Strict Single Crossing and the Strict Spence-Mirrlees Condition:

A Comment on Monotone Comparative Statics[1]

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Abstract

Milgrom and Shannon [1994] assert that under appropriate conditions the Spence-Mirrlees condition is equivalent to their single crossing property, and that the strict versions are also equivalent. In this note, however, we give counterexamples which show that their strict single crossing property may hold even though the strict Spence-Mirrlees condition fails. In fact, we show that the strict single crossing property may hold even though the strict Spence-Mirrlees condition holds only on a set of arbitrarily small measure. We also give a correct statement of the relationship between the Spence-Mirrlees condition and the single crossing property. Finally, we illustrate the fact that the strict single crossing property can allow both pooling and separating equilibria while the strict Spence-Mirrlees condition eliminates the possibility of pooling equilibria.

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1. Introduction

Milgrom and Shannon [1994] clarify the relationship between order-theoretic methods for comparative statics and more traditional differential techniques by developing relationships between the differential Spence-Mirrlees single crossing property and the order-theoretic single crossing property. Both conditions are central for monotone comparative statics analysis in a number of settings. In particular, Milgrom and Shannon show that the order-theoretic single crossing property is necessary and sufficient for the set of optimal choices to be nondecreasing in certain choice problems, and that a strict form of the single crossing property guarantees the stronger conclusion that every selection from the set of maximizers is non-decreasing in such problems. Milgrom and Shannon assert that under appropriate conditions the Spence-Mirrlees condition is equivalent to their single crossing property, and that the strict versions are also equivalent. In this note, however, we give counterexamples which show that their strict single crossing property may hold even though the strict Spence-Mirrlees condition fails. In fact, we show that the strict single crossing property may hold even though the strict Spence-Mirrlees condition holds only on a set of arbitrarily small measure. We also give a correct statement of the relationship between the Spence-Mirrlees condition and the single crossing property.

These counterexamples explain the discrepancy between the monotonicity conclusions that Milgrom and Shannon [1994] derive from the strict single crossing property and the *strict* monotonicity conclusions that Edlin and Shannon [1997] derive from the strict Spence-Mirrlees condition. In section 3 we also use these counterexamples to illustrate the fact that the strict single crossing property can allow both pooling and separating equilibria while the strict Spence-Mirrlees condition eliminates the possibility of pooling equilibria. The elimination of pooling equilibria in signalling and screening models is more subtle than Edlin and Shannon's [1997] strict monotonicity conclusions because agents need not face a differentiable constraint.

2. Results

To state our result and examples, we require two definitions of single crossing: the order-theoretic single crossing property of Milgrom and Shannon [1994] and the differential Spence-Mirrlees condition.

Definition 1. *Let X and T be partially ordered sets. A function $f : X \times T \to \mathbf{R}$ is said to satisfy the* ***single crossing property*** *in $(x;t)$ if for all $x' > x^*$:*

1. *whenever $f(x', t^*) \geq f(x^*, t^*)$, then $f(x', t') \geq f(x^*, t')$ for all $t' > t^*$; and*

2. *whenever $f(x', t^*) > f(x^*, t^*)$, then $f(x', t') > f(x^*, t')$ for all $t' > t^*$.*

The function is said to satisfy the ***strict single crossing property*** *in $(x;t)$ if for all $x' > x^*$, whenever $f(x', t^*) \geq f(x^*, t^*)$ then $f(x', t') > f(x^*, t')$ for all $t' > t^*$.*

Definition 2. *Let $f : X \times T \to \mathbf{R}$ be continuously differentiable, where $X \subset \mathbf{R}^2$. Then f is said to satisfy the* ***(strict) Spence-Mirrlees condition*** *if $\frac{f_x}{|f_y|}$ is (increasing) nondecreasing in t, and $f_y \neq 0$ and has the same sign for every (x, y, t).*

Milgrom and Shannon [1994, Theorem 3] assert that these conditions are equivalent as long as $T = \mathbf{R}$ and the function f is continuously differentiable and what they call **completely regular**, which means that the level sets are path-connected. A correct version of their theorem can be stated as follows.

