Expected Utility and Risk

Federico Echenique
Caltech – SS205a

November 24, 2021

Daniel Bernoulli

Before Bernoulli, it was thought risky prospects should be evaluated by its expected value.

"Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats."

"...the determination of the value of an item must not be based on its price, but rather on the utility it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount."

St. Petersburg paradox

Proposed by Daniel Bernoulli's cousin, Nicolas Bernoulli.

Consider a gamble where a casino tosses a fair coin until heads comes up.

If heads comes comes at the *nth* toss for the first time, the payoff is 2^n ducats (the payout duplicates for each time tails comes up).

How much is such a gamble worth?

The expected payoff is

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\frac{1}{2})^n 2^n = \infty,$$

Nicolas thought such a gamble was never worth more than 20 ducats.

St. Petersburg paradox

Daniel suggested that marginal utility should be inversely proportional to wealth:

$$Du(x) = \frac{a}{x}.$$

So

$$u(x) = C + \int_1^x \frac{a}{t} dt = C + a \log(x).$$

Or, taking an affine transformation, $u(x) = \log(x)$.
Then expected utility

$$\lim_{n \to \infty} \sum_{i=1}^{n} (\frac{1}{2})^n \log(2^n)$$

is finite.

Wait, what is a ducat???

1724 Dutch dukaat (Wikipedia).

Digression: probability distributions

Let $X = [a, b]$ and let $\Delta(X)$ be the set of all (Borel) probability measures on X.
For each $\mu \in \Delta(X)$, can define a function $F_\mu : X \to \mathrm{R}$ by

$$F_\mu(t) = \mu\big(\{x \in X : x \leq t\}\big)$$

called the *cumulative distribution function* (cdf) associated to μ.

Proposition

Any cdf F_μ satisfies:

1. F_μ is (weakly) monotone increasing.
2. F_μ is right-continuous.
3. $F_\mu(b) = 1$.
4. $\lim_{t \to a} F(t) = 0$.

Conversely, a function that satisfies these properties is the cdf of some probability distribution on X.

Digression: probability distributions

In fact, if $F : [a, b] \to R$ satisfies the properties in the proposition, we may define

$$X(t) = \inf\{x \in [a, b] : F(x) \geq t\}.$$

(So if F is st. increasing and cont. then $X(t) = F^{-1}(t)$.)

Consider X as a r.v. defined on the prob. space $[0, 1]$ with the uniform distribution.

Then we have:

Proposition

The distribution of X has cdf F.

Expectation

When $\mu \in \Delta(X)$, we write

$$\mathrm{E}_\mu = \int_X x d\mu(x) = \int_X x dF_\mu(x)$$

for its expectation.

And for an integrable function $g : X \to \mathrm{R}$,

$$\mathrm{E}_\mu g(x) = \int_X g(x) d\mu(x)$$

Digression: Jensen's inequality.

Consider an integrable function $u : X \to \mathrm{R}$.

Theorem (Jensen's inequality)

u is concave iff

$$\int_X u(x)d\mu(x) \leq u\Big(\int_X x d\mu(x)\Big)$$

for all $\mu \in \Delta(X)$

Digression: Jensen's inequality – Proof.

First we prove ($\Longrightarrow$).

Let $x^* = \mathrm{E}_\mu$. Since u is concave there exists b, a supergradient of u at x^*. Thus

$$u(x) \leq u(x^*) + b(x - x^*)$$

for all $x \in X$.

Integrating, we obtain that

$$\begin{aligned}\int_X u(x) d\mu(x) &\leq \int_X \Big(u(x^*) + b(x - x^*)\Big) d\mu(x) \\ &= u(x^*) + b \int_X (x - x^*) d\mu(x) \\ &= u(x^*)\end{aligned}$$

Digression: Jensen's inequality – Proof.

Now turn to ($\Longleftarrow$).

Let $x, y \in X$ and $\lambda \in (0,1)$.

Let $\mu \in \Delta(X)$ be the probability dist. that assigns probability λ to x and $(1-\lambda)$ to y.

Then Jensen's inequality is the defining inequality of concave functions. □

Monetary lotteries.

