Economics 202A
Suggested Solutions to Problem Set 3

David Romer/Galina Hale

Spring 1999

1 Romer, 2.5. The productivity slowdown and saving

In this problem we consider a permanent fall in g. Recall our equations of motion in the Ramsey model:

$$\begin{aligned}\dot{c} &= c\,\frac{f'(k)-\rho-\theta g}{\theta}\\ \dot{k} &= f(k)-(n+g)k-c.\end{aligned}$$

(a) The RHS of the $\dot{k}$ will go up as g goes down. Thus for every k, c must go up as well for k to remain constant. Thus the $\dot{k}=0$ locus will shift up (see Figure 1).

(b) The RHS of the $\dot{c}$ equation goes up as g goes down, therefore $f'(k)$ should go down in order to compensate for the change. This means that k should increase for every c. Thus the $\dot{c}=0$ locus shifts to the right (see Figure 1).

(c) At the time of change the capital will remain constant, because it is the stock predetermined by the past behavior. However, because of the unexpected permanent change in g, consumption can jump. Since the change is permanent, we know that eventually economy will converge to a new BGP following the new saddle path. So at the time of change consumption will jump on the new saddle path. However, we can not tell whether the new saddle path is below (point A) or above (point B) the old BGP, therefore we can not tell whether consumption jumps up or down. In fact, if by coincidence new saddle path is going through old BGP (point E), consumption will not jump at all. After (possible) initial jump consumption will increase and so will capital along the new saddle path to the new BGP.

The intuition is that now when technology grows slower, we can maintain higher levels of the variables per unit of effective labor on the BGP.

(d) First we have to determine what is the saving rate (fraction of output that is saved) on the balanced growth path. Recall that on the BGP $f(k^*) - c^* = (n+g)k^*$, $s = \frac{f(k^*)-c^*}{f(k^*)} = \frac{(n+g)k*}{f(k^*)}$. Differentiating both sides with respect to g and taking into account that $k^* = k^*(g)$:

$$\frac{\partial s}{\partial g} = \frac{f(k^*)[k^* + (n+g)\frac{\partial k^*}{\partial g}] - f'(k^*)\frac{\partial k^*}{\partial g}(n+g)k^*}{f(k^*)^2}.$$

We know that k^* is determined by $f'(k^*) = \rho + \theta g$, thus $f''(k^*)\frac{\partial k^*}{\partial g} = \theta$ and $\frac{\partial k^*}{\partial g} = \frac{\theta}{f''(k^*)} \equiv \varphi < 0$. Substituting this into our equation, we get

$$\begin{aligned} \frac{\partial s}{\partial g} &= \frac{f(k^*)k^* + f(k^*)(n+g)\varphi - f'(k^*)\varphi(n+g)k^*}{f(k^*)^2} = \\ &= \frac{k + (n+g)\varphi - \alpha_K(k^*)\varphi(n+g)}{f(k^*)} = \frac{k + (n+g)\varphi(1 - \alpha_K(k^*))}{f(k^*)}. \end{aligned}$$

The denominator of this equation is positive, the first term in the numerator is positive, the second is negative because φ is negative. So the sign of the derivative is ambiguous. This is intuitive, we know that c increased and k increased (and thus y increased) but the size of the change depends on parameters, and therefore we can not sign the difference between y and c.

(e) For the Cobb-Douglas production function $f(k) = k^\alpha$, $f'(k^*) = \alpha k^{*\alpha-1} = \rho + \theta g$. We also know that

$$s = \frac{(n+g)k^*}{k^{*\alpha}} = \frac{(n+g)}{k^{*\alpha-1}} = \alpha\frac{(n+g)}{\rho+\theta g},$$

and therefore
$$\frac{\partial s}{\partial g} = \alpha \frac{\rho + \theta g - \theta n - \theta g}{(\rho + \theta g)^2} = \alpha \frac{\rho - \theta n}{(\rho + \theta g)^2}.$$
Thus the sign of the derivative depends on the sign of $\rho - \theta n$.

We determined that the productivity slowdown has the ambiguous effect on the saving rate, but that it increases the capital and consumption per unit of effective labor on the BGP.

2 Romer, 2.8. Capital taxation

At time $t = 0$ the capital income is taxed at a rate τ: $r(t) = (1 - \tau)f'(k(t))$ and the government returns the money in lump-sum way. This change in the policy is unexpected.

(a) For the time after new policy implementation, the new equation of motion of consumption will describe the economy:

$$\begin{aligned} \dot{c} &= c\,\frac{(1-\tau)f'(k) - \rho - \theta g}{\theta} \\ \dot{k} &= f(k) - (n+g)k - c. \end{aligned}$$

The equation of motion for capital will not be affected since the total income of households is not affected (tax revenues are returned). The change only affects the household intertemporal consumption decision. Thus the $\dot{k} = 0$ locus is not affected. The $\dot{k} = 0$ locus shifts to the left, because the decrease in capital is necessary to compensate for the decrease in after-tax capital revenue and keep it equal to $\rho + \theta g$ (see Figure 2).

(b) (c) The capital will not change at $t = 0$, because it is predetermined. Since the change is unanticipated, consumption will immediately jump up to a new saddle path. After that both consumption and capital will decrease till they reach new BGP levels that are lower then before the change (see Figure 2).

The intuition for this result: since the interest rates are now lower, households want to save less, which means that capital stock will decrease which eventually will decrease output. As a result of the decrease in output consumption also decreases.

(d) Many countries with different tax rates.

