BASIC DEFINITIONS AND NOTATION — ECON 201B

FEDERICO ECHENIQUE

0.1. **Sets and cartesian products.** Let A and B be two sets.

- A^B denotes the set of all functions $f : B \to A$.
- $A \times B$ is the set of ordered pairs (a, b) with $a \in A$ and $b \in B$, the *cartesian product* of A and B.
- For n sets $A_1, \dots, A_n$, we write

  $$\prod_{i=1}^{n} A_i = \times_{i=1}^{n} A_i = \{(a_1, \dots, a_n) : a_i \in A_i \quad 1 \leq i \leq n\}$$

  for the set of all n-tuples $(a_1, \dots, a_n)$ with $a_1 \in A_1, \dots, a_n \in A_n$.
- When $A_i = A$ for all i, $\prod_{i=1}^{n} A_i = A^n$.
- $\mathbf{N}$ denotes the natural numbers: $1, 2, \dots$
- For $n \in \mathbf{N}$, $[n] = \{1, \dots, n\}$ is the set consisting of the first n natural numbers.

If I is a set, and A_i is a set for each $i \in I$, we write $\prod_{i \in I} A_i$ for the set of all functions f with domain I and $f(i) \in A_i$ for all $i \in I$. Note that, when I is finite, this convention is consistent with the notation we introduced for finite cartesian products above.

When $A = \prod_{i \in I} A_i$ and $i \in I$ we often write $A = (A_i, A_{-i})$, where $A_{-i} = \prod_{j \in I \setminus \{i\}} A_j$. Similarly, if $a \in \prod_{i \in I}$ we may write $a = (a_i, a_{-i})$, with $a_i \in A_i$ and $a_{-i} \in A_{-i}$. For example, when $a = (a_1, \dots, a_n) \in A$ then we write $a = (a_i, a_{-i})$, with $a_{-i} \in A_{-i} = \times_{j \neq i} A_j$. This notation is very convenient when we want to discuss changes in a_i holding fixed a_{-i}. For example, when $f : A \to \mathbf{R}$, we may be interested in the property that $f(a_i, a_{-i}) > f(a_i', a_{-i})$ for all $a_{-i} \in A_{-i}$.

0.2. **Vector spaces.**

- $\mathbf{R}$ denotes the real numbers;
- $\mathbf{R}^n = \mathbf{R}^{[n]}$ is the n-dimensional Euclidean space; the elements of $\mathbf{R}^n$ are *vectors*.
- $x \cdot y = \sum_{i=1}^{n} x_i y_i$ is the *inner product* of $x, y \in \mathbf{R}^n$.

- If $(p,\alpha) \in \mathbf{R}^n \times \mathbf{R}$, $H(p,\alpha) = \{x \in \mathbf{R}^n : p \cdot x = \alpha\}$ is a *hyperplane*.
- $\|x\| = \sqrt{x \cdot x}$ is the *Euclidean norm* of $x \in \mathbf{R}^n$.

If $x, y \in \mathbf{R}^n$ then we say that $x \leq y$ if $x_i \leq y_i$ (as real numbers) for $1 \leq i \leq n$. We also have $x \ll y$ if $x_i < y_i$ for $1 \leq i \leq n$, and $x < y$ if $x \leq y$ and $x \neq y$. $\mathbf{R}^n_+$ is the set of vectors $x \in \mathbf{R}^n$ with $x_i \geq 0$.

If $A, B \subseteq \mathbf{R}^n$ then $A + B = \{x + y : x \in A, y \in B\}$ is their sum. Make sure that you understand, for example, $\{x\} + \mathbf{R}^n_+$ for $x \in \mathbf{R}^n$.

A subset $A \subseteq \mathbf{R}^n$ is *convex* if $\lambda x + (1-\lambda)y \in A$ for all $x, y \in A$ and $\lambda \in [0,1]$. $\lambda x + (1-\lambda)y$ is a *convex combination* of x and y. The *convex hull* of a set A is the intersection of all convex sets that contain A; or, equivalently, the set of all convex combinations of elements from A.

0.3. **Probability and expectation.** Suppose that A is a finite set.