Interpret $x \in X$ as a monetary payment, and $\mu \in \Delta(X)$ as a *lottery* over monetary payments.

We assume an agent who chooses among lotteries according to a utility function $V : \Delta(X) \to \mathrm{R}$. (She has a preference over lotteries that has utility representation V.)

Assume V has the *expected utility form*, meaning that there is an integrable $u : X \to \mathrm{R}$ s.t

$$V(\mu) = \int_X u(x) d\mu(x).$$

Monetary lotteries.

The function u is called the *Bernoulli* utility associated with V (also called a vonNeumann-Morgenstern utility).

In ss201(b) you will investigate the foundations behind expected utility.

Note: we shall also write

$$V(F_\mu) = \int_a^b u(t)dF_\mu(t)$$

Monetary lotteries.

Finally, the Bernoulli utility is only unique up to a positive affine transformation.

This means that the preferences over lotteries represented by

$$V(\mu) = \int_X u(x) d\mu(x).$$

are the same as those represented by

$$W(\mu) = \int_X (\alpha + \beta u(x)) d\mu(x),$$

for $\beta > 0$.

We shall take advantage of this fact some times and use a particular "normalization" of a Bernoulli utility.

For example if pick a particular $x_0 \in X$ we can wlog assume that $u(x_0) = 0$.

Monetary lotteries: FOSD.

First-order stochastic dominance (FOSD): a partial order on lotteries based on the idea that more money is preferred to less money.

What do all agents who prefer more money to less agree on?

We say that a lottery μ first-order stochastically dominates ν if all agents who prefer more money to less would rather have μ than ν.

Monetary lotteries: FOSD.

Let U_1 be the set of all monotone increasing functions $u : X \to \mathbb{R}$.

Define a binary relation $\geq_1$ on $\Delta(X)$ by

$$\mu \geq_1 \nu \text{ iff } \int u d\mu \geq \int u d\nu \quad \forall u \in U_1.$$

In words: we say that a probability distribution F_μ *first-order stochastically dominates* F_ν if $\mu \geq_1 \nu$.

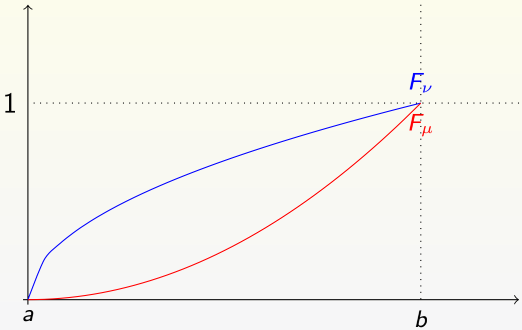
The identity function is monotone increasing, so $\mu \geq_1 \nu$ implies that $\mathrm{E}_\mu \geq \mathrm{E}_\nu$.

Monetary lotteries: FOSD.

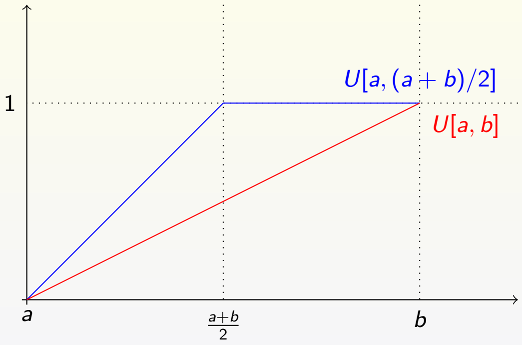
Theorem

$\mu \geq_1 \nu$ iff $F_\nu(x) \geq F_\mu(x)$ for all $x \in X$.

Ex



Ex



Proof:

($\Longrightarrow$) The functions

$$t \mapsto \begin{cases} 0 & \text{if } t < x \\ 1 & \text{if } t \geq x \end{cases}$$

are monotone increasing.
So $\mu \geq_1 \nu$ implies that $1 - F_\nu(x) \leq 1 - F_\mu(x)$, and hence that $F_\nu(x) \geq F_\mu(x)$.

