Monotone comparative statics

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Comparative statics

Comparative statics

Consider a firm with production function $f : \mathrm{R} \to \mathrm{R}$, hiring labor at salary w and selling its product at price p.

Profit of the firm when it employs l units of labor is

$$g(w, l) = pf(l) - wl.$$

Labor demand is $l(w) = \operatorname{argmax}_{l \geq 0} pf(l) - wl$.

To do comparative statics, suppose we can apply the implicit function theorem.

We need f to be smooth, and assume an interior solution to the profit maximization problem.

From the first order condition $pf'(l(w)) - w = 0$ we obtain: $pf''(l(w))l'(w) - 1 =$. Meaning that $l'(w) = 1/pf''(l)$. Then it would seem that downward-sloping labor demand hinges on f being concave.

As we shall see this idea is misleading.

Comparative statics

Proposition

Let $f : \mathrm{R}^2 \to \mathrm{R}$ be twice differentiable, with

$$\frac{\partial^2 f(x,t)}{\partial x \partial t} \geq 0,$$

and suppose that $a, b : \mathrm{R} \to \mathrm{R}$ are monotone increasing, with $a(t) < b(t)$ for all t. Then there is

$$x^*(t) \in \operatorname{argmax}\{f(x,t) : x \in [a(t), b(t)]\}$$

that is monotone increasing.

Partial orders

A binary relation $\leq$ on a set X is:

- *reflexive* if $(\forall x \in X)(x \leq x)$;
- *antisymmetric* if $(\forall x, y \in X)(x \leq y \text{ and } y \leq x \implies x = y)$;
- *transitive* if

$$(\forall x, y, z \in X)(y \leq x \text{ and } z \leq y \implies z \leq x);$$

- a *partial order* if it is reflexive, antisymmetric and transitive;
- a *linear order* if $(\forall x, y \in X)(x \leq y \text{ or } y \leq x)$.

Obs. linear orders are also called complete.

A pair $(X, \leq)$, where X is a set and $\leq$ is a partial order on X, is called a *partially ordered set*, or a PO set.

Examples

The set $(\mathrm{R}^n, \leq)$ is a PO set.

Let $C_b(A)$ be the set all continuous and bounded functions $f : A \subseteq \mathrm{R} \to \mathrm{R}$, and define $f \leq g$ if $f(x) \leq g(x)$ for all x. Then $(C_b(A), \leq)$ is a PO set.

Let Ω be a non-empty set, and denote by 2^Ω the set of all subsets of Ω. Then $(2^\Omega, \subseteq)$ is a PO set.

The English alphabet ordered lexicographically.

On the set Z_+ of positive integers, let $a \leq b$ if b is divisible by a. Then $(Z_+, \leq)$ is a PO set.

Examples

Let V be a real vector space and P be a pointed convex cone in V (meaning that if $x, y \in P$ $\alpha \in (0,1)$ and $\lambda > 0$ then $\lambda x \in P$ and $\alpha x + (1-\alpha)y \in P$; and that if $x, -x \in P$ then $x = 0$).
Then P defines a partial order by $x \leq y$ iff $y - x \in P$.

Exercise

Verify that indeed this defines a partial order.

Later in the class, we'll work with a special case where P is defined via a collection of linear functions $\mathcal{F}$:

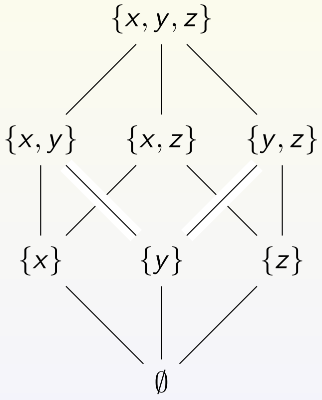
$$P = \{x \in V : f(x) \geq 0 \text{ for all } f \in \mathcal{F}\}$$

Exercise

Show that if $\mathcal{F}$ is a collection of linear functions such that $0 = \cap\{f^{-1}(0) : f \in \mathcal{F}\}$, then P as defined is a pointed convex cone.

Partial orders

Finite PO sets are represented by a *Hasse diagram*. For ex. c ($2^X, \subseteq$), by set inclusion:



Lattices

Let $(X, \leq)$ be a PO set and $A \subseteq X$ be a subset of X.
Then $A^u = \{x \in X : \forall z \in A, z \leq x\}$ is the set of *upper bounds* of A; and $A^l = \{x \in X : \forall z \in A, x \leq z\}$ is the set of *lower bounds* of A.

