Economics 202A
Suggested Solutions to Problem Set 4

David Romer/Galina Hale

Spring 1999

1 The "exam" question

As in the previous problem set, we have

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{(1-\tau)f'(k) - \rho - \theta g}{\theta} \\ \dot{k} &= f(k) - (n+g)k - c, \end{aligned}$$

except now τ is a function of c. In particular, $\tau(\bar{c}^*) = \bar{\tau}$, so one thing we know for sure is that the economy's new BGP coincides with the old one, i.e. $\dot{c} = 0$ and $\dot{k} = 0$ lines still intersect at the same point.

(a) What happens to the $\dot{c} = 0$ locus? Since τ is determined by total consumption only, each household takes the tax rate as given in their maximization problem. Therefore the only change to the Euler equation is the new tax rate.

$$\frac{\dot{c}}{c} = \frac{(1-\tau(c))f'(k) - \rho - \theta g}{\theta}$$

and the steady state condition is given by

$$(1-\tau(c))f'(k) = \rho + \theta g = \text{constant},$$

from which follows

$$\begin{aligned}(1-\tau(c)) &= \frac{\rho+\theta g}{f'(k)} \\ \tau(c) &= 1-\frac{\rho+\theta g}{f'(k)}.\end{aligned}$$

We also know that τ is monotonically increasing function of consumption and that it is bound between 0 and 1. So, we know

$$\begin{aligned}0 &< 1-\frac{\rho+\theta g}{f'(k)} < 1, \\ -1 &< -\frac{\rho+\theta g}{f'(k)} < 0, \\ 0 &< -\frac{\rho+\theta g}{f'(k)} < 1, \\ \rho &+ \theta g < f'(k) < \infty, \\ 0 &< k < k_0.\end{aligned}$$

The LHS of the last inequality results from Inada conditions. k_0 on the RHS is the BGP level of capital in the absence of any taxation (as we know from last problem set, it is to the right of the $\bar{k}^*$ and to the left of GR). Thus we know that $\dot{c}=0$ line is bound between vertical axis and k_0.

What about the slope? We know from our previous problem set that when the tax rate increases, the BGP level of consumption decreases ($\dot{c}=0$ locus shifts to the left). Since in this case the tax rate is higher the higher the consumption, the $\dot{c}=0$ will be a downward sloping line.

To summarize, $\dot{c}=0$ is a downward sloping line bound between $k=0$ and $k=k_0$ and it goes through $(\bar{c}^*, \bar{k}^*)$ point. The example is on Figure 1.

(b) What happens to the $\dot{k}=0$ locus? Nothing, since as before all the tax revenues are rebated to the consumers, the aggregate budget constraint (the $\dot{k}$ equation) is not affected.

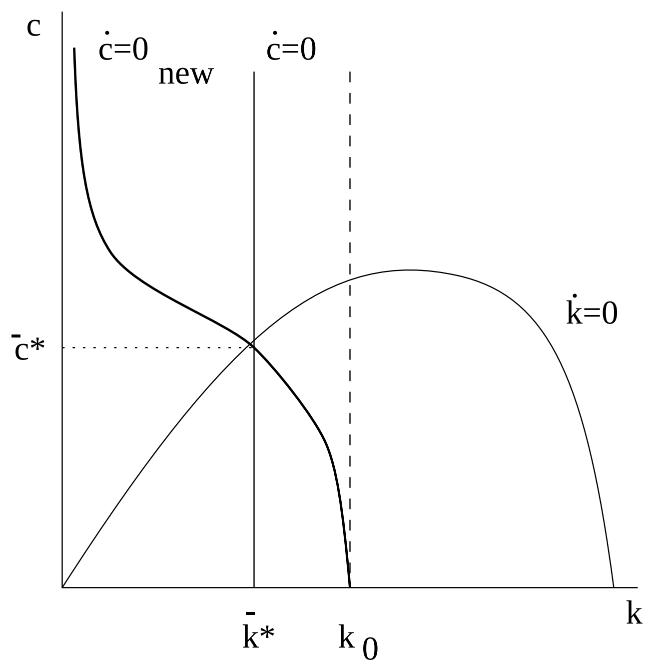

Figure 1: Capital taxation as a function of c

2 Romer, 2.13. Playing with Diamond model

We know from class and the textbook that in the log utility – Cobb-Douglas case

$$k_{t+1} = \frac{1}{(1+n)(1+g)(2+\rho)}(1-\alpha)k_t^{\alpha}.$$

(a) When n increases, k_{t+1} goes down for every k_t, except for $k_t = 0$. Therefore the answer is as shown on Figure 2. Note that the increase of the rate of growth of population leads to the decrease in the BGP level of capital (same result we saw in the Solow model).