Proof:

($\Longleftarrow$) Suppose that $F_\nu(x) \geq F_\mu(x)$ for all $x \in [a, b]$ and define

$$X(t) = \inf\{x \in [a, b] : F_\mu(x) \geq t\}$$
$$Y(t) = \inf\{x \in [a, b] : F_\nu(x) \geq t\}.$$

By our prior result, we know that $X \sim F_\mu$ and $Y \sim F_\nu$. By hypothesis, $X(t) \geq Y(t)$.
Fix any monotone increasing function $u \in U_1$. Then $X(t) \geq Y(t)$ for all t implies that

$$\begin{aligned} \int u dF_\mu &= \int u(X(t))dt \\ &\geq \int u(Y(t))dt \\ &= \int u dF_\nu \end{aligned}$$

Monetary lotteries: FOSD.

Corollary

$\mu \geq_1 \nu$ iff there are three r.v: X, Y, and Z with $X \sim F_\mu$, $Y \sim F_\nu$, $Z \geq 0$ s.t

$$X = Y + Z$$

To prove the corollary, construct X and Y as in the proof of the theorem and note that $Z = X - Y \geq 0$.

Monetary lotteries: Risk aversion

What is risk aversion?

Would you like \$100 for sure, or a lottery that pays \$0 or \$200 with equal probability?

We'll equate risk aversion with a preference for the mean value of a lottery over the lottery itself.

If you think back to the result on Jensen's inequality, risk aversion thus defined characterizes concave Bernoulli utility.

Monetary lotteries: Risk aversion

Consider an agent with expected utility V and associated Bernoulli utility u.

We say that the agent is *risk averse* if, for all $\mu \in \Delta(X)$,

$$\int_X u(x) d\mu(x) \leq u(\mathsf{E}_\mu).$$

We can now re-state our result on Jensen's inequality:

Proposition

An agent with Bernoulli utility u is risk averse iff u is concave.

Monetary lotteries: Risk

Let U_2 be the set of all concave functions $u : X \to \mathbb{R}$.

Define $\geq_2$ on $\Delta(X)$ by

$$\mu \geq_2 \nu \text{ iff } \int u d\mu \geq \int u d\nu \quad \forall u \in U_2.$$

Recall that if g is convex, then $-g$ is concave, so $\mu \geq_2 \nu$ iff $E_\mu g(x) \leq E_\nu g(x)$ for all convex g.

The identity function is concave and convex, so $\mu \geq_2 \nu$ implies that $E_\mu = E_\nu$.

And the function x^2 is convex, so $\mu \geq_2 \nu$ implies that ν has higher variance than μ.

Monetary lotteries: Risk

Theorem

$\mu \geq_2 \nu$ iff

$$\mathrm{E}_\mu = \mathrm{E}_\nu$$

$$\text{and} \qquad \int_a^x F_\nu(s)ds \geq \int_a^x F_\mu(s)ds$$

for all $x \in X$.

Proof

For any cdf F:

$$\begin{aligned}\int_a^x F(t)dt &= tF(t)\Big|_a^x - \int_a^x t dF(t) \\ &= x\int_a^x dF(t) - \int_a^x t dF(t) \\ &= \int_a^x (x-t) dF(t) \\ &= \int_a^b \max\{x-t, 0\} dF(t).\end{aligned}$$

Proof

But the function $t \mapsto \max\{x - t, 0\}$ is convex, as it is the max of two affine functions.

Proof

This means that $t \mapsto -\max\{x-t,0\}$ is concave, and thus $\mu \geq_2 \nu$ implies

$$\int_a^b (-1)\max\{x-t,0\}dF_\mu(t) \geq \int_a^b (-1)\max\{x-t,0\}dF_\nu(t).$$

Then:

$$\begin{aligned}\int_a^x F_\mu(t)dt &= \int_a^b \max\{x-t,0\}dF_\mu(t)\\ &\leq \int_a^b \max\{x-t,0\}dF_\nu(t)\\ &= \int_a^x F_\nu(t)dt.\end{aligned}$$

The identity function is both convex and concave, so $\int x dF_\mu(x) = \int x dF_\nu(x)$.

Proof

I'm not going to prove the converse.

The idea is simple: any concave function can be approximated by positive linear combinations of functions of the form $t \mapsto (-1)\max\{x - t, 0\}$, constants, and the identity function.

