WARP and the Law of Demand

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Law of demand

Weak Axiom of Revealed Preference (WARP)

Let $X \subset \mathrm{R}^n_+$ be the consumption set.

For an ordinary demand function $x^*\colon \mathrm{R}^n_{++} \times \mathrm{R}_{++} \to X$, define the binary relation S on X by

$$x \, S \, y \text{ if } (\, \exists (p, w) \,) \, [\, x = x^*(p, w) \;\&\; y \neq x \;\&\; p \cdot y \leq w \,].$$

That is, x is demanded when y is in the budget set but not demanded, so x is *revealed preferred* to y.

Weak Axiom of Revealed Preference (WARP)

The demand function x^* obeys *Samuelson's Weak Axiom of Revealed Preference (SWARP)* if S is an asymmetric relation.

That is, if for every $x, y \in X$,

$$x \, S \, y \implies \neg y \, S \, x.$$

In other words, if x is revealed preferred to y, then y is never revealed preferred to x.

The demand function x^* satisfies *budget exhaustion* if $\forall (p, w)$,

$$p \cdot x^*(p, w) = w.$$

WARP

Assuming budget exhaustion, We may now rewrite SWARP in the form that Samuelson used.

Let x^0 and x^1 belong to the range of x^*. That is, let

$$x^0 = x^*(p^0, w^0) = x^*(p^0, p^0 \cdot x^0) \text{ and } x^1 = x^*(p^1, w^1) = x^*(p^1, p^1 \cdot x^1).$$

Then $p^1 \cdot x^0 \leq p^1 \cdot x^1$ and $x^0 \neq x^1$ imply $x^1 \; S \; x^0$; while $x^0 \neq x^1$ and $\neg x^0 \; S \; x^1$ imply $p^0 \cdot x^1 > p^0 \cdot x^0$.
SWARP becomes:

$$x^0 \neq x^1 \text{ and } p^1 \cdot x^0 \leq p^1 \cdot x^1 \quad \Longrightarrow \quad p^0 \cdot x^1 > p^0 \cdot x^0.$$

Slutsky compensated demand

Define the *Slutsky compensated demand* s from the ordinary demand function x^* as

$$s(p, \bar{x}) = x^*(p, p \cdot \bar{x}).$$

So if $\bar{x} = x^*(\bar{p}, \bar{w})$, then $s(p, \bar{x})$ is the demand $x^*(p, w)$ where w has been adjusted (compensated) so that consumption $\bar{x}$ is still just affordable at price vector p.

Slutsky compensated demand

Lemma

Let x^* satisfy the budget exhaustion condition and SWARP. Let

$$x^0 = x^*(p^0, w^0) \text{ and } x^1 = x^*(p^1, p^1 \cdot x^0).$$

Then

$$(p^1 - p^0) \cdot (x^1 - x^0) \leq 0,$$

with equality if and only if $x^1 = x^0$.

This property is a version of the law of demand: A LOD for *compensated* price changes.

Slutsky compensated demand

Proof.

If $x^1 = x^0$, then the conclusion is true as an equality. So assume $x^1 \neq x^0$. By budget exhaustion

$$p^1 \cdot x^1 = p^1 \cdot x^0. \tag{1}$$

Since $x^1 \neq x^0$, this says that $x^1 \; S \; x^0$. So by SWARP, we have $\neg x^0 \; S \; x^1$, that is,

$$p^0 \cdot x^1 > w^0 = p^0 \cdot x^0, \tag{2}$$

where the second equality follows from budget exhaustion. Subtracting inequality (2) from equality (1) gives

$$(p^1 - p^0) \cdot x^1 < (p^1 - p^0) \cdot x^0,$$

which proves the conclusion of the lemma.

Slutsky substitution matrix

Suppose that the ordinary demand function is C^1.

Note that

$$\frac{\partial s_i(p,\bar{x})}{\partial p_j} = \frac{\partial x_i^*(p, p\cdot\bar{x})}{\partial p_j} + \bar{x}_j \frac{\partial x_i^*(p, p\cdot\bar{x})}{\partial w}.$$

In particular, by setting $\bar{x} = x^*(p,w)$ we may define the *Slutsky substitution term*

$$\begin{aligned}\sigma_{i,j}(p,w) &= \frac{\partial s_i\big(p, x^*(p,w)\big)}{\partial p_j} \\ &= \frac{\partial x_i^*(p,w)}{\partial p_j} + x_j^*(p,w)\frac{\partial x_i^*(p,w)}{\partial w}.\end{aligned}$$

Slutsky substitution matrix

Interpret:

$$\sigma_{i,j}(p, w) = \frac{\partial x_i^*(p, w)}{\partial p_j} + x_j^*(p, w)\frac{\partial x_i^*(p, w)}{\partial w}.$$

Obs. that $x_j^*(p, w)\frac{\partial x_i^*(p,w)}{\partial w}$ captures the effect of a differential change in p_j on demand for i through the change in income needed to compensate for the change in price.

WARP and the substitution matrix

Theorem

Let $x^*\colon \mathrm{R}^n_{++} \times \mathrm{R}_{++} \to \mathrm{R}^n_+$ be differentiable and satisfy the budget exhaustion condition and SWARP. Then for every $(p, w) \in \mathrm{R}^n_{++} \times \mathrm{R}_{++}$, and every $v \in \mathrm{R}^n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_i v_j \leq 0.$$

That is, the matrix of Slutsky substitution terms is negative semidefinite.

WARP and the substitution matrix

Proof:

Fix $(p, w) \in R^n_{++} \times R_{++}$ and $v \in R^n$. By homogeneity of degree 2 of the quadratic form in v, without loss of generality we may scale v so that $p \pm v \gg 0$.

Define the function x on $[-1, 1]$ via

$$x(t) = s\big(p + tv, x^*(p, w)\big). \tag{3}$$

Note that this function is differentiable, and $x(0) = x^*(p, w)$.

By Lemma 1 (with $p + tv$ playing the rôle of p^1 and p playing the rôle of p^0),

$$(p + tv - p) \cdot \big(x(t) - x(0)\big) = tv \cdot \big(x(t) - x(0)\big) \leq 0.$$

For nonzero t, dividing by $t^2 > 0$ gives

$$v \cdot \frac{x(t) - x(0)}{t} \leq 0.$$

WARP and the substitution matrix

Taking limits as $t \to 0$ gives

$$v \cdot x'(0) \leq 0. \tag{4}$$

By the Chain Rule applied to (3),

$$x_i'(t) = \sum_{j=1}^{n} \frac{\partial s_i\big(p + tv, x^*(p,w)\big)}{\partial p_j} v_j. \tag{5}$$

Evaluating (5) at $t = 0$ yields

$$\begin{aligned} x_i'(0) &= \sum_{j=1}^{n} \frac{\partial s_i\big(p, x^*(p,w)\big)}{\partial p_j} v_j \\ &= \sum_{j=1}^{n} \sigma_{i,j}(p,w) v_j, \end{aligned}$$

where the second equality is just the definition of $\sigma_{i,j}(p,w)$.

Combining this with (4) gives

$$0 \geq v \cdot x'(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_i v_j,$$

which completes the proof.

WARP and the substitution matrix

Corollary

Under the assumptions of the theorem,

$$\sigma_{i,i}(p, w) \leq 0$$

That is, compensated own-price changes are negative.

Substitution matrix and utility maximization

We obtained the negative-semidefiniteness of the Slutzky substitution matrix under the assumptions of WARP and budget exhaustion, but not utility maximization.

In fact a “rational” demand satisies in addition that the matrix is symmetric.