Appendix for:
Price Setting in Forward-Looking Customer Markets

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Appendix A. Second Order Approximations

Appendix A.1. A Derivation of a 2nd Order Approximation to the Firm's Value

It is straightforward to show that the steady state price with full commitment is

$$p(z) = \frac{\theta}{\theta - 1} S,$$

where variables without subscripts denote steady state values. Notice furthermore that equation (1) in the paper implies that $C = (1-\gamma)c(z)$ and equation (2) in the paper implies that $(1-\gamma\beta)P = p(z)$. A second order Taylor series approximation of the value of the firm around the steady state of the solution to the firm's problem with commitment is given by

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \Bigg[& c(z)(p_t(z) - p(z)) + \frac{1}{\theta} p(z)(c_t(z) - c(z)) + (p_t(z) - p(z))(c_t(z) - c(z)) \\ & - \frac{\theta - 1}{\theta} \frac{p(z)}{S} (S_t - S)(c_t(z) - c(z)) + \frac{c(z)}{\beta^t} (p_t(z) - p(z))(M_{0,t} - \beta^t) \\ & + \frac{1}{\theta} \frac{p(z)}{\beta^t} (c_t(z) - c(z))(M_{0,t} - \beta^t) \Bigg] + \text{ ex. terms} + \mathcal{O}(||\xi||^3), \end{aligned} \quad \text{(A.1)}$$

where "ex. terms" stands for terms that are exogenous to the firm's decision problem, ξ stands for a vector of the exogenous variables and $\mathcal{O}(||\xi||^3)$ denotes higher order terms.

The exposition of our results is simplified if we make a change of variables. Let $\hat{c}_t(z) = \log(c_t(z)/c(z))$ and define hatted versions of all other variables in the same way. Making use of the fact that

$$c_t(z) = c(z) \left(1 + \hat{c}_t(z) + \frac{1}{2} \hat{c}_t(z) \right) + \mathcal{O}(||\xi||^3),$$

we can rewrite equation (A.1) as

$$E_0 \sum_{t=0}^{\infty} p(z)c(z)\beta^t \left[\left(\hat{p}_t(z) + \frac{1}{2}\hat{p}_t(z)^2 \right) + \frac{1}{\theta}\left(\hat{c}_t(z) + \frac{1}{2}\hat{c}_t(z)^2 \right) + \hat{p}_t(z)\hat{c}_t(z) - \frac{\theta-1}{\theta}\hat{S}_t\hat{c}_t(z) + \hat{p}_t(z)\hat{M}_{0,t} + \frac{1}{\theta}\hat{c}_t(z)\hat{M}_{0,t} \right] + \text{ex. terms} + \mathcal{O}(||\xi||^3). \quad \text{(A.2)}$$

Appendix A.2. A Derivation of a 2nd Order Approximation to the Consumer Demand Curve

Notice that consumer demand—given by equation (2) in the paper—may be rewritten as

$$P_t C_t^{\frac{1}{\theta_t}} (c_t(z) - \gamma c_{t-1}(z))^{-\frac{1}{\theta_t}} = p_t(z) + E_t \left[\gamma M_{t,t+1} P_{t+1} C_{t+1}^{\frac{1}{\theta_{t+1}}} (c_{t+1}(z) - \gamma c_t(z))^{-\frac{1}{\theta_{t+1}}} \right].$$

A second order Taylor series approximation of this equation around the steady state of the solution to the firm's problem with commitment is given by

$$\begin{aligned}
&(P_t - P) + \frac{1}{\theta}PC^{-1}(C_t - C) - \frac{1}{\theta}Pc(z)^{-1}(c_t(z) - c(z)) + \frac{\gamma}{\theta}Pc(z)^{-1}(c_{t-1}(z) - c(z)) \\
&-\frac{1}{\theta}Pc(z)^{-1}(P_t - P)(c_t(z) - c(z)) + \frac{1+\theta}{2\theta^2}Pc(z)^{-2}(c_t(z) - c(z))^2 \\
&-\frac{1}{\theta^2}PC^{-1}c(z)^{-1}(C_t - C)(c_t(z) - c(z)) + \frac{1}{\theta^2}Pc(z)^{-1}(c_t(z) - c(z))(\theta_t - \theta) \\
&= (p_t(z) - p(z)) - \frac{\gamma\beta}{\theta}Pc(z)^{-1}E_t(c_{t+1}(z) - c(z)) + \text{s.o.ex.terms} + \mathcal{O}(||\xi, \gamma||^3), \quad \text{(A.3)}
\end{aligned}$$