Monetary lotteries: Risk

Theorem

$\mu \geq_2 \nu$ iff there are three r.v: X, Y, and Z (defined on the same probability space) with $X \sim F_\mu$, $Y \sim F_\nu$ and $\mathrm{E}(Z|X) = 0$ s.t

$$Y = X + Z$$

This is a lot harder to prove and I'll omit the proof.

Monetary lotteries: SOSD

We're mainly interested in agents who are both risk-averse and prefer more money to less.

We've seen that FOSD is the answer when we ask what all agents who prefere more money to less agree on.

So what do all agents who, 1) prefer more money to less, and 2) are risk averse, agree on?

The answer will be second-order stochastic dominance (SOSD).

Monetary lotteries: SOSD

Let $U_{12} = U_1 \cap U_2$.

Define $\geq_{12}$ on $\Delta(X)$ by

$$\mu \geq_{12} \nu \text{ iff } \int u d\mu \geq \int u d\nu \quad \forall u \in U_{12}.$$

Monetary lotteries: SOSD

In words: we say that a probability distribution F_μ *second-order stochastically dominates* F_ν if $\mu \geq_{12} \nu$.

Theorem

$\mu \geq_{12} \nu$ iff

$$\int_a^x F_\nu(s)ds \geq \int_a^x F_\mu(s)ds$$

for all $x \in X$.

Monetary lotteries: SOSD

Theorem

$\mu \geq_{12} \nu$ iff there are three r.v: X, Y, and Z (defined on the same probability space) with $X \sim F_\mu$, $Y \sim F_\nu$ and $\mathrm{E}(Z|X) \leq 0$ s.t

$$Y = X + Z$$

Mathematical aside

Note that the binary relations $\geq_i$ for $i \in \{1, 2, 12\}$ have been defined through a positive cone that results as the positive vectors of a collection of linear functions.

$\Delta(X)$ is not a vector space, but it can be embedded into a vector space of *signed measures*.

Then $\mu \geq_i \nu$ iff $\mu - \nu \in P_i$.

Where

$$P_i = \{\mu : f(\mu) \geq 0 \text{ for all } f \in \mathcal{F}_i\}$$

$\mathcal{F}_i$ is the family $\mu \mapsto \int u d\mu$ for $u \in U_i$. A collection of linear functions.

This is as in our discussion of partial orders. Indeed $\geq_i$ is a partial order

Risk aversion

For the rest of our discussion, we consider only Bernoulli utilities that are cont. and strictly monotonically increasing.

For an agent with a Bernoulli utility u, the number $c(\mu, u)$ is a *certainty equivalent* for lottery μ and agent u if it satisfies that:

$$\int u d\mu = u(c(\mu, u))$$

Proposition

If u is strictly monotonically increasing and continuous, $c(\mu, u)$ exists and is unique.

Risk aversion

Proposition

An agent with Bernoulli utility u is risk averse iff $c(\mu, u) \leq \mathsf{E}_\mu$ for all lotteries $\mu \in \Delta(X)$.

Proof.

The following are equivalent:

$$c(\mu, u) \leq \mathsf{E}_\mu$$
$$u(c(\mu, u)) \leq u(\mathsf{E}_\mu)$$
$$\int u d\mu \leq u(\mathsf{E}_\mu)$$

First, because u is st. increasing, and second by defn. of certainty equivalent.

Risk aversion

The *risk premium* of μ for u is

$$R(\mu, u) = \mathsf{E}_\mu - c(\mu, u).$$

So $R(\mu, u) \geq 0$ when u is risk averse.

Comparative risk aversion

Consider two agents, one with Bernoulli utility u and the other with v. What does it mean to say that one is more risk averse than the other?

Say that u is *at least as risk averse* as v if whenever $E_\mu u(x) \geq u(\theta)$ then $E_\mu v(x) \geq v(\theta)$.

Proposition

u is at least as risk averse as v iff there exists a strictly increasing and concave function g with $u = g \circ v$.

Say that u is a *concave transformation* of v.

Proof:

Let g be such a function and let μ and θ be such that $\mathsf{E}_\mu u(x) \geq u(\theta)$.

