta_i}(R,W)\Delta g^n_{\alpha_i}(W \mid R, s'_i, s_i) - u_{\theta_i}(L,W)\Delta g^n_{\alpha_i}(W \mid L, {s'}_i, s_i)\right) dW \\
&= \int_\Omega \left(\frac{du_{\theta_i}}{d\omega}(R,W)\Delta G^n_{\alpha_i}(W \mid R, s_i, s'_i) - \frac{du_{\theta_i}}{d\omega}(L,W)\Delta G^n_{\alpha_i}(W \mid L, s_i, s'_i)\right) dW \\
&\geq \int_{\Omega^n \subset \Omega} \frac{du_{\theta_i}}{d\omega}(R,W)\Delta G^n_{\alpha_i}(W \mid R, s_i, s'_i) dW \\
&\geq c_M \int_{\Omega^n \subset \Omega} \frac{du_{\theta_i}}{d\omega}(R,W) dW \\
&\geq c_m \cdot c_M \inf_{W \in \bar{\Omega}} \frac{du_{\theta_i}}{d\omega}(R,W) \\
&\equiv c > 0
\end{aligned}$$

for all $n \geq n'$ (where Ω^n, $c_m \cdot c_M > 0$, and n' are all defined in Claim OA.3.1 below), where the first line follows by definition, the second by integration by parts (note how the signals are inverted), the third by Claim OA.3.1(i) (see below) and the facts that that $\frac{du_{\theta_i}}{d\omega}(R,\omega) > 0$ and $\frac{du_{\theta_i}}{d\omega}(L,\omega) < 0$ for all ω, the fourth by Claim OA.3.1(ii).

Finally, for the fifth line, let $\bar{\Omega} = \Omega \setminus \cup_{i=1}^{N}(\omega_i - \epsilon, \omega_i + \epsilon)$ where $(\omega_1, ..., \omega_N)$ are the discontinuity points of $\frac{du_{\theta_i}}{d\omega}(R, \cdot)$; by assumption there are finitely many, so $N < \infty$ and $\epsilon > 0$ is chosen such that $\epsilon < \min_{i \neq j} |\omega_i - \omega_j|$. It is easy to see that $\bar{\Omega}$ is compact and over it, $\frac{du_{\theta_i}}{d\omega}(R, \cdot)$ is well-defined and continuous. Since $c_m \cdot c_M > 0$ and $\inf_{\omega \in \bar{\Omega}} \frac{du_{\theta_i}}{d\omega}(R, \omega) = \min_{\omega \in \bar{\Omega}} \frac{du_{\theta_i}}{d\omega}(R, \omega) > 0$ where (because u_{θ_i} is continuously differentiable in $\bar{\Omega}$ and $\frac{du_{\theta_i}}{d\omega}(R, \omega) > 0$ for all ω). □

Claim OA.3.1: For all i and $s_i' > s_i$ such that $\alpha_i^n(s_i) = \alpha_i^n(s_i')$: (i) For all n, $\Delta G_{\alpha_i}^n(\omega \mid O, s_i, s_i') \geq 0$ for all ω and $O \in \{R, L\}$; (ii) There exists n', $c_M > 0$, and $(\Omega^n)_n$ with $\Omega^n = [l_n, u_n] \subseteq \Omega$ and $\liminf_{n \to \infty} u_n - l_n = \beta_2 > 0$ such that, for all $n \geq n'$ and all $\tilde{\omega} \in \Omega^n \setminus \{-1, 1\}$,

$$\Delta G_{\alpha_i}^n(\tilde{\omega} \mid R, s_i, s_i') \geq c_M.$$

Proof of Claim OA.3.1. There exists $z > 0$ such that for all n and all $\omega' > \omega$,

$$\begin{aligned}
&g_{\alpha_i}^n(\omega' \mid O, s_i')g_{\alpha_i}^n(\omega \mid O, s_i) - g_{\alpha_i}^n(\omega' \mid O, s_i)g_{\alpha_i}^n(\omega \mid O, s_i') \\
&\quad = \frac{P_{\alpha_i}^n(O \mid \omega', s_i)P_{\alpha_i}^n(O \mid \omega, s_i)g(\omega')g(\omega)}{P_{\alpha_i}^n(O, s_i')P_{\alpha_i}^n(O, s_i)} \left[q_{\theta_i}(s_i' \mid \omega')\, q_{\theta_i}(s_i \mid \omega) - q_{\theta_i}(s_i \mid \omega')\, q_{\theta_i}(s_i' \mid \omega)\right] \\
&\quad \geq z \frac{P_{\alpha_i}^n(O \mid \omega', s_i)P_{\alpha_i}^n(O \mid \omega, s_i)g(\omega')g(\omega)q_{\theta_i}(s_i' \mid \omega)\, q_{\theta_i}(s_i \mid \omega)\,(\omega' - \omega)}{P_{\alpha_i}^n(O, s_i')P_{\alpha_i}^n(O, s_i)} \\
&\quad \geq 0
\end{aligned} \tag{24}$$