where s.o.ex.terms denotes "second order exogenous terms" and $\mathcal{O}(||\xi, \gamma||^3)$ denotes terms that are third order (or higher) in $||\xi, \gamma||$. The norm $||\xi, \gamma||$ is simply meant to denote the standard Euclidian distance norm in (ξ, γ) space. As in the case of the expression for the value of the firm, we find it convenient to rewrite equation (A.3) in terms of the hatted variables and subsitute the $E_t\hat{c}_{t+1}(z)$ term our for a $E_t\hat{p}_{t+1}(z)$ term. This yields

$$\begin{aligned}
&\left(\hat{c}_t(z) + \frac{1}{2}\hat{c}_t(z)^2 \right) - \gamma\hat{c}_{t-1}(z) - \frac{1+\theta}{2\theta(1-\gamma)}\hat{c}_t(z)^2 - \theta\hat{P}_t - \hat{C}_t + \hat{P}_t\hat{c}_t(z) + \frac{1}{\theta}\hat{C}_t\hat{c}_t(z) - \hat{c}_t(z)\hat{\theta}_t \\
&= -\theta(1-\gamma-\gamma\beta)\hat{p}_t(z) - \frac{\theta}{2}\hat{p}_t(z)^2 - \theta\gamma\beta E_t\hat{p}_{t+1}(z) + \text{s.o.ex.terms} + \mathcal{O}(||\xi, \gamma||^3), \quad \text{(A.4)}
\end{aligned}$$

Appendix B. Proofs of Propositions

Appendix B.1. Proposition 1

Proof: We can rearrange equation (5) in the paper so that it says that

$$\frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z) - \frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_{t-1}(z) - \frac{\gamma\beta}{1-\gamma-\gamma\beta}E_t\hat{c}_{t+1}(z) = -\theta\hat{p}_t(z) - \frac{1}{2}\frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z)^2$$
$$+\frac{1}{2}\frac{1+\theta}{\theta(1-\gamma)(1-\gamma-\gamma\beta)}\hat{c}_t(z)^2 - \frac{\theta-1}{1-\gamma-\gamma\beta}\hat{c}_t(z)\hat{\Upsilon}_t - \frac{1}{2}\frac{\theta}{1-\gamma-\gamma\beta}\hat{p}_t(z)^2$$
$$-\frac{1}{1-\gamma-\gamma\beta}\left(\theta\hat{P}_t + \hat{C}_t - \hat{P}_t\hat{c}_t(z) - \frac{1}{\theta}\hat{C}_t\hat{c}_t(z)\right) + \text{ s.o.ex.terms} + \mathcal{O}(||\xi,\gamma||^3).$$

Now notice that equation (4) in the paper may be written

$$E_0\sum_{t=0}^{\infty} p(z)c(z)\beta^t\left[\frac{1}{\theta}\left(\frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z) - \frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_{t-1}(z) - \frac{\gamma\beta}{1-\gamma-\gamma\beta}\hat{c}_{t+1}(z)\right)\right.$$
$$+\left(\hat{p}_t(z) + \frac{1}{2}\hat{p}_t(z)^2\right) + \frac{1}{\theta}\frac{1}{2}\hat{c}_t(z)^2 + \hat{p}_t(z)\hat{c}_t(z) - \frac{\theta-1}{\theta}\hat{S}_t\hat{c}_t(z)$$
$$\left.+\hat{p}_t(z)\hat{M}_{0,t} + \frac{1}{\theta}\hat{c}_t(z)\hat{M}_{0,t}\right] - p(z)c(z)\frac{1}{\theta}\frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_0(z) + \text{ ex. terms} + \mathcal{O}(||\xi,\gamma||^3).$$

Substituting consumer demand into this expression now yields

$$E_0\sum_{t=0}^{\infty} p(z)c(z)\beta^t\left[\frac{1}{\theta}\left(-\frac{1}{2}\frac{1}{1-\gamma-\gamma\beta}\left(\hat{c}_t(z)^2 - \frac{1+\theta}{\theta(1-\gamma)}\hat{c}_t(z)^2 + 2(\theta-1)\hat{c}_t(z)\hat{\Upsilon}_t + \theta\hat{p}_t(z)^2\right.\right.\right.$$
$$\left.\left.-2\hat{P}_t\hat{c}_t(z) - 2\frac{1}{\theta}\hat{C}_t\hat{c}_t(z)\right)\right) + \frac{1}{2}\hat{p}_t(z)^2 + \frac{1}{\theta}\frac{1}{2}\hat{c}_t(z)^2 + \hat{p}_t(z)\hat{c}_t(z) - \frac{\theta-1}{\theta}\hat{S}_t\hat{c}_t(z)$$
$$\left.+\hat{p}_t(z)\hat{M}_{0,t} + \frac{1}{\theta}\hat{c}_t(z)\hat{M}_{0,t}\right] - p(z)c(z)\frac{1}{\theta}\frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_0(z) + \text{ ex. terms} + \mathcal{O}(||\xi,\gamma||^3).$$