Then $u = g \circ v$ implies that $\mathsf{E}_\mu g(v(x)) \geq g(v(\theta))$, or $g^{-1}(\mathsf{E}_\mu g(v(x))) \geq v(\theta)$. ($g$ is st. inc. so has an inverse.)

Moreover, by concavity of g,

$$\mathsf{E}_\mu g(v(x)) \leq g(\mathsf{E}_\mu v(x)),$$

which means that

$$g^{-1}(\mathsf{E}_\mu g(v(x))) \leq g^{-1}(g(\mathsf{E}_\mu v(x))) = \mathsf{E}_\mu v(x)$$

Hence,

$$\begin{aligned} \mathsf{E}_\mu v(x) &\geq g^{-1}(\mathsf{E}_\mu g(v(x))) \\ &\geq v(\theta). \end{aligned}$$

Proof:

Conversely, let $g : v(X) \to \mathbb{R}$ be a strictly increasing function s.t $u(x) = g(v(x))$. Such g exists because both u and v are strictly increasing (and thus are ordinally equivalent).

Obs. that the range of v is an interval and hence convex.

Suppose (towards a contradiction) that g is not concave. Then there's z, y and λ s.t

$$g(\lambda z + (1-\lambda)y) < \lambda g(z) + (1-\lambda)g(y)$$

Let $z = v(z')$ and $y = v(y')$ and consider the lottery that assigns z' probability λ and y' probability $1-\lambda$.

Proof:

Then

$$g(\mathsf{E}_\mu v(x)) < \mathsf{E}_\mu g(v(x)) = \mathsf{E}_\mu u(x) = u(c(u,\mu))$$

Thus

$$\mathsf{E}_\mu v(x) < g^{-1}(u(c(u,\mu))) = v(c(u,\mu))$$

A contradiction because u is willing to give up the sure amount $c(u,\mu)$ for μ (actually indifferent between the two), while v is not willing to do that. □

Comparative risk aversion

Exercise

Show that u is at least as risk averse as v iff for all μ,

$$c(\mu, u) \leq c(\mu, v)$$

Comparative risk aversion

Suppose now that u and v are C^2. By the proposition we proved, we know that u is at least as risk averse as v iff there's a concave g with $u = g \circ v$.

Then we have

$$u'(x) = g'(v(x))v'(x)$$
$$u''(x) = g''(v(x))[v'(x)]^2 + g'(v(x))v''(x)$$

Recall that $g' > 0$ as g is st. inc. and the concavity of g is equivalent to $g'' \leq 0$.

Comparative risk aversion

Thus

$$\frac{-u''(x)}{u'(x)} = \underbrace{\frac{-g''(v(x))[v'(x)]^2}{g'(v(x))v'(x)}}_{\geq 0} - \frac{g'(v(x))v''(x)}{g'(v(x))v'(x)}$$

So we can say in this case that u is at least as risk averse as v if

$$\frac{-u''(x)}{u'(x)} \geq \frac{-v''(x)}{v'(x)}$$

The magnitude

$$r_A(x,u) = \frac{-u''(x)}{u'(x)}$$

is a (local) measure of the curvature of u; of the risk aversion of the agent with utility u.

It is called the *Arrow-Pratt coefficient of absolute risk aversion*

Absolute risk aversion

$r_A(x, u)$ depends on x. When is it the same for all x?

$$r_A(x, u) = \frac{-u''(x)}{u'(x)} = \rho$$

defines a differential equation.

Solving it yields:

$$u(x) = C_1 - C_2 e^{-\rho x},$$

which is equivalent to $-e^{-\rho x}$.

This is known as the *constant absolute risk aversion (CARA)* utility.

Risk aversion and wealth

Remember the Bernoulli quote.

Daniel thought that the value of a gamble depends on how wealthy a person is.

Like a 50-50 lottery that gives 20,000 ducats or nothing is worth less to a poor person than to a rich person.

Fix a cont. and st. inc. utility function $u : \mathrm{R} \to \mathrm{R}$.

Consider the family of utilities u_w, where

$$u_w = u(w + x).$$

Interpret u_w as the utility of an agent. with wealth $w \in \mathrm{R}_+$.