(i) We already know that the saving rate is equal to $\frac{(n+g)k^*(\tau)}{f(k^*(\tau))}$. We also know from the phase diagram (and it is not hard to prove formally) that $\frac{\partial k^*}{\partial \tau} \equiv \phi < 0$. Then

$$\begin{aligned}
\frac{\partial s}{\partial \tau} &= (n+g)\,\frac{f(k^*)\frac{\partial k^*}{\partial \tau} - k^* f'(k^*)\frac{\partial k^*}{\partial \tau}}{f(k^*)^2} = \\
&= (n+g)\,\frac{\phi}{f(k^*)} - \frac{\alpha_K(k^*)\phi}{f(k^*)} = (1-\alpha_K(k^*))\frac{\phi}{f(k^*)} < 0,
\end{aligned}$$

since $(1-\alpha_K(k^*)) > 0, \phi < 0, f(k^*) > 0$. Thus we showed that the saving rate on the BGP is decreasing with the tax rate: the higher is the tax rate, the lower is the return on capital, the less we want to save.

(ii) **NO**. The interest rates in all the countries will equalize in the absence of international capital flows. The reason is that the saving behavior will offset the effect of the capital revenues taxation: if the taxes are higher, savings and thus capital stock are lower and therefore the marginal product of capital is higher offsetting the tax effect.

Formally, since in all the countries on the BGP $\dot{c} = 0$ implies that $r = (1-\tau)f'(k^*) = \rho + \theta g$ and assuming that preferences and technologies are the same, the after-tax interest rates must be the same in all countries and therefore investors from high-saving countries do not have an incentive to invest in low-saving country.

(e) **NO**. Any distortion is bad for the economy. The initial consumption path was the one maximizing utility. If now the government introduces the policy which is the reverse of the one we just analyzed, it will not change the budget constraint and the new consumption path (shown on Figure 3 for $k^*_{NEW} < k^{GR}$) can not be better then the initial one.

Although the BGP level of consumption increases, it does not mean that it will increase the lifetime utility: the initial utility loss will outweigh the gain. If $k^*_{NEW} < k^{GR}$, then initial sacrifice is bigger and level of new BGP consumption is lower, so we are even worse off.

The initial consumption path was maximizing the lifetime utility and thus is better then any other path. Therefore there will be no utility gain from the distortionary subsidy that is paid completely by lump-sum taxes thus preserving the present value of the wealth.

(f) Now when the tax revenues are not rebated but spent, we combine the result in (a) and (b) with the effect of government spending. The only differences with the government spending considered in class is that it depends on capital: $G = \tau f'(k)k$, so that $G = 0$ when $k = 0$. Thus, the dynamic system, describing the economy is

$$\begin{aligned}\dot{c} &= c\,\frac{(1-\tau)f'(k) - \rho - \theta g}{\theta} \\ \dot{k} &= f(k) - (n+g)k - c - \tau f'(k)k\end{aligned}$$

and thus the $\dot{c} = 0$ locus shifts to the left and $\dot{k} = 0$ shifts down.

The dynamics of the economy is illustrated on Figure 4. In this case the new BGP level of consumption is lower then in part (b) by the amount of government spending. The direction of the initial jump in consumption is undetermined – it depends on the location of the new saddle path.

3 Romer, 2.10. Temporary changes

(a) At $t = 0$ government announces that $\tau(0 \leq t < t_1) > 0$ and $\tau = 0$ for all other dates. Tax revenues are rebated. This is an unanticipated temporary change in τ. In fact, we can think of it as two changes: first, an unanticipated increase in τ and then the anticipated decrease in τ on date t_1. From the previous problem we know that the increase in τ when rebated, shifts the $\dot{c} = 0$ line to the left leaving the $\dot{k} = 0$ line unaffected.

It is easier to start from the end. On date t_1, nothing can jump because the change is anticipated. So on date t_1 when the $\dot{c} = 0$ shifts back we should already be on the old saddle path (see Figure 5). Between times 0 and t_1, economy obeys the dynamics dictated by the new system (shown with arrows) so that it is on the old saddle path precisely on date t_1. How can we insure that? This is insured by the size of initial jump in c at time $t = 0$. In particular, the longer is the period before t_1, the bigger will be the initial jump in c. Note that c must jump up, otherwise the economy will never cross its old saddle path. Note also that the path can never cross or hit the new saddle path for the same reason[1].

[1] Unless the new saddle path is intersecting the old one which is possible but unlikely event.

Summary paths of c and k are shown on Figure 6.

What is the economics behind all this? Households want to decrease their savings temporary because of the decrease in the after-tax return on capital. They choose a consumption path knowing that soon the tax will be eliminated (described by Euler equation in new situation). Then, to satisfy the after-news budget constraint, they had to make a discrete change in their consumption.

(b) Now both changes are anticipated. At time 0 we know that at t_1 τ will go up and at t_2 it will come back down. The changes in the phase diagram are the same as before. Nothing can jump at t_1 or t_2. Again, the economy will not reach new saddle path because it has to come back to the old one. At t_2 and before it, the dynamics is the same as before t_1 in the previous case. The difference now is that there is no jump at t_1. The jumps can only occur at the time of a news. In this case this is time 0. Importantly, after the upward jump in consumption at time $t = 0$, the dynamics will still obey the old Euler equation (dashed arrows on Figure 7) till the date t_1 and thereafter the dynamics will be the same as in part (a). Summary is on Figure 8. Note that the initial jump is smaller then with unanticipated change because there is time for adjustment. The longer is the time before t_1, the smaller will be the initial jump.