If we now multiply this expression by $(1-\gamma)(1-\gamma-\gamma\beta)$, use consumer demand to substitute for $\hat{c}_t(z)$ and simplify, we get that

$$E_0\sum_{t=0}^{\infty} p(z)c(z)\beta^t\left[\frac{1-\theta}{2}\hat{p}_t(z)^2 + (\theta-1)\hat{\Upsilon}_t\hat{p}_t(z) + (\theta-1)\hat{S}_t\hat{p}_t(z)\right] + p(z)c(z)\gamma\hat{p}_0(z)$$
$$+\text{ex. terms} + \mathcal{O}(||\xi,\gamma||^3). \qquad \text{(B.1)}$$

Setting the derivative of this with respect to $\hat{p}_t(z)$ for $t \geq 1$ equal to zero shows that the firm's optimal pricing policy under full commitment to a state-contingent rule for $t \geq 1$ is

$$\hat{p}_t(z) = \hat{S}_t + \hat{\Upsilon}_t$$

up to an error of order $\mathcal{O}(||\xi,\gamma||^2)$. ■

Appendix B.2. Proposition 2

Proof: A derivation analogous to the derivation of expression (B.1) yields that the objective of the firm at time t can be written as

$$E_t \sum_{j=0}^{\infty} p(z)c(z)\beta^j \left[\frac{1-\theta}{2}\hat{p}_{t+j}(z)^2 + (\theta-1)\hat{\Upsilon}_{t+j}\hat{p}_{t+j}(z) + (\theta-1)\hat{S}_{t+j}\hat{p}_{t+j}(z) \right] + p(z)c(z)\gamma\hat{p}_t(z)$$
$$+\text{ex. terms} + \mathcal{O}(||\xi,\gamma||^3) \quad \text{(B.2)}$$

We seek a Markov perfect equilibrium that is accurate up to a residual of order $\mathcal{O}(||\xi,\gamma||^2)$. The definition of a Markov perfect equilibrium implies that the strategies of the firms are functions of only the "pay-off relevant" state of the economy. In our model, the state of the economy at time t is $(\hat{c}_{t-1}(z), \hat{S}_t, \hat{\Upsilon}_t)$. However, given our approximation, $\hat{c}_{t-1}(z)$ contributes only terms of order $\mathcal{O}(||\xi,\gamma||^2)$ since it is multiplied by γ in consumer demand. This implies that, up to a residual of order $\mathcal{O}(||\xi,\gamma||^2)$, the stretegies of the firm are functions of only $(\hat{S}_t, \hat{\Upsilon}_t)$. Since these two variables are i.i.d., the firm can correctly assume that its decisions at time t have no effect on outcomes in any period $T \geq t+1$. The firm can therefore simply maximize expression (B.2) with respect to the current period price $\hat{p}_t(z)$. This yields

$$\hat{p}_t(z) = \frac{\gamma}{\theta-1} + \hat{S}_t + \hat{\Upsilon}_t, \quad \text{(B.3)}$$

up to an error of order $\mathcal{O}(||\xi,\gamma||^2)$. ■

Appendix B.3. Proposition 3

Proof: We must show that the firm does not have a profitable deviation when it is setting $p_t(z) = p_t^c(z)$. The potential benefit from deviating at this point is a higher price in the current period. Denote this benefit by $\Pi_t^d - \Pi_t^c$ (which could be negative). The loss is the change in future profits associated with playing the Markov perfect equilibrium in future periods rather than $p_{t+j}(z) = p_{t+j}^c(z)$. Denote the expected loss in the period after the deviation by $E[\Pi_{+1}^c - \Pi_{+1}^m]$ and the per period loss in subsequent periods by $E[\Pi^c - \Pi^m]$. The firm will refrain from deviating in the current period if $\Pi_t^d - \Pi_t^c < \beta E[\Pi_{+1}^c - \Pi_{+1}^m] + \beta^2 E[\Pi^c - \Pi^m]/(1-\beta)$. This condition will hold for all $\beta > \underline{\beta}_t$ where $\underline{\beta}_t$ is implicitely defined by $\Pi_t^d - \Pi_t^c = \underline{\beta}_t E[\Pi_{+1}^c - \Pi_{+1}^m] + \underline{\beta}_t^2 E[\Pi^c - \Pi^m]/(1-\underline{\beta}_t)$. The firm will never deviate if