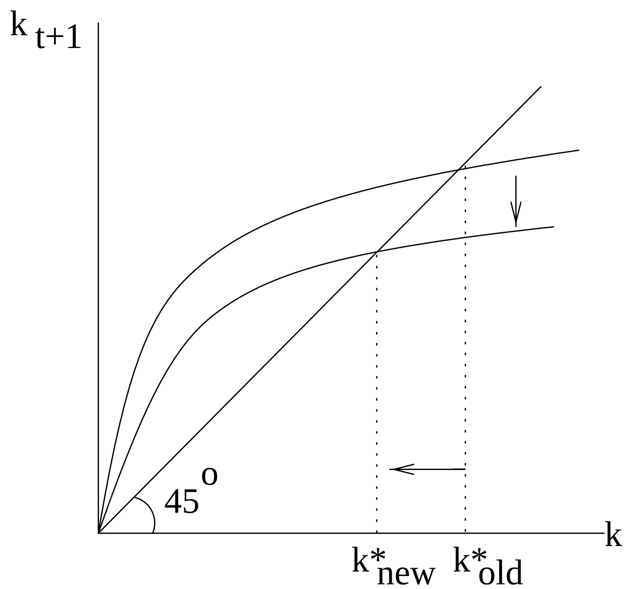

Figure 2: 2.13 a),b); 2.16 a)

(b) We modify the production function in order to be specific about what the "shift" means. Now

$$k_{t+1} = \frac{1}{(1+n)(1+g)(2+\rho)}(1-\alpha)Bk_t^{\alpha}$$

and B falls. This moves the k_{t+1} function down for every k_t in the same manner as an increase in the population growth. So the answer is given by Figure 2 again. Note that the downward shift in the production function decreases the BGP level of capital (same result as in the Solow model again).

(c) The share of capital goes up.

Note that for every function $f(x)$ the following is true

$$\frac{\partial \ln f(x)}{\partial x} = \frac{\partial f(x)}{\partial x f(x)} \Longrightarrow \frac{\partial f(x)}{\partial x} = f(x)\frac{\partial \ln f(x)}{\partial x}.$$

We can apply it now to the $\frac{\partial(k_t^\alpha)}{\partial\alpha}$.

$$\frac{\partial(k_t^\alpha)}{\partial\alpha} = k_t^\alpha \frac{\partial(\alpha \ln k_t)}{\partial\alpha} = k_t^\alpha \ln k_t.$$

Now we can take a derivative of k_{t+1} with respect to α. To simplify notation, denote $\frac{1}{(1+n)(1+g)(2+\rho)} \equiv \varphi$, with $0 < \varphi < 1$. Then

$$\frac{\partial k_{t+1}}{\partial\alpha} = \varphi[-k_t^\alpha + (1-\alpha)k_t^\alpha \ln k_t] = \varphi[k_t^\alpha((1-\alpha)\ln k_t - 1)].$$

Thus for $\ln k_t > \frac{1}{1-\alpha}$, k_{t+1} increases, for $\ln k_t < \frac{1}{1-\alpha}$, k_{t+1} decreases. This means that the curve becomes "less curved" — see Figure 3.

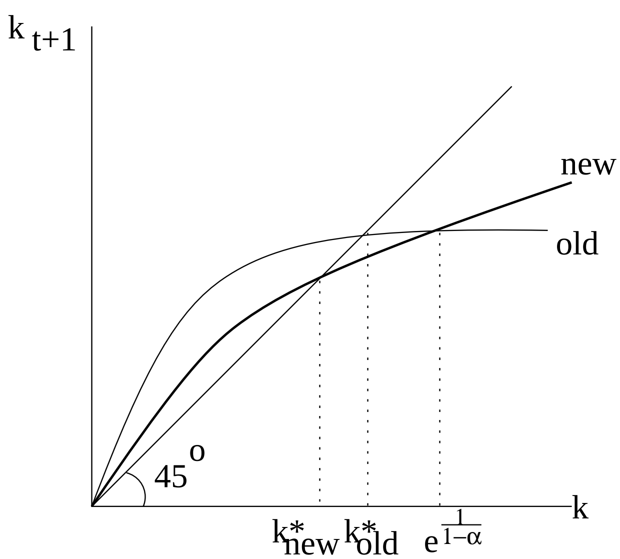

Figure 3: The increase in the capital share

What happens to the BGP level of capital? We know that on the BGP

$$k^* = [\varphi(1-\alpha)]^{\frac{1}{1-\alpha}},$$

therefore

$$\frac{\partial k^*}{\partial\alpha} = \frac{1}{1-\alpha}[\varphi(1-\alpha)]^{\frac{\alpha}{1-\alpha}}(-\varphi) < 0.$$

k^* is decreasing when α goes up.

3 Romer, 2.16. Social Security

Since the budget constraint changes, we have to solve the Diamond model again.