Risk aversion and wealth

Now the agent with utility u exhibits *decreasing risk version* if u_w is at least as risk averse as $u_{w'}$ when $w < w'$.

Under the smoothness conditions we discussed, this happens with $r_A(x, u)$ is monotone decreasing in x.

Risk aversion and wealth

Recall that $\Delta([a,b])$. Now assume $a>0$.

Consider a multiplicative risk relative to wealth w; think for ex. about risk related to a rate of return.

Let $\tilde{u}_w(x) = u(wx)$.

Let $w < w'$ and suppose that $\tilde{u}_w$ is at least as risk averse as $\tilde{u}_{w'}$. Then there is a concave g s.t $u(wx) = g(u(w'x))$.

Risk aversion and wealth

This gives

$$\begin{aligned} wu'(wx) &= g'(u(w'x))u'(w'x)w' \\ w^2u''(wx) &= g''(u(w'x))[u'(w'x)w']^2 + g'(u(w'x))u''(w'x)(w')^2 \end{aligned}$$

Then

$$\frac{w^2u''(wx)}{wu'(wx)} = \frac{g''(u(w'x))[u'(w'x)w']^2}{g'(u(w'x))u'(w'x)w'} + \frac{g'(u(w'x))u''(w'x)(w')^2}{g'(u(w'x))u'(w'x)w'},$$

which implies that

$$\frac{-wu''_w(x)}{u'_w(x)} \geq \frac{-w'u''(w'x)}{u'(w'x)}$$

Risk aversion and wealth

This suggests the *coefficient of relative risk aversion:*

$$r_R(x,u) = \frac{-xu''(x)}{u'(x)}$$

So that when $r_R(x,u)$ is a decreasing function in x then u exhibits decreasing risk aversion.
The boundary case when $r_R(x,u)$ is constant implies ($\rho \neq 1$) that

$$\frac{-xu''(x)}{u'(x)} = \rho,$$

which results (fixing a normalization) in

$$u(x) = \frac{x^{1-\rho}}{1-\rho};$$

the *constant relative risk aversion (CRRA)* utility.

Application: demand for insurance

Consider an agent with Bernoulli utility u and wealth w who considers buying fire insurance.

A fire happens with prob. π and when it happens the loss is D.

One "unit" of insurance costs q and pays out one dollar if there is a fire.

When the agent buys a quantity $\alpha \geq 0$ of insurance, their wealth is $w - \alpha q$ if there is no fire, and $w - D + \alpha - \alpha q$ if there is a fire.

So the agent's problem is to maximize

$$\pi u(w - D + \alpha - \alpha q) + (1 - \pi)u(w - \alpha q)$$

Application: demand for insurance

Now suppose that $q = \pi$, meaning that the insurance contract is *actuarially fair*. Then the agent's wealth is $w - \alpha\pi$ if there is no fire, and $w - D + \alpha - \alpha\pi$ if there is a fire.

The expected wealth is then

$$\pi(w - D + \alpha(1 - \pi)) + (1 - \pi)(w - \alpha\pi) = w - \pi D$$

By setting $\alpha = D$ the agent can make sure to get $w - \pi D$ in each state.

By Jensen's inequality, this is the best that the agent can do.

Punchline: when offered actuarially fair insurance, a risk averse agent will *fully insure*.

Application: demand for insurance

Assuming u is smooth, we may also calculate the first order conditions:

$$\pi u'(w - D + \alpha(1-q))(1-q) + (1-\pi)u'(w - \alpha q)q = \begin{cases} \leq 0 \text{ if } \alpha = 0 \\ = 0 \text{ if } \alpha = 0 \end{cases}$$

And you can show the prev. result using the FOC (see the book).

Expected Utility: A critical assessment.

First choose between

- Receiving $ 1,000,000 for sure.
- A lottery that pays
  - $ 5,000,000 w/prob. 0.10,
  - $ 1,000,000 w/prob. 0.89,
  - $ 0 w/prob. 0.01

Second, choose between

- A lottery that pays
  - $ 1,000,000 w/prob. 0.11
  - $ 0 w/prob. 0.89
- A lottery that pays
  - $ 5,000,000 w/prob. 0.10
  - $ 0 w/prob. 0.90.