$\beta > \underline{\beta} = \max \underline{\beta}_t$. If $\Pi_t^d - \Pi_t^c$ is negative, $\underline{\beta} = 0$. It is relatively easy to show that $\underline{\beta}$ is independent of γ. ■

Appendix B.4. Proposition 4

Proof: First, we show that the profit maximizing price rule assuming that prices can only be changed in odd numbered periods is given by equation (10) in the paper. The value of the firm's expected profits from period t on are given by expression (B.2). The firm maximizes this expression with respect to $\hat{p}_t$ subject to the constraint $\hat{p}_t(z) = \hat{p}_{t+1}(z)$. We can use this constraint to eliminate $\hat{p}_{t+1}(z)$ in expression (B.2) and rewrite it ignoring terms that are exogenous to the firm's time t problem. This yields

$$p(z)c(z)\left[\frac{(1-\theta)(1+\beta)}{2}\hat{p}_t(z)^2 + (\theta-1)\hat{\Upsilon}_t\hat{p}_t(z) + (\theta-1)\hat{S}_t\hat{p}_t(z) + \gamma\hat{p}_t(z)\right] + \text{ex. terms} + \mathcal{O}(||\xi,\gamma||^3) \quad \text{(B.4)}$$

Maximizing this with respect to $\hat{p}_t(z)$ yields equation (10) in the paper.

Next, we show that the value expected profits in future periods from adhering to the price path stated in the proposition is greater than from playing the Markov perfect equilibrium. The value of the expected future profits is equal to

$$E_t\sum_{j=1}^{\infty} p(z)c(z)\beta^j\left[\frac{1-\theta}{2}\hat{p}_{t+j}(z)^2 + (\theta-1)(\hat{S}_{t+j} + \hat{\Upsilon}_{t+j})\hat{p}_{t+j}(z)\right] + \text{ex. terms} + \mathcal{O}(||\xi,\gamma||^3).$$

Using equation (8) in the paper we can derive that, ignoring exogenous terms and terms of higher than second order, the value of the expected profits of a firm if it plays the Markov perfect equilibrium in all future periods is

$$E_t\Pi_{t+1}^m = p(z)c(z)\frac{\beta}{1-\beta}\frac{\theta-1}{2}\left[-\left(\frac{\gamma}{\theta-1}\right)^2 + \text{var}(\hat{S}_t + \hat{\Upsilon}_t)\right].$$

Similarly, using equation (10) in the paper we can derive that the value of the expected profits of a firm that prices according the the rule in the proposition is larger than or equal to

$$\begin{aligned}\min_{\hat{S}_t+\hat{\Upsilon}_t} E_t\Pi^F_{t+1} &= \min_{\hat{S}_t+\hat{\Upsilon}_t}\Bigg[p(z)c(z)\beta\frac{\theta-1}{2}\left[-\left(\frac{\gamma}{(\theta-1)(1+\beta)}\right)^2+\frac{1+2\beta}{(1+\beta)^2}(\hat{S}_t+\hat{\Upsilon}_t)^2\right]\\ &\quad +p(z)c(z)\frac{\beta^2}{1-\beta}\frac{\theta-1}{2}\left[-\left(\frac{\gamma}{(\theta-1)(1+\beta)}\right)^2+\frac{1+2\beta}{(1+\beta)^2}\mathrm{var}(\hat{S}_t+\hat{\Upsilon}_t)\right]\Bigg]\\ &= p(z)c(z)\frac{\beta}{1-\beta}\frac{\theta-1}{2}\left[-\left(\frac{\gamma}{(\theta-1)(1+\beta)}\right)^2+\frac{\beta(1+2\beta)}{(1+\beta)^2}\mathrm{var}(\hat{S}_t+\hat{\Upsilon}_t)\right]\end{aligned}$$

Comparing these expressions we get that $\min_{\hat{S}_t+\hat{\Upsilon}_t} E_t\Pi^F_{t+1} > E_t\Pi^m_{t+1}$ if

$$\gamma^2 > (\theta-1)^2\frac{1+\beta-\beta^2}{2\beta+\beta^2}\mathrm{var}(\hat{S}_t+\hat{\Upsilon}_t).$$

Given this condition an argument analogous to the one given in the proof of Proposition 3 implies that the firm does not have a profitable deviation while it is setting its price according to equation (10) in the paper as long as $\beta > \underline{\beta}$. ■

Appendix B.5. Proposition 5

We must show that the portion of the price path described in the proposition that is played in equilibrium is the best feasible price path from the firm's perspective. Once we have shown this, an argument analogous to the proof of Proposition 3 implies that the firm does not have a profitable deviation from this price path.