The *least upper bound*, or *supremum* of A, if it exists, is a smallest element of A^u. So the supremum of A, denoted $\sup A$, is $x \in A^u$ s.t $x \leq z$ for all $z \in A^u$.

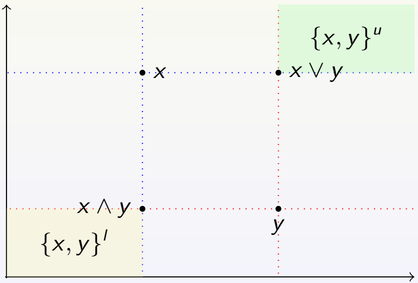
The *greatest lower bound*, $\inf A$, is $x \in A^l$ such that $z \leq x$ for all $z \in A^l$.

Lattices

For sets with two elements, we use a special notation to denote infimum and supremum.

For $x, y \in X$, we let $x \vee y = \sup\{x, y\}$ $x \wedge y = \inf\{x, y\}$.

We term $x \vee y$ the *join* of x and y; and $x \wedge y$ the *meet* of x and y.



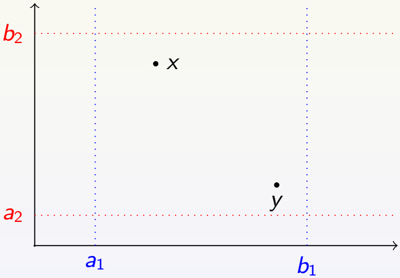
Lattices

Definition

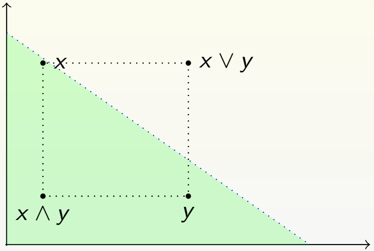
A PO set $(X, \leq)$ is a *lattice* if, for all $x, y \in X$, $x \wedge y$ and $x \vee y$ exist in X.

Example:

A *rectangle* in R^n is a subset $\prod_{i=1}^{n}[a_i, b_i]$ of R^n, with $a_i < b_i$ for all $i = 1, \ldots, n$. A rectangle is a lattice.



Not a lattice



Lattices

A lattice $(X, \leq)$ is *complete* if for all $B \subseteq X$, $\inf B$ and $\sup B$ exists in X.[1]

[1] This terminology is potentially confusing. We often use complete order to talk about a linear order.

Sublattice

Let $(X, \leq)$ be a lattice and $A \subseteq X$. We say that A is a *sublattice* of $(X, \leq)$ if, for all $x, y \in A$, $x \wedge y, x \vee y \in A$.

Let A be a sublattice of $(X, \leq)$. We say that A is *subcomplete* if, for all $B \subseteq A$, $\inf B, \sup B \in A$.

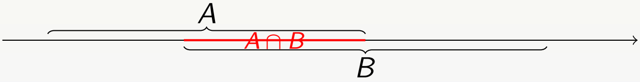
Strong set order

Let $(X, \leq)$ be a lattice. For $A, B \subseteq X$, say that A is *smaller than B in the strong set order* (or *induced set order*; and denoted by $A \sqsubseteq B$) if

$$x \in A \text{ and } y \in B \Longrightarrow x \wedge y \in A \text{ and } x \vee y \in B.$$

Strong set order

Example with $X = \mathbb{R}$.
$A \sqsubseteq B$:



Strong set order

Lemma

$\sqsubseteq$ is antisymmetric and transitive on $2^X \setminus \{\emptyset\}$.

Proof

Obs. $x \wedge y \leq x$ so $(x \wedge y) \vee x = x$

And $x \leq x \vee y$ so $(x \vee y) \wedge x = x$

Proof

First anti-symmetry. Let $A \sqsubseteq B$ and $B \sqsubseteq A$. Let $x \in A$ and $y \in B$. Then $x \wedge y \in A$ and $x \vee y \in B$, as $A \sqsubseteq B$. Then $x = (x \vee y) \wedge x \in B$ and $y = y \vee (x \wedge y) \in A$, as $B \sqsubseteq A$. Thus $A = B$.