(a) **PAYG social security**

The utility function is logarithmic

$$U_t = \ln C_{1,t} + \frac{1}{1+\rho} \ln C_{2,t+1},$$

the budget constraint becomes

$$\begin{aligned} C_{1,t} &= (Aw_t - T) - S_t \\ C_{2,t+1} &= (1 + r_{t+1})S_t + (1+n)T, \end{aligned}$$

where S_t is the savings of young. We can solve it to derive the life-time budget constraint

$$C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + \frac{(1+n)T}{1+r_{t+1}} = \overbrace{Aw_t - \frac{r_{t+1}-n}{1+r_{t+1}}T}^{\text{Discounted life-time income}} .$$

Since the LHS of the budget constraint did not change as compared to the standard Diamond model, the Euler equation will be the same:

$$C_{2,t+1} = \frac{1+r_{t+1}}{1+\rho} C_{1,t}.$$

We can now plug it in the life-time budget constraint to get

$$C_{1,t} = \frac{1+\rho}{2+\rho}[Aw_t - \frac{r_{t+1}-n}{1+r_{t+1}}T]$$

and then we can express S_t from the first period constraint[1]:

$$S_t \;=\; Aw_t - T - C_{1,t} = \left[1 - \frac{1+\rho}{2+\rho}\right] Aw_t - \left[1 - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)\right] =$$

[1]Note that savings in this case can actually be negative if social security payment is higher than what people would save by themselves.

$$
\begin{aligned}
&= \frac{1}{2+\rho}Aw_t - \left[\frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}\right]T = \\
&= \frac{1}{2+\rho}Aw_t - \overbrace{\left[\frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})}\right]}^{\equiv Z_t > 0}T.
\end{aligned}
$$

(i) Now recall that old people "eat" their capital and therefore K_{t+1} consists only of the savings of young people: $K_{t+1} = S_t L - T$, or per unit of effective labor, recalling that for the Cobb-Douglas production function $w = (1-\alpha)k^\alpha$ and that in this problem A is constant,

$$
k_{t+1} = \frac{K_{t+1}}{A(1+n)L_t} = \frac{1}{1+n}\frac{S_t}{A} = \frac{1}{1+n}\left(\frac{1-\alpha}{2+\rho}k_t^\alpha - \frac{Z_t T}{A}\right).
$$

Note that if $T = 0$ we get our original equation from the standard Diamond model. Since $Z_t > 0$, we know that k_{t+1} decreases for every k_t.

(ii) We can look again at Figure 2 to convince ourselves that this leads to the decrease in the BGP level of capital. Intuitively, less money is available for savings every period, therefore the lower level of capital per unit of effective labor can be sustained on the BGP.

(iii) If the economy is dynamically efficient ($k^* < k^{GR}$), the (marginal) decrease in k^* will decrease consumption for all future generations, only current generation of old people will benefit from the introduction of the PAYG system, all future generations will be worse off. However, if the economy was dynamically inefficient ($k^* > k^{GR}$), the (marginal) decrease in k^* will increase consumption for all future generations and current generation of old will still gain. Therefore PAYG system can improve welfare in this case.

(b) **FF social security**

With this policy, the budget constraint becomes

$$
\begin{aligned}
C_{1,t} &= (Aw_t - T) - S_t \\
C_{2,t+1} &= (1+r_{t+1})S_t + (1+r_{t+1})T,
\end{aligned}
$$

We can solve it to derive the life-time budget constraint

$$
C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + T = Aw_t,
$$

which is the same as in the standard model. So the Euler equation will be as before:

$$C_{2,t+1} = \frac{1 + r_{t+1}}{1 + \rho} C_{1,t}$$

and we can plug it in the budget constraint to get

$$\begin{aligned} C_{1,t} &= \frac{1+\rho}{2+\rho} A w_t \\ S_t &= A w_t - \frac{1+\rho}{2+\rho} A w_t - T = \frac{1}{2+\rho} A w_t - T, \end{aligned}$$

which says that the FF social security system causes the one-to-one reduction in personal savings. This makes sense because the social security brings the same return as private savings and therefore is a perfect substitute for private savings. We can now guess the answer to this question, because all this implies that total savings (private plus government) are not affected by the introduction of FF social security system and therefore the behavior and the BGP level of capital will not change. But let us derive this more formally.

(i) (ii) Now the capital stock will be held not only by households but also by government and thus

$$K_{t+1} = S_t L_t + T L_T,$$

or, per unit of effective labor,

$$k_{t+1} = \frac{1}{1+n} \left[\frac{1}{2+\rho} w_t - \frac{T}{A} \right] + \frac{1}{1+n} \frac{T}{A} = \frac{1}{1+n} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha},$$

which is exactly the same as without social security (there is no T in the equation) and our guess was indeed correct: nothing happens to k_{t+1} equation and therefore nothing happens to the BGP level of capital.