We begin by deriving the function in our model that correponds to $R(x_t, \mu_t, \theta_t)$ in Athey et al. (2004). Given the consumer's demand curve—equation (5)—one can view the decision the consumer makes at each point in time as a decision about what he expects prices to be in the next period. To see this, notice that we can rewrite equation (5) as

$$\begin{aligned}-\theta\hat{p}_t(z) = \frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z) - \frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_{t-1}(z) - \frac{\gamma\beta}{1-\gamma-\gamma\beta}E_t\hat{c}_{t+1}(z) + \frac{1}{2}\frac{\theta}{1-\gamma-\gamma\beta}\hat{p}_t(z)^2\\ +\frac{1}{2}\frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z)^2 - \frac{1}{2}\frac{1+\theta}{\theta(1-\gamma)(1-\gamma-\gamma\beta)}\hat{c}_t(z)^2 - \theta\hat{P}_t - \hat{C}_t\\ +\frac{1}{(1-\gamma-\gamma\beta)}\left(\hat{P}_t + \frac{1}{\theta}\hat{C}_t + (\theta-1)\hat{\Upsilon}_t\right)\hat{c}_t(z) + \text{ s.o.ex.terms} + \mathcal{O}(||\xi,\gamma||^3).\end{aligned}$$

Notice furthermore that

$$\hat{c}_t(z) = -\theta\hat{p}_t(z) + \theta\hat{P}_t + \hat{C}_t + \mathcal{O}(||\xi,\gamma||^2). \tag{B.5}$$

Using this fact, we can rewrite the consumer's demand curve as

$$
\begin{aligned}
-\theta\hat{p}_t(z) = {} & \frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z) - \frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_{t-1}(z) + \frac{\gamma\beta\theta}{1-\gamma-\gamma\beta}E_t\hat{p}_{t+1}(z) + \frac{1}{2}\frac{\theta}{1-\gamma-\gamma\beta}\hat{p}_t(z)^2 \\
& + \frac{1}{2}\frac{\theta^2}{1-\gamma-\gamma\beta}(\hat{p}_t(z)^2 - \hat{P}_t\hat{p}_t(z) - \frac{1}{\theta}\hat{C}_t\hat{p}_t(z)) - \frac{1}{2}\frac{(1+\theta)\theta}{(1-\gamma)(1-\gamma-\gamma\beta)}(\hat{p}_t(z)^2 - \hat{P}_t\hat{p}_t(z) - \frac{1}{\theta}\hat{C}_t\hat{p}_t(z)) \\
& - \theta\hat{P}_t - \hat{C}_t - \frac{\theta}{(1-\gamma-\gamma\beta)}\left(\hat{P}_t + \frac{1}{\theta}\hat{C}_t + (\theta-1)\hat{\Upsilon}_t\right)\hat{p}_t(z) + \text{ s.o.ex.terms} + \mathcal{O}(||\xi,\gamma||^3).
\end{aligned}
$$

Notice that once the consumer has chosen what to expect about the firm's price in period $t+1$, this equation determines his demand. One can therefore view the consumer's decision at each point given the form of the demand curve as a choice about what to expect about the firm's price in the next period. In equilibrium, the consumer will have rational expectations. We have used this fact by writing the consumer's expectation as $E_t\hat{p}_{t+1}$. However, more generally, we can denote the consumer's expectation about $\hat{p}_t$ at time $t-1$ as x_t. Using this notation, the consumer's demand curve becomes