- $\Delta(A)$ denotes the set of probability distributions on A. That is $\Delta(A) = \{p \in \mathbf{R}^A_+ : \sum_{a \in A} p(a) = 1\}$.
- If $p \in \Delta(A)$, for any subset $B \subseteq A$ we can calculate
$$p(B) = \sum_{a \in B} p(a),$$
with $p(\emptyset) = 0$ and $p(A) = 1$.
- We could alternatively define probabilities as functions $p : 2^A \to \mathbf{R}_+$, so that $p(\emptyset) = 0$, $p(A) = 1$, and $P(C \cup B) = p(C) + p(B)$ when $C \cap B = \emptyset$.
- For $p \in \Delta(A)$, the *support* of p is the set
$$\text{supp}(p) = \{a \in A : p(a) > 0\}$$
of elements with strictly positive probability.
- If $f : A \to \mathbf{R}$ and $p \in \Delta(A)$ then we write the mathematical expectation of f under p as
$$\mathbf{E}_p f = \sum_{a \in A} f(a)p(a) = \int_A f(a)\,\mathrm{d}p(a).$$
To distinguish the variable of integration, we may on occasion write $\mathbf{E}_p f(\tilde{a})$.

Often, as a notational shortcut, we write $f(p) = \mathbf{E}_p f$. When $A = \prod_{i=1}^n A_i$ and $p = (p_1, \ldots, p_n) \in \prod_{i=1}^n \Delta(A_i)$ then we interpret p as the probability on A obtained as the independent mixture of each marginal

p_i. That is, for each $B = \prod_{i=1}^n B_i$, with $B_i \subseteq A_i$, we have $p(B) = p(B_1) \cdot p(B_2) \cdots p(B_n)$. And we write

$$\begin{aligned} f(p) = f(p_1, \ldots, p_n) &= \sum_{a_1 \in A_1} \cdots \sum_{a_n \in A_n} p(a_1) \cdots p(a_n) f(a_1, \ldots, a_n) \\ &= \sum_{a_i \in A_i} p(a_i) f(a_i, p_{-i}) \\ &= \mathbf{E}_{p_i} f(\tilde{a}_i, p_{-i}) = f(p_i, p_{-i}). \end{aligned}$$

When $f : A \to \mathbf{R}^n$, then $\mathbf{E}_p f = (\mathbf{E}_p f_1, \ldots, \mathbf{E}_p f_n)$.

If $x, y \in \mathbf{R}^n$ and $\lambda \in [0, 1]$, we may interpret $(\lambda, 1 - \lambda)$ as a probability on $\{x, y\}$, and the convex combination $\lambda x + (1 - \lambda) y$ as its expectation. In the same spirit, if $U = \{u_1, \ldots, u_m\} \subseteq \mathbf{R}^n$ is a finite set with m distinct elements, and $\lambda_i \in [0, 1]$ $1 \leq i \leq m$, with $\sum_i \lambda_i = 1$, then we can interpret $(\lambda_1, \ldots, \lambda_m)$ as a probability on U, and the convex combination $\sum_i \lambda u_i \in \mathbf{R}^n$ as its expectation. Then the convex hull of U is the set of expected values of probabilities on U.

In Econ 201b, we will most of the time work with probabilities over finite sets. On occasion, when $A \subseteq \mathbf{R}^n$ may not be a finite set, we write $\Delta(A)$ for the set of all Borel probability measures on A.

0.4. **Sequences.** Fix a set A.

- A function mapping $[n] = \{1, \ldots, n\} \to A$ is a *sequence of length* n in A. We write it as
$$h = (a_1, \ldots, a_n).$$
- A^n denotes the set of all sequences of length n.
- $\cup_{n \in \mathbf{N}} A^n$ denotes all finite sequences in A.
- A function from $\mathbf{N}$ to A is an *infinite* sequence.

When
$$h' = (a'_1, \ldots, a'_n, a'_{n+1}, \ldots)$$
is a sequence (finite or infinite), and $h = (a_1, \ldots, a_n)$ is a sequence with $a'_i = a_i$, we say that h' is a *continuation* of h and that h is a *prefix* of h'.