Expected Utility: A critical assessment.

First choose between

- Receiving $ 1,000,000 for sure.
- A lottery that pays
  - $ 5,000,000 w/prob. 0.10,
  - $ 1,000,000 w/prob. 0.89,
  - $ 0 w/prob. 0.01

Second, choose between

- A lottery that pays
  - $ 1,000,000 w/prob. 0.11
  - $ 0 w/prob. 0.89
- A lottery that pays
  - $ 5,000,000 w/prob. 0.10
  - $ 0 w/prob. 0.90.

Expected Utility: A critical assessment.

The Allais paradox: These choices are incompatible with expected utility theory. Indeed:

The first choice implies that

$$u(10^6) > 0.1u(5 \times 10^6) + 0.89u(10^6) + 0.01u(0),$$

or

$$0.11u(10^6) > 0.1u(5 \times 10^6) + 0.01u(0).$$

The second choice implies that

$$0.11u(10^6) + 0.89u(0) < 0.1u(5 \times 10^6) + 0.9u(0),$$

or

$$0.11u(10^6) < 0.1u(5 \times 10^6) + 0.01u(0).$$

Expected Utility: A critical assessment.

I'll mention another critique of EU that's closely related to the theory we have discussed, of choice over monetary lotteries. It was most clearly formulated by Matthew Rabin.

Idea: suppose an agent rejects a 50-50 lottery to win \$11 or lose \$10 at all wealth levels.

Then this agent must reject a 50-50 lottery to win G and lose \$100, **no matter how high G is.**

Expected Utility: A critical assessment.

Suppose an agent who rejects a 50-50 lottery that gains g and looses l, for all wealth levels $w \in [a, b]$. Suppose also (for simplicity) that $b = a + k(l + g)$ for $k \in Z_+$.

We'll see what this rejection implies for a 50-50 lottery to win G or lose L.

Rejecting lottery at wealth $w + l$ means that

$$u(w) + u(w + g + l) \leq 2u(w + l)$$

Rabin calibration

Hence,

$$u(w) - u(w + l) \leq u(w + l) - u(w + g + l)$$

By concavity:

$$u(w + l) - u(w) \leq u'(w)l$$

Thus, using concavity again:

$$u'(w + l + g) \leq \frac{u(w + l + g) - u(w + l)}{g} \leq \frac{u(w + l) - u(w)}{g}.$$

Rabin calibration

Hence,

$$u'(w+l+g) \leq \frac{u(w+l)-u(w)}{l}\frac{l}{g} \leq u'(w)(l/g)$$

This means that

$$u'(b) \leq u'(a)(l/g)^k$$

Normalize so that $u(a)=0$ and $u'(a)=1$. Then

$$u'(b) \leq (l/g)^k \text{ and thus } u(b+G) \leq u(b)+(l/g)^k G$$

Notice that when k is large, even a very large gain G will not give much bigger utility than b. In consequence, the agent may reject lotteries that seem very favorable.

Rabin calibration

TABLE I

IF AVERSE TO 50-50 LOSE \$100 / GAIN g BETS FOR ALL WEALTH LEVELS, WILL TURN DOWN 50-50 LOSE L / GAIN G BETS; G'S ENTERED IN TABLE.

		g		
L	\$101	\$105	\$110	\$125
\$400	400	420	550	1,250
\$600	600	730	990	∞
\$800	800	1,050	2,090	∞
\$1,000	1,010	1,570	∞	∞
\$2,000	2,320	∞	∞	∞
\$4,000	5,750	∞	∞	∞
\$6,000	11,810	∞	∞	∞
\$8,000	34,940	∞	∞	∞
\$10,000	∞	∞	∞	∞
\$20,000	∞	∞	∞	∞

Expected Utility: A critical assessment.

Does this mean that EU theory should be abandoned? You should furnish your own answer.

In EU's defense, it's a useful tool in economic models, and gives the "right" economic behavior in models of markets, at least to a first-order approximation. For example making agents respond to prices in the ways that roughly resemble observed data.

And the hypothesis in Rabin's calibration that small-stakes gambled are rejected for a wide range of wealth levels has been questioned empirically.