Now to show transitivity: Let $A \sqsubseteq B$ and $B \sqsubseteq C$. Let $x \in A$ and $y \in C$. Choose $z \in B$. Note:

$$x \vee y = x \vee ((y \wedge z \vee y)) = (x \vee (y \wedge z)) \vee y.$$

Proof cont.

Now, $y \wedge z \in B$, as $B \sqsubseteq C$ Then $x \vee (y \wedge z) \in B$, as $A \sqsubseteq B$. So $x \vee y \in C$, as $B \sqsubseteq C$.

Similarly,

$$x \wedge y = (x \wedge (x \vee z)) \wedge y = x \wedge ((x \vee z) \wedge y).$$

Now, $(x \vee z) \in B$, so $(x \vee z) \wedge y \in B$. Then $x \wedge y \in A$

Strong set order

Observe that $A \sqsubseteq A$ iff A is a sublattice of $(X, \leq)$.

If we denote by $L(X)$ the set of all nonempty sublattices of $(X, \leq)$, we obtain:

Theorem

$(L(X), \sqsubseteq)$ is a PO set.

Increasing differences

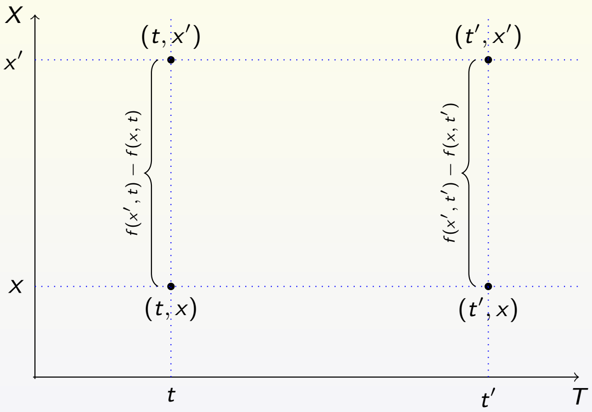
Let $(X, \leq)$ be a lattice and $(T, \leq')$ be a PO set.

A function $f : X \times T \to \mathrm{R}$ has *increasing differences* if, for all $x, x' \in X$ and $t, t' \in T$ with $x < x'$ and $t < t'$

$$f(x', t) - f(x, t) \leq f(x', t') - f(x, t').$$

(Put differently, if the function $t \mapsto f(x', t) - f(x, t)$ is monotone increasing, for each $x, x' \in X$ with $x < x'$.)

Increasing differences



Supermodulariy

A function $f : X \times T \to \mathbb{R}$ has *strictly increasing differences* if, for all $x, x' \in X$ and $t, t' \in T$ with $x < x'$ and $t < t'$

$$f(x', t) - f(x, t) < f(x', t') - f(x, t').$$

We say that the function has increasing differences "in (x, t)."

Obs. that a function has increasing differences in (x, t) iff it has increasing differences in (t, x).

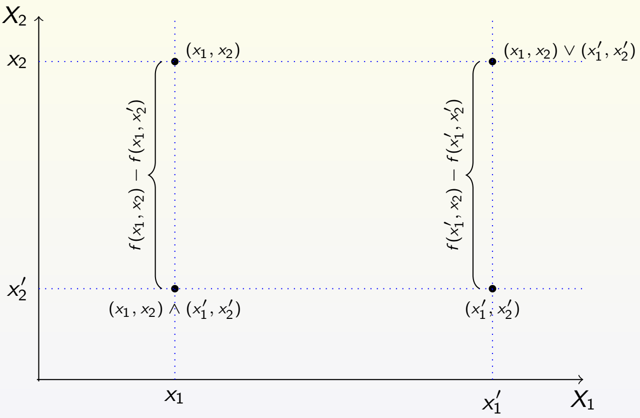
Supermodulariy

Definition

A function $f : X \to R$ is *supermodular* if, for all $x, x' \in X$,

$$f(x) + f(x') \leq f(x \vee x') + f(x \wedge x').$$

Supermodularity



Supermodularity requires that

$$f(x) - f(x \wedge x') \leq f(x \vee x') - f(x'),$$

which in the figure means that

$$f(x_1, x_2) - f(x_1, x_2') \leq f(x_1', x_2) - f(x_1', x_2').$$

So in this case, supermodularity is the same as increasing differences in each dimension.

Supermodulariy

Supermodularity is often viewed in economics as a notion of complementarity because of the connection to increasing differences. In the figure, think of f as production from two inputs, or the utility of consumption from two goods.