$$
\begin{aligned}
-\theta\hat{p}_t(z) = {} & \frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z) - \frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_{t-1}(z) + \frac{\gamma\beta\theta}{1-\gamma-\gamma\beta}x_{t+1} + \frac{1}{2}\frac{\theta}{1-\gamma-\gamma\beta}\hat{p}_t(z)^2 \\
& + \frac{1}{2}\frac{\theta^2}{1-\gamma-\gamma\beta}(\hat{p}_t(z)^2 - \hat{P}_t\hat{p}_t(z) - \frac{1}{\theta}\hat{C}_t\hat{p}_t(z)) - \frac{1}{2}\frac{(1+\theta)\theta}{(1-\gamma)(1-\gamma-\gamma\beta)}(\hat{p}_t(z)^2 - \hat{P}_t\hat{p}_t(z) - \frac{1}{\theta}\hat{C}_t\hat{p}_t(z)) \\
& - \theta\hat{P}_t - \hat{C}_t - \frac{\theta}{(1-\gamma-\gamma\beta)}\left(\hat{P}_t + \frac{1}{\theta}\hat{C}_t + (\theta-1)\hat{\Upsilon}_t\right)\hat{p}_t(z) + \text{ s.o.ex.terms} + \mathcal{O}(||\xi,\gamma||^3). \qquad \text{(B.6)}
\end{aligned}
$$

Next, notice that the second order approximation of the value of the firm—equation (4)—may be written

$$
\begin{aligned}
E_0\sum_{t=0}^{\infty} p(z)c(z)\beta^t \Bigg[& \hat{p}_t(z) + \frac{1}{\theta}\left(\frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z) - \frac{\gamma}{1-\gamma-\gamma\beta}\hat{c}_{t-1}(z)\right) - \frac{\gamma}{\theta}\frac{1}{1-\gamma-\gamma\beta}\hat{c}_t(z) \\
& + \frac{1}{2}\hat{p}_t(z)^2 + \frac{1}{\theta}\frac{1}{2}\hat{c}_t(z)^2 + \hat{p}_t(z)\hat{c}_t(z) - \frac{\theta-1}{\theta}\hat{S}_t\hat{c}_t(z) + \hat{p}_t(z)\hat{M}_{0,t} + \frac{1}{\theta}\hat{c}_t(z)\hat{M}_{0,t}\Bigg] \\
& + \text{ex. terms} + \mathcal{O}(||\xi||^3)
\end{aligned}
$$

Using equation (B.6) to eliminate the first two terms in this expression, equation (B.5) to eliminate $\hat{c}_t(z)$ and multiplying the resulting expression by $(1-\gamma-\gamma\beta)(1-\gamma)$ gives

$$
E_0\sum_{t=0}^{\infty} p(z)c(z)\beta^t \left[\gamma\hat{p}_t(z) - \gamma\beta x_{t+1} + \frac{1-\theta}{2}\hat{p}_t^2 + (\theta-1)(\hat{S}_t + \hat{\Upsilon}_t)\hat{p}_t(z)\right] + \text{ex. terms} + \mathcal{O}(||\xi,\gamma||^3). \qquad \text{(B.7)}
$$

Collecting the terms in the sum that involve $\hat{p}_t(z)$, x_t and $\hat{\Phi}_t = \hat{S}_t + \hat{\Upsilon}_t$ we can define

$$R(x_t, \hat{p}_t(z), \hat{S}_t) = \gamma\hat{p}_t(z) - \gamma x_t - \frac{1}{2}(\theta-1)\hat{p}_t^2(z) + (\theta-1)\hat{\Phi}_t\hat{p}_t(z) \quad \text{(B.8)}$$

for $t \geq 1$. This is the function in our model that corresponds to $R(x_t, \mu_t, \theta_t)$ in Athey et al. (2004). Mapping our notation into the notation used by Athey et al. (2004) we get that: $x_t = x_t$, $\mu_t = \hat{p}_t(z)$ and $\theta_t = \hat{\Phi}_t$. In the notation used by Athey et al. (2004), the firm's objective function is

$$R(x_t, \mu_t, \theta_t) = \gamma\mu_t - \gamma x_t - \frac{1}{2}(\theta-1)\mu_t^2 + (\theta-1)\theta_t\mu_t.$$

Notice that this function satisfies all the conditions required for the propositions in Athey et al. (2004) to be valid. Specifically, $R_x(x_t, \mu_t, \theta_t) = -\gamma < 0$, $R_{\mu\theta}(x_t, \mu_t, \theta_t) = \theta - 1 > 0$ and $R_{\mu\mu}(x_t, \mu_t, \theta_t) = -(\theta-1) < 0$.