Theorem 2.1. *Let $\mathbf{R}^2$ be given the lexicographic order, with $(x, y) \geq_l (x', y')$ if either $x > x'$ or $x = x'$ and $y \geq y'$. Suppose that $U(x, y, t) : \mathbf{R}^3 \to \mathbf{R}$ is completely regular and continuously differentiable with $U_y \neq 0$. Then $U(x, y; t)$ satisfies the single crossing property in $(x, y; t)$ if and only if it satisfies the Spence-Mirrlees condition. Moreover, $U(x, y; t)$ satisfies the strict single crossing property in $(x, y; t)$ if it satisfies the strict Spence-Mirrlees condition.*

Although the lexicographic order may appear to have come out of the blue here, for sufficiently well-behaved preferences the single crossing property under the lexicographic order is equivalent to the more familiar assumption that indifference curves cross at most once, and always from the same direction. See Athey, Milgrom and Roberts [1996] for a discussion of this point.

Milgrom and Shannon's proof establishes that under these regularity conditions, the strict single crossing property holds whenever the strict Spence-Mirrlees condition holds, and that the non-strict versions of these properties are equivalent. That the strict properties are not equivalent is demonstrated by the following example. Let $T = \{t^*, t'\}$ with $t' > t^*$, and let $f(x, y, t^*) = y - x^2$ and $f(x, y, t') = y - x^2 + x^3/10$, as illustrated in Figure 2.1, which graphs $f(\cdot, 0, t')$ and $f(\cdot, 0, t^*)$. Then f satisfies the strict single crossing property in $(x, y; t)$, since $f(x, y, t') - f(x, y, t^*)$ is increasing in x, but the strict Spence-Mirrlees condition fails whenever $x = 0$, since

$$\frac{f_x}{|f_y|}(0, y, t') = 0 = \frac{f_x}{|f_y|}(0, y, t^*).$$

Insert Figure 2.1 here

Since the strict Spence-Mirrlees condition holds almost everywhere in the above example, one might conjecture that Milgrom and Shannon were almost correct. That is, perhaps for the class of differentiable functions the strict single crossing property implies that the strict Spence-Mirrlees condition holds almost everywhere. Surprisingly, however, continuously differentiable functions can violate the strict Spence-Mirrlees condition over most of their domains and still satisfy the strict single crossing property everywhere. To establish this fact, we first show that an analogous conjecture for one-dimensional problems is also false by constructing a function $g(x, t)$ that has strictly increasing differences, but that has increasing marginal returns only on a set of arbitrarily small measure. For this example, we require several additional definitions.

Definition 3. *A function $g : X \times T \to \mathbf{R}$ is said to have **strictly increasing differences** if $g(x', t') - g(x^*, t') > g(x', t^*) - g(x^*, t^*)$ whenever $x' > x^*$ and $t' > t^*$.*

Definition 4. *A function $g : X \times T \to \mathbf{R}$ is said to have **increasing marginal returns** at $\tilde{x}$ if $g_x(\tilde{x}, t)$ is increasing in t.*

The key to the following counterexamples is the fact that for any given $\varepsilon \in (0, 1)$, there exists a closed, nowhere dense subset of $[0, 1]$ having measure ε, called the ε-Cantor set and denoted C_ε. Like the Cantor set, it is constructed by sequentially removing open intervals from $[0, 1]$. First, the interval $[0, 1]$ is split by removing an open interval from its center, leaving two closed intervals of equal length. These closed intervals are likewise split by removing open intervals from their centers, and this process is continued *ad infinitum.* The 2^{n-1} intervals removed in the n^{th} iteration are each of length $\frac{1-\varepsilon}{2^{2n-1}}$, so that the total length removed is $(1-\varepsilon)\sum_{n=1}^{\infty}\frac{1}{2^n}=1-\varepsilon$. What remains is the ε-Cantor set, which has measure ε.[2]

Consider the function

$$g(x,t)\equiv t\int_0^x h(s)ds, \text{ where } h(s)\equiv\inf_{z\in C_\varepsilon}|z-s|.$$