Supermodularity says that any "marginal" increase along the X_2 dimension is aided by increases in the X_1 dimension. The two inputs, or the two consumption goods, are therefore complements.

Supermodulariy

Two elements $x, x' \in X$ are *unordered* if $x \not\leq x'$ and $x' \not\leq x$.

A function $f : X \to R$ is *strictly supermodular* if, for all unordered $x, x' \in X$,

$$f(x) + f(x') < f(x \vee x') + f(x \wedge x').$$

Remark

If f and g are supermodular functions, and $\lambda > 0$, then $f + g$ and λf are supermodular functions.

There is, however, a supermodular f and strictly increasing $h : \mathrm{R} \to \mathrm{R}$ such that $h \circ f$ is not supermodular. *Supermodularity is a cardinal property.*

Quasi-supermodulariy

Let $(X, \leq)$ and $(T, \leq)$ be PO sets.

A function $f : X \times T \to \mathrm{R}$ satisfies the *single-crossing property* in (x, t) ($x \in X$ and $t \in T$) if, for all $x, x' \in X$ and $t, t' \in T$.

$$f(x,t) \leq f(x',t) \Longrightarrow f(x,t') \leq f(x',t')$$
$$f(x,t) < f(x',t) \Longrightarrow f(x,t') < f(x',t')$$

when $x \leq x'$ and $t \leq t'$.

Moreover, if

$$f(x,t) \leq f(x',t) \Longrightarrow f(x,t') < f(x',t')$$

for $t < t'$ then we say that f satisfies the *strict single crossing property*.

Remark

If $f : X \times T \to \mathbb{R}$ has increasing differences in (x, t) then it satisfies the single crossing property.

Let $(X, \leq)$ be a lattice. A function $f : X \to \mathbb{R}$ is *quasi-supermodular* if, for all $x, x' \in X$

$$f(x \wedge x') \leq f(x') \Longrightarrow f(x) \leq f(x \vee x')$$
$$f(x \wedge x') < f(x') \Longrightarrow f(x) < f(x \vee x')$$

Quasi-supermodulariy

Remark

If $f : X \to R$ is supermodular then it is quasi-supermodular.

Theorem

Let $(X_i, \leq_i)$ be a lattice for $i = 1, \ldots, n$. If $f : \times_{i=1}^n X_i \to \mathrm{R}$ has increasing differences in (x_i, x_j), for $i \neq j$, and $x_i \mapsto f(x_i, x_{-i})$, for all i is supermodular, then f is supermodular.

Corollary

Let $X \subseteq \mathrm{R}^n$ be a lattice under the usual order on R^n. If $f : X \to \mathrm{R}$ has increasing differences in any two variables then it is supermodular.

Supermodulariy

Proposition

Let $X_i \subseteq \mathrm{R}$ be an open set, and $X = \times_{i=1}^{n} X_i \subseteq \mathrm{R}^n$. Let $f : X \to \mathrm{R}$ be twice continuously differentiable. If

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$$

for $i \neq j$ then f has increasing differences in (x_i, x_j); and if this inequality holds for all $i \neq j$ then f is supermodular.

Supermodulariy: Examples

- $f(x_1, x_2) = x_1 x_2$
- $f(x_1, \ldots, x_n) = K \prod x_i^{a_i}$, $K > 0$ and $a_i > 0$
- $\log[D_1(p_1, p_2)(p_1 - c)]$, when

$$p_2 \mapsto \frac{\partial \log D_1(p_1, p_2)}{\partial p_1}$$

is monotone increasing. When D_1 is a demand function (for "differentiated products") this means that demand elasticity

$$\epsilon = -\frac{\partial \log D_1(p_1, p_2)}{\partial log p_1}$$

is decreasing in p_2.

Supermodulariy: Examples

- $f(x_1, x_2) = g(x_1 - x_2)$, when g is a concave function.
  In particular, in the previous example, $\log[D_1(p_1, p_2)(p_1 - c)]$ is supermodular in (p_1, c) as the log is concave.

Monotoniciy theorem

Let $(X, \leq)$ be a lattice, $(T, \leq')$ a PO set, and $f : X \times T \to \mathrm{R}$ a function. Denote by

$$M(t, S) = \mathrm{argmax}\{f(x, t) : x \in S\}$$

the set of maximizers of f over the set $S \subseteq X$ for fixed $t \in T$.