The main difference between our results and the results in Athey et al. (2004) is that they consider a model in which $R(x_t, \mu_t, \theta_t)$ is the social welfare function, i.e. it is the objective of all the agents in the model. The fact that $R(x_t, \mu_t, \theta_t)$ in Athey et al. (2004) is the social welfare function entails that the resulting policy is socially optimal. Here we use the objective of the firm as our $R(x_t, \mu_t, \theta_t)$, which means that the resulting policy is not socially optimal but rather the best policy from the perspective of the firm. The proofs in Athey et al. (2004) do not rely on $R(x_t, \mu_t, \theta_t)$ being a social welfare function. Only their interpretation as solving for the socially optimal policy relies on this.

Given equation (B.8) and the following monotone hazard conditions: $(1-P(\hat{\Phi}_t))/p(\hat{\Phi}_t)$ is strictly decreasing in $\hat{\Phi}_t$ and $P(\hat{\Phi}_t)/p(\hat{\Phi}_t)$ is strictly increasing in $\hat{\Phi}_t$, Proposition 1 in Athey et al. (2004) shows that the pricing policy that is optimal from the perspective of the firm is static. Here $p(\hat{\Phi}_t)$ and $P(\hat{\Phi}_t)$ denote the pdf and cdf of $\hat{\Phi}_t$, respectively. We assume that $\hat{\Phi}_t \in [\underline{\hat{\Phi}}, \bar{\hat{\Phi}}]$.

Furthermore, Proposition 2 in Athey et al. (2004) shows that the firm's best pricing policy is either a constant price or it is a policy of bounded discretion, i.e.,

$$\hat{p}(z) = \begin{cases} \hat{p}^*(\hat{\Phi}; z) \text{ if } \hat{\Phi} \in [\underline{\hat{\Phi}}, \hat{\Phi}^*] \\ \hat{p}^*(\hat{\Phi}^*; z) \text{ if } \hat{\Phi} \in [\hat{\Phi}^*, \bar{\hat{\Phi}}] \end{cases} \quad \text{(B.9)}$$

where $\hat{p}^*(\hat{\Phi}; z)$ denotes the static best response of a firm with a desired price equal to $\hat{\Phi}$ and $\underline{\hat{\Phi}} \leq \hat{\Phi}^* \leq \bar{\hat{\Phi}}$.

To complete the description of the policy most prefered by the firm, we must calculate four things: 1) Under what conditions does the firm prefer a constant price? 2) What is the optimal constant price from the firm's perspective? 3-4) When the firm prefers to set its price according to equation (B.9), what is the optimal cutoff point $\hat{\Phi}^*$ and what is the firm's static best response $\hat{p}^*(\hat{\Phi}; z)$?

The remainder of this section draws heavily on appendix D in Athey et al. (2004). First, notice that the static best reponse of the firm solves $R_{\hat{p}(z)}(E\hat{p}(z), \hat{p}(z), \hat{\Phi}) = 0$. The solution is

$$\hat{p}^*(\hat{\Phi}, z) = \frac{\gamma}{\theta - 1} + \hat{\Phi}. \tag{B.10}$$

If the firm's pricing policy is of the form (B.9), then

$$E\hat{p}(z) = \int_{\underline{\hat{\Phi}}}^{\hat{\Phi}^*} \hat{p}^*(\hat{\Phi}, z)p(\hat{\Phi})d\hat{\Phi} + \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} \hat{p}^*(\hat{\Phi}^*, z)p(\hat{\Phi})d\hat{\Phi}.$$

Using equation (B.10) to plug in for $\hat{p}^*(\hat{\Phi}, z)$ in this equation we get that

$$E\hat{p}(z) = \frac{\gamma}{\theta - 1} - \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} (\hat{\Phi} - \hat{\Phi}^*)p(\hat{\Phi})d\hat{\Phi}.$$

Athey et al. (2004) show that the objective of the firm, $\int R(E\hat{p}(z), \hat{p}(z), \hat{\Phi})p(\hat{\Phi})d\hat{\Phi}$ may be written

$$R(E\hat{p}(z), \hat{p}^*(\underline{\hat{\Phi}}, z), \underline{\hat{\Phi}}) + \int_{\underline{\hat{\Phi}}}^{\hat{\Phi}^*} R_{\hat{\Phi}}(E\hat{p}(z), \hat{p}^*(\hat{\Phi}, z), \hat{\Phi})[1 - P(\hat{\Phi})]d\hat{\Phi}$$
$$+ \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} R_{\hat{\Phi}}(E\hat{p}(z), \hat{p}^*(\hat{\Phi}^*, z), \hat{\Phi})[1 - P(\hat{\Phi})]d\hat{\Phi}.$$

