Demand Duality

Federico Echenique
Caltech – SS205a

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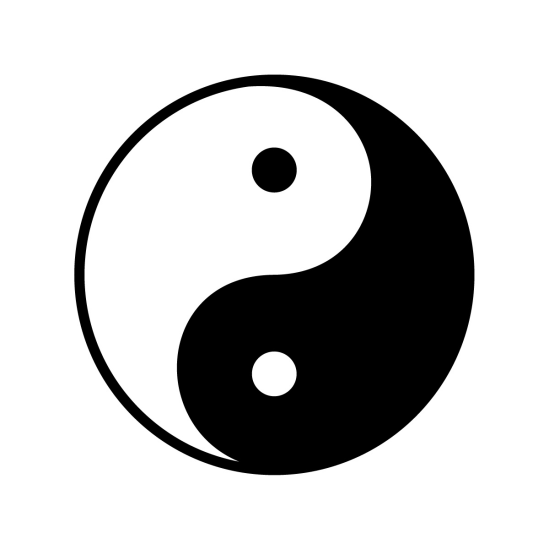
Duality



Duality

Utility maximization

$$\mathcal{U}(p, m)$$

$$\begin{array}{cc} \max_{x \in \mathbf{R}^n_+} & u(x) \\ \text{s.t} & p \cdot x \leq m \end{array}$$

Value: $v(p, m)$

Solutions: (Marshallian) demand.

Expenditure minimization

$$\mathcal{E}(p, \theta)$$

$$\begin{array}{cc} \min_{x \in \mathbf{R}^n_+} & p \cdot x \\ \text{s.t} & u(x) \geq \theta \end{array}$$

Value: $e(p, \theta)$

Solutions: (Hicksian) Compensated demand.

Two basic facts:

Let $X = \mathbb{R}^n_+$, u be continuous and locally non-satiated.

- If x solves $\mathcal{U}(p, m)$, and $\theta = u(x)$, then x solves $\mathcal{E}(p, \theta)$.
- If x solves $\mathcal{E}(p, \theta)$ and $m = p \cdot x > 0$, then x solves $\mathcal{U}(p, m)$.

Press pause here and attempt a proof of these two before following the rest of the lecture.

Duality

Suppose that u is cont. and LNS.

Claim

If x solves $\mathcal{U}(p, m)$, and $\theta = u(x)$, then x solves $\mathcal{E}(p, \theta)$

Proof.

Suppose (towards a contradiction) that $u(x') \geq \theta$ and $p \cdot x' < p \cdot x = m$ (by LNS). Let $\varepsilon > 0$ be small enough that $\|x'' - x'\| < \varepsilon$ implies that $p \cdot x'' \leq m$. By LNS there exists x'' with $\|x'' - x'\| < \varepsilon$ and $u(x'') > u(x') \geq \theta$. This would contradict that x solves $\mathcal{U}(p, m)$.

Duality

Suppose that u is cont. and LNS.

Claim

If x solves $\mathcal{E}(p, \theta)$ and $m = p \cdot x > 0$, then x solves $\mathcal{U}(p, m)$

Proof.

Suppose that $u(x') > \theta = u(x)$. Since x solves $\mathcal{E}(p, \theta)$ we must have $p \cdot x' \geq m$. Suppose (towards a contradiction) that $p \cdot x' = m$. Let $\delta > 0$ be small enough that $u((1 - \delta)x') > u(x)$. Then

$$p \cdot (1 - \delta)x' = (1 - \delta)p \cdot x' = (1 - \delta)m < m,$$

as $m > 0$. This is absurd as x solves $\mathcal{E}(p, m)$.

Make a note!!

Remember the last claim next Winter when you prove the second welfare theorem!

Duality

Let u be cont. LNS, and quasi-concave.

$$
\begin{aligned}
m &= e(p, v(p,m)) \\
\theta &= v(p, e(p,\theta)) \\
x^*(p,m) &= x^h(p, v(p,m)) \\
x^h(p,\theta) &= x^*(p, e(p,\theta))
\end{aligned}
$$

Digression on convex analysis

- Separating hyperplane theorem
- Support functions and their supergradients

Separating hyperplane Theorem

Theorem

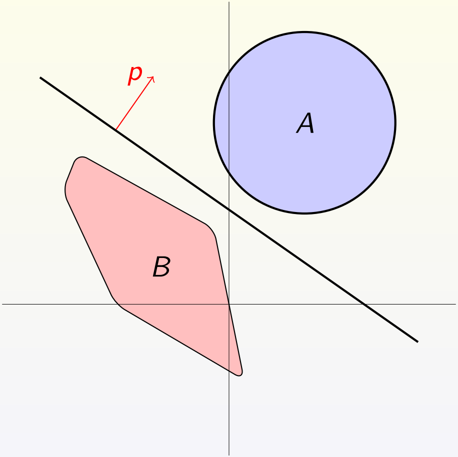
Let $A, B \subseteq \mathrm{R}^m$ be non-empty, disjoint, convex sets. There is $p \in \mathrm{R}^m$, and $\alpha \in \mathrm{R}$, such that

$$p \cdot b \leq \alpha \leq p \cdot a$$

for all $a \in A$ and $b \in B$, and at least one of inequalities is strict for some $a \in A$ and $b \in B$ (in particular, $p \neq 0$).