Monotoniciy theorem

Theorem (Milgrom and Shannon's Monotonicity Theorem)

$M(t, S)$ is monotone increasing iff

- $x \mapsto f(x, t)$ is quasi-supermodular;
- f satisfies the single crossing propery in (x, t)

Monotoniciy theorem: Proof

($\Longleftarrow$) Let $t \leq t'$ and $S \sqsubseteq S'$.

Let $x \in M(t, S)$ and $x' \in M(t', S')$. Since $S \sqsubseteq S'$, $x \wedge x' \in S$ and $x \vee x' \in S'$. Now, $x \wedge x' \in S$ and $x \in M(t, S)$ means that $f(x \wedge x', t) \leq f(x, t)$.

By SCP, $f(x \wedge x', t') \leq f(x, t')$.

QSM implies that $f(x', t') \leq f(x \vee x', t')$.

Since $x' \in M(t', S')$ and $x \vee x' \in S'$ we obtain that $x \vee x' \in M(t', S')$.

Now: $x', x \vee x' \in M(t', S')$ imply that $f(x', t') = f(x \vee x', t')$. So we must have, by single crossing and supermodularity, that $f(x \wedge x', t) = f(x, t)$. Thus $x \wedge x' \in M(t, S)$.

Monotoniciy theorem: Proof

($\Longrightarrow$) Conversely, suppose that $M(t,S)$ is monotone increasing. Let $x, x' \in X$ and $t, t' \in T$.

First consider $S = \{x, x \wedge x'\}$ and $S' = \{x', x \vee x'\}$.

Note that $S \sqsubseteq S'$.

Suppose that $f(x \wedge x', t) \leq f(x,t)$; so $x \in M(t,S)$.

Then $M(t,S) \sqsubseteq M(t,S')$ implies that we must have $f(x',t) \leq f(x \vee x', t)$ (as $f(x',t) > f(x \vee x', t)$ would imply $x' \in M(t,S')$ and $x \vee x' \notin M(t,S')$)). Similarly, $f(x \wedge x', t) < f(x,t)$ implies $f(x',t) < f(x \vee x', t)$.

Monotoniciy theorem: Proof

Let $x < x'$ and $t < t'$. Consider $S = \{x, x'\}$. If $f(x,t) \leq f(x',t)$, then $x' \in M(t,S) \sqsubseteq M(t',S)$ implies that $f(x,t') \leq f(x',t')$. Similarly, $f(x,t) < f(x',t)$ implies that $f(x,t') < f(x',t')$.

Monotoniciy theorem

The following result follows directly from the more general Milgrom and Shannon result.

Corollary (Topkis's monotonicity theorem)

Let $x \mapsto f(x,t)$ be supermodular and satisfy increasing differences in (x,t). Then $M(t,S)$ is monotone increasing.

Application: labor demand

Let $g(l, w) = pf(l) - wl$, where p, w, l are all real variables: p is price, w is wage and l is labor.

$f : \mathrm{R} \to \mathrm{R}$ is a production function. Note that π satisfies the single-crossing property in $(-w, l)$.

So labor demand is monotone decreasing. This result holds without any assumptions on f; it is purely a consequence of the interaction between wages and labor: at higher wages, any given increase in labor use gives a higher cost increase.

Suppose you want to generalize the result to more than one factor.

Let $g(z, w) = pf(z) - w \cdot z$, where $z = (z_1, z_2)$ is a vector of production factors and $w = (w_1, w_2)$ is a vector of factor prices.

As before, p is price and f a production function.

Now factor demand is decreasing when f is supermodular. In this case we do need some assumptions on production, but it is a natural assumption because we want the increase in one factor go "move" the demand for other factors in the same direction.

Application: differentiated products

Let $\pi_i(p_i, p_{-i}, c_i) = (p_i - c_i)D_i(p_i, p_{-i})$ be the profit function of a firm i.

D_i is the demand facing firm i when prices are $p = (p_1, \ldots, p_n) = (p_i, p_{-i})$ and c_i is the (constant) marginal cost.

Suppose that demand elasticity

$$\epsilon = -\frac{\partial \log D_1(p_1, p_2)}{\partial logp_1}$$

is decreasing in p_2.

Application: differentiated products

By a similar calculation to what we already did, we see that $g_i(p_i, p_{-i}, c_i)$ has increasing differences in (p_i, c_i) and in (p_i, p_{-i}).

So the optimal price will be increasing in c_i and also in competitors' prices.