Since $R_{\hat{\Phi}}(E\hat{p}(z), \hat{p}(z), \hat{\Phi}) = (\theta - 1)\hat{p}(z)$, this expression simplifies to

$$\gamma \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} (\hat{\Phi} - \hat{\Phi}^*)p(\hat{\Phi})d\hat{\Phi} + (\theta - 1) \int_{\underline{\hat{\Phi}}}^{\hat{\Phi}^*} \hat{\Phi}[1 - P(\hat{\Phi})]d\hat{\Phi} + (\theta - 1) \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} \hat{\Phi}^*[1 - P(\hat{\Phi})]d\hat{\Phi} + \text{ex. terms.}$$

Differentiating this with respect to $\hat{\Phi}^*$ and setting the resulting expression equal to zero yields

$$-\gamma \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} p(\hat{\Phi})d\hat{\Phi} + (\theta - 1) \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} [1 - P(\hat{\Phi})]d\hat{\Phi} = 0,$$

which is equivalent to

$$-\gamma[1 - P(\hat{\Phi}^*)] + (\theta - 1) \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} [1 - P(\hat{\Phi})] d\hat{\Phi} = 0.$$

When $\hat{\Phi}^* < \bar{\hat{\Phi}}$, $1 - P(\hat{\Phi}^*) > 0$, so this last equation is equivalent to

$$-\gamma + (\theta - 1) \int_{\hat{\Phi}^*}^{\bar{\hat{\Phi}}} \frac{1 - P(\hat{\Phi})}{p(\hat{\Phi})} \frac{p(\hat{\Phi})}{1 - P(\hat{\Phi}^*)} d\hat{\Phi} = 0. \tag{B.11}$$

Notice that the second term on the left hand side of this equaiton is the conditional mean of $(1 - P(\hat{\Phi}))/p(\hat{\Phi})$ over the interval $[\hat{\Phi}^*, \bar{\hat{\Phi}}]$. Since $(1 - P(\hat{\Phi}))/p(\hat{\Phi})$ is strictly decreasing in $\hat{\Phi}^*$ (monotone hazard assumption), its conditional mean is also strictly decreasing in $\hat{\Phi}^*$. This implies that equation (B.11) has at most one interior solution. Since the expression on the left hand side of equation (B.11) is decreasing in both γ and $\hat{\Phi}^*$, it is furthermore the case that $\hat{\Phi}^*$ is decreasing in γ.

We have shown that equation (B.11) has at most one interior solutions. To show that such a solution in fact exists we must show that the left hand side of this equation is negative for $\hat{\Phi}^*$ close $\bar{\hat{\Phi}}$ and positive for $\hat{\Phi}^* = \underline{\hat{\Phi}}$. Notice that when $\hat{\Phi}^* \to \bar{\hat{\Phi}}$, $(1 - P(\hat{\Phi}))/p(\hat{\Phi}) \to 0$. This implies that for $\gamma > 0$ and $\hat{\Phi}^*$ close enough to $\bar{\hat{\Phi}}$, the left hand side of equation (B.11) is strictly less than zero. When $\hat{\Phi}^* = \bar{\hat{\Phi}}$, equation (B.11) is not defined. However, $\hat{\Phi}^* = \bar{\hat{\Phi}}$ is a solution to the equation above equation (B.11). However, since the expression on the left hand side of that equation is strictly negative for $\hat{\Phi}^* < \bar{\hat{\Phi}}$ in the neighborhood of $\bar{\hat{\Phi}}$, this is not a local maximum.

Athey et al. (2004) show that at $\hat{\Phi}^* = \underline{\hat{\Phi}}$ the left hand side of equation (B.11) becomes $-\gamma - \underline{\hat{\Phi}}$. Since $\underline{\hat{\Phi}} < 0$, this is positive for $\gamma \in (0, -\underline{\hat{\Phi}})$. So, there is an interior solution in this case. When $\gamma > -\underline{\hat{\Phi}}$ there is no interior solution to equation (B.11). This implies that for this range of γ the firm's best policy is a constant price.

Finally, when $\gamma > -\underline{\hat{\Phi}}$ the firm chooses its constant price to maximize

$$\int_{\underline{\hat{\Phi}}}^{\bar{\hat{\Phi}}} R(E\hat{p}(z), \hat{p}(z), \hat{\Phi}) p(\hat{\Phi}) d\hat{\Phi}$$

subject to $E\hat{p}(z) = \hat{p}(z)$. The solution to this problem is $\hat{p}(z) = 0$.

References

Athey, S., Atkeson, A., Kehoe, P. J., 2005. The optimal degree of discretion in monetary policy. Econometrica 73 (5), 1431–1475.