Here the set $\{x : p \cdot x = \alpha\}$ is a *hyperplane*. It defines two half-spaces: $\{x : p \cdot x \leq \alpha\}$ and $\{x : p \cdot x \geq \alpha\}$.

Separating hyperplane



Digression on convex analysis

Consider a function $f : \mathrm{R}^n \to \mathrm{R} \cup \{-\infty\}$.

So it's allowed to take the value $-\infty$.

Say that the *domain* of f is

$$\{x \in \mathrm{R}^n : -\infty < f(x)\}.$$

We can still write the definition of concavity, and it will make sense under standard conventions regarding $-\infty$: f is concave if

$$\lambda f(x) + (1-\lambda)f(y) \leq f(\lambda x + (1-\lambda)y)$$

for all $\lambda \in (0,1)$.

Supergradients

Let $f\colon \mathrm{R}^n \to \mathrm{R} \cup \{-\infty\}$ be concave.

A vector p is a *supergradient* of f at the point x if for every y it satisfies the *supergradient inequality*,

$$f(x) + p \cdot (y - x) \geq f(y).$$

Note that this is the overestimation property of the gradient of a concave function. So, when f is differentiable at x, $Df(x)$ is a supergradient at x.

Support function

The *support function* of a non-empty set A is defined as

$$\mu_A(p) = \inf\{p \cdot x : x \in A\};$$

where $\mu_A(p) = -\infty$ is possible.

If A is compact, then μ_A is always finite, and there is some point in A where the infimum is achieved.

Support function

Think of the support function as the value of the optimization problem

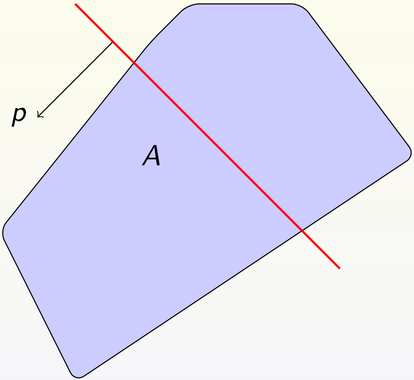
$$\begin{array}{ll} \min & p \cdot x \\ \text{s.t} & x \in A. \end{array}$$

Where we need to use inf instead of min.

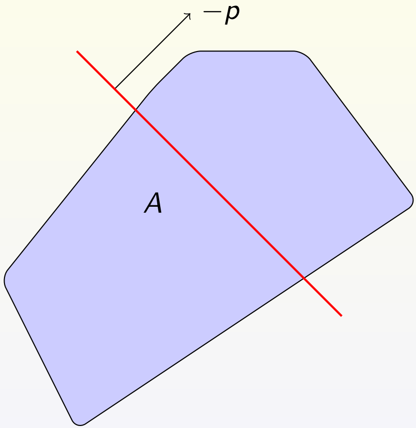
Many applications in economics:

- Expenditure function (this lecture).
- Profit function (w/max instead of min).
- Decision under uncertainty.

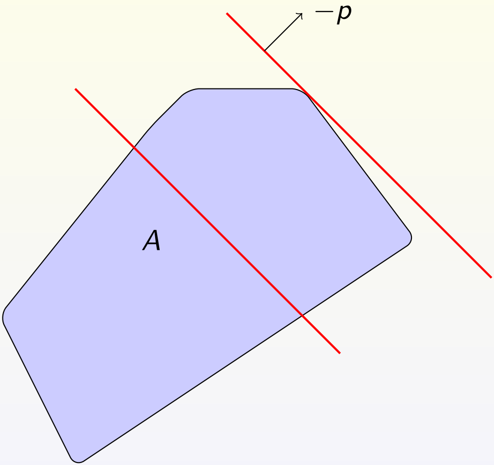
Support function



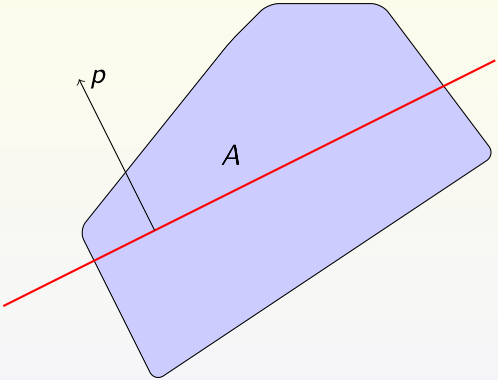
Support function