Since $h(\cdot)$ is continuous, $g(\cdot)$ is well-defined and continuously differentiable. Furthermore, $g(\cdot)$ has strictly increasing differences. To see this, note first that

$$g(x',t)-g(x^*,t)=t\int_{x^*}^{x'}h(s)ds.$$

Since C_ε is closed, $h(s)>0\ \forall s\notin C_\varepsilon$. Hence this integral is positive whenever $x'>x^*$, since C_ε is closed and nowhere dense.[3] However, $g(\cdot)$ only has increasing marginal returns for $x\notin C_\varepsilon$, since $g_x(x,t)=th(x)$, which implies that $g_x(x,t)=0$ if $x\in C_\varepsilon$, and $g_x(x,t)>0$ if $x\notin C_\varepsilon$. Since C_ε has measure ε, which can be set arbitrarily close to 1, this implies that $g(\cdot)$ may very rarely have increasing marginal returns.

Next, notice that if $r(x,t)$ is any function with strictly increasing differences and we define $w(x,y,t)=r(x,t)+y$, then w satisfies the strict single crossing property in $(x,y;t)$ with respect to the lexicographic order on $\mathbf{R}^2$. To see this, suppose that $(x',y')>_\ell(x,y)$ and $w(x',y',t^*)\geq w(x,y,t^*)$. Either $x'>x$, or $x'=x$ and $y'>y$. If $x'=x$, then

$$w(x',y',t')-w(x,y,t')=y'-y$$

which is positive since $y'>y$. If $x'>x$, then since $w(x',y',t^*)\geq w(x,y,t^*)$, we know that

$$\begin{aligned} y-y' &\leq r(x',t^*)-r(x,t^*) \\ &< r(x',t')-r(x,t')\ \ \forall t'>t^* \end{aligned}$$

since $r(x,t)$ has strictly increasing differences. Thus $w(x',y',t')>w(x,y,t')$ for all $t'>t^*$ in either case, which shows that w satisfies the strict single crossing property.

[2]See, for example, Aliprantis and Burkinshaw [1981, p. 113] for a further discussion of the construction of this set and some of its properties.

[3]Since C_ε is nowhere dense, if $x'\neq x^*$, there exists $\tilde{x}\notin C_\varepsilon$ between x^* and x'. Since C_ε is closed, there is an open interval around $\tilde{x}$ contained in the complement of C_ε, and $h(\cdot)$ must be positive on this interval since $h(s)>0\ \forall s\notin C_\varepsilon$.

From this discussion, it follows that $f(x, y, t) \equiv g(x, t) + y$ satisfies the strict single crossing property. However, if $x \in C_\varepsilon$, then f fails to satisfy the strict Spence-Mirrlees condition at (x, y) for any y, because

$$\frac{f_x}{|f_y|}(x, y, t^*) = \frac{\partial g}{\partial x}(x, t^*) = 0 = \frac{\partial g}{\partial x}(x, t') = \frac{f_x}{|f_y|}(x, y, t').$$

Notably, this failure occurs on a set of measure ε, which again can be arbitrarily close to 1.

Milgrom and Shannon [1994] err by presuming that if the strict Spence-Mirrlees condition fails, then it must fail on a set of positive measure with nonempty interior. They then integrate along an indifference curve in this interior to show that if the strict Spence-Mirrlees condition fails, then so too must the strict single crossing property. As these examples illustrate, however, their presumption can be wrong: even though the strict single crossing property holds, the Spence-Mirrlees condition can fail, and can fail on a set of positive measure, as long as that set has an empty interior. In our examples, their integration argument cannot work because there is no path along an indifference curve where the strict Spence-Mirrlees condition fails.

3. An Illustrative Example

As the results of the previous section indicate, the strict single crossing property is weaker than the strict Spence-Mirrlees condition, and thus the monotone comparative statics results obtained by Milgrom and Shannon are actually stronger than they claimed. The fact that these properties differ also explains why Edlin and Shannon [1997] are able to derive strict comparative statics results from the strict Spence-Mirrlees condition, while such conclusions cannot be drawn from the strict single crossing property. Milgrom and Shannon [1994] show that the strict single crossing property is sufficient to guarantee that every selection from the set of maximizers is nondecreasing, yet this conclusion allows the possibility that some selections may remain constant over some range of parameters. This difference is illustrated in Figure 3.1.