where the first line uses the fact that $P_{\alpha_i}^n(O \mid \hat{\omega}, s_i) = P_{\alpha_i}^n(O \mid \hat{\omega}, s_i')$ for all $\hat{\omega}$ (because of conditional independence and the fact that $\alpha_i^n(s_i) = \alpha_i^n(s_i')$), the second line follows from A6, and the third line follows because $z > 0$ and $\omega' > \omega$. Therefore, it follows from Milgrom (1981, Proposition 1) that, for all n, $\Delta G_{\alpha_i}^n(\omega \mid O, s_i, s_i') \geq 0$ for all ω.

(ii) From the proof of Claim OA.2, there exists n' and $c' > 0$ such that, for all $n \geq n'$,

$$\int_{\Omega} P_{\alpha_i}^n(o = R \mid W, s_i)G(dW) \geq c'$$

for all i, α_i, s_i. For $a \in (0, 1)$, let

$$\omega_a^n = min\left\{\omega' : \int_{W \leq \omega'} P_{\alpha_i}^n(o = R \mid W, s_i)G(dW) \geq a \cdot c'\right\} \in \Omega.$$

Fix any $n \geq n'$. Then

$$c'/4 = \int_{\omega^n_{0.25} \leq W \leq \omega^n_{0.50}} P^n_{\alpha_i}(o = R \mid W, s_i) G(dW) \leq G\left(\omega^n_{0.50}\right) - G\left(\omega^n_{0.25}\right).$$

Therefore, the fact that G has no mass points implies that there exists $c_L > 0$ such that $\omega^n_{0.50} - \omega^n_{0.25} \geq c_L$. A similar argument establishes that here exists $c_R > 0$ such that $\omega^n_{0.75} - \omega^n_{0.50} \geq c_R$.

Let $\Omega^n = [\omega^n_{0.50} - c_m/2, \omega^n_{0.50} + c_m/2]$, where $c_m \equiv min\{c_L, c_R\} > 0$. Then, $u_n - l_n = c_m > 0$. In addition, fix any $\tilde{\omega} \in \Omega^n$. Then, by construction,

$$\int_{\omega < \tilde{\omega} - c_m/2} P^n_{\alpha_i}(o = R \mid W, s_i) G(dW) \geq c'/4 \tag{25}$$

and

$$\int_{\omega > \tilde{\omega} + c_m/2} P^n_{\alpha_i}(o = R \mid W, s_i) G(dW) \geq c'/4. \tag{26}$$

By integrating each side of (24) twice, first with respect to $G(d\omega)$ over $\omega \leq \tilde{\omega}$ and second with respect to $G(d\omega')$ over $\omega' > \tilde{\omega}$, we obtain

$$\begin{aligned}
&\Delta G^n_{\alpha_i}(\tilde{\omega} \mid R, s_i, s'_i) = \\
&= \frac{z}{P^n_{\alpha_i}(R, s'_i) P^n_{\alpha_i}(R, s_i)} \int_{W' > \tilde{\omega}} \int_{W < \tilde{\omega}} P^n_{\alpha_i}(R \mid \omega', s_i) P^n_{\alpha_i}(R \mid W, s_i) g(W') g(W) q_{\theta_i}\left(s'_i \mid W\right) q_{\theta_i}\left(s_i \mid W\right)(W' - W) dG(W) dG(W') \\
&\geq z \int_{W' > \tilde{\omega} + \frac{c_m}{2}} \int_{W < \tilde{\omega} - \frac{c_m}{2}} P^n_{\alpha_i}(R \mid W', s_i) P^n_{\alpha_i}(R \mid W, s_i) g(W') g(W) q_{\theta_i}\left(s'_i \mid W\right) q_{\theta_i}\left(s_i \mid W\right)(W' - W) dG(W) dG(W') \\
&\geq z \cdot c_m \cdot d^2 \int_{W' > \tilde{\omega} + \frac{c_m}{2}} P^n_{\alpha_i}(R \mid W', s_i) G(dW') \int_{W < \tilde{\omega} - \frac{c_m}{2}} P^n_{\alpha_i}(R \mid W, s_i) G(dW) \\
&\geq z \cdot c_m \cdot d^2 \cdot \left(\frac{c'}{4}\right)^2 \equiv c_M > 0,
\end{aligned}$$

where the first inequality follows from $P^n_{\alpha_i}(R, s'_i) P^n_{\alpha_i}(R, s_i) \leq 1$, the second from A3, and the third from (25) and (26).

□