Insert Figure 3.1 here

Signaling and screening models provide another example of the importance of distinguishing between the strict single crossing property and the strict Spence-Mirrlees condition. In such models, the strict single crossing property allows both pooling and separating behavior in equilibrium, while the strict Spence-Mirrlees condition rules out pooling equilibria. These models are more complex than the optimization problems considered in Edlin and Shannon [1997]: here, a screener need not offer agents a differentiable choice set, so separating behavior cannot be inferred simply by comparing solutions to agents' optimization problems. Instead separation results from equilibrium considerations.

As an illustration, consider the menu of price-quality contracts a monopoly will choose to offer to consumers. Let $T = \{\ell, h\}$ with $h > \ell$. Consumers of types ℓ and h have preferences given by

$$U_\ell(q,p) = \begin{cases} 2q - (q-1)^2 - p, & \text{if } q \in [0,2]; \\ 3 - p, & \text{if } q > 2 \end{cases}$$

and

$$U_h(q,p) = \begin{cases} 2q - (q-1)^2 - p + \frac{1}{3}(q-1)^3, & \text{if } q \in [0,2]; \\ \frac{10}{3} - p, & \text{if } q > 2, \end{cases}$$

where q denotes quality and p denotes price. The monopoly cannot observe a consumer's type. Let $\mathbf{R}^2$ be given the lexicographic order on $(q, -p)$, that is, the order in which $(q', p') \geq (q, p)$ if either $q' > q$ or $q' = q$ and $-p' \geq -p$. By the same argument given in the previous example, these preferences satisfy the strict single crossing property on $[0,2] \times \mathbf{R}_+ \times T$ since

$$U_h(q,p) - U_\ell(q,p) = \frac{1}{3}(q-1)^3$$

is increasing in q. The strict Spence-Mirrlees condition fails whenever $q = 1$, however, since

$$\frac{\partial U_h}{\partial q} \Big/ \left|\frac{\partial U_h}{\partial p}\right| (1,p) = 2 = \frac{\partial U_\ell}{\partial q} \Big/ \left|\frac{\partial U_\ell}{\partial p}\right| (1,p).$$

When the production cost is 2 per unit, the unique equilibrium is a pooling equilibrium in which the profit-maximizing monopoly will offer only one contract, $q_h = q_\ell = 1$, $p_h = p_\ell = 3$, as depicted in figure 3.2(a).[4] The pooling equlibrium is possible because the strict Spence-Mirrlees condition fails at $q = 1$.

[4] The monopoly's profit maximization problem is

$$\begin{array}{lll} \max & p_\ell + p_h - 2(q_\ell + q_h) & \\ \text{s.t.} & U_\ell(q_\ell, p_\ell) \geq U_\ell(0,0) & (IR_\ell) \\ & U_h(q_h, p_h) \geq U_h(0,0) & (IR_h) \\ & U_\ell(q_\ell, p_\ell) \geq U_\ell(q_h, p_h) & (IC_\ell) \end{array}$$

Insert Figure 3.2 here

In contrast, suppose instead that the preferences of the high type are given by

$$\overline{U}_h(q,p) = \begin{cases} 3q - (q-1)^2 - p, & \text{if } q \in [0,2]; \\ 5 - p, & \text{if } q > 2. \end{cases}$$

In this case, preferences satisfy not only the strict single crossing property but also the stronger strict Spence-Mirrlees condition on $[0,2] \times \mathbf{R}_+ \times T$. Here it is optimal for the monopoly to offer a separating contract which involves selling a higher quality level to the high type than to the low type. Offering two distinct contracts is optimal here because by standard arguments, $\frac{\partial U_\ell}{\partial q}(q_\ell^*, p_\ell^*) \geq 2$,[5] so that by the strict Spence-Mirrlees condition, $\frac{\partial \overline{U}_h}{\partial q}(q_\ell^*, p_\ell^*) > 2$. Since the high type's marginal willingness to pay at (q_ℓ^*, p_ℓ^*) exceeds marginal cost, unlike the previous example, the monopoly will offer

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$$U_h(q_h, p_h) \geq U_h(q_\ell, p_\ell). \quad (IC_h)$$

To find the solution $(q_\ell^*, p_\ell^*); (q_h^*, p_h^*)$ to the monopoly's problem, observe first that (IR_ℓ) must bind, so that $U_\ell(q_\ell^*, p_\ell^*) = U_\ell(0,0) = -1$; equivalently, $p_\ell^* = 2q_\ell^* - (q_\ell^* - 1)^2 + 1$. Thus we can restrict attention to contracts (q_ℓ, p_ℓ) such that $p_\ell = 2q_\ell - (q_\ell - 1)^2 + 1$.

Given any such contract, the optimal contract to offer to the high type solves

$$\begin{aligned} \max_{(q_h, p_h)} \quad & -2q_h + p_h \\ \text{s.t.} \quad & U_h(q_h, p_h) \geq U_h(0,0) \\ & U_h(q_h, p_h) \geq U_h(q_\ell, p_\ell) \\ & U_\ell(q_\ell, p_\ell) \geq U_\ell(q_h, p_h). \end{aligned}$$

The most profitable contract satisfying (IC_h) is $q_h = 1, p_h = 3 - \frac{1}{3}(q_\ell - 1)^3$. When $q_\ell < 1$, this contract also satisfies (IC_ℓ) and (IR_h) because the strict single crossing property and (IR_ℓ) hold. Hence this contract is optimal when $q_\ell < 1$. In contrast, when $q_\ell \geq 1$, the optimal contract is $(q_h, p_h) = (q_\ell, p_\ell)$. Thus given any level q_ℓ offered to the low type, the monopoly's maximum profits will be

$$\pi(q_\ell) = \begin{cases} 2 - (q_\ell - 1)^2 - \frac{1}{3}(q_\ell - 1)^3, & \text{if } q_\ell < 1; \\ 2 - 2(q_\ell - 1)^2, & \text{if } q_\ell \geq 1. \end{cases}$$

The solution to this profit maximization problem occurs at $q_\ell^* = 1$, and hence the unique solution to the monopoly's problem is to offer the pooling contract $(q_\ell^*, p_\ell^*) = (q_h^*, p_h^*) = (1,3)$.

[5] If $\frac{\partial U_\ell}{\partial q}(q_\ell^*, p_\ell^*) < 2$, or equivalently if $q_\ell^* > 1$, then the monopoly can increase profits by selling slightly less to the low type. More precisely, consider changing the low offer to $(\tilde{q}_\ell, \tilde{p}_\ell)$ where $U_\ell(\tilde{q}_\ell, \tilde{p}_\ell) = U_\ell(q_\ell^*, p_\ell^*)$ and $\tilde{q}_\ell = q_\ell^* - \epsilon$ for some $\epsilon > 0$. Clearly (IR_ℓ) and (IC_ℓ) continue to hold. This new contract also satisfies (IC_h) because: (1) the low type is indifferent between (q_ℓ^*, p_ℓ^*) and $(\tilde{q}_\ell, \tilde{p}_\ell)$, so by the strict single crossing property the high type prefers (q_ℓ^*, p_ℓ^*) to $(\tilde{q}_\ell, \tilde{p}_\ell)$; and (2) (IC_h) holds for the original contract. Under the new contract, monopoly profits change by

$$2\epsilon - 2\epsilon - (q_\ell^* - \epsilon - 1)^2 + (q_\ell^* - 1)^2 = 2\epsilon(q_\ell^* - 1) - \epsilon^2,$$

which is positive for ϵ sufficiently small since $q_\ell^* > 1$, indicating that the monopoly is not optimizing. Hence $\frac{\partial U_\ell}{\partial q}(q_\ell^*, p_\ell^*) \geq 2$.

a second bundle with a higher quality level intended for the high type, as illustrated in figure 3.2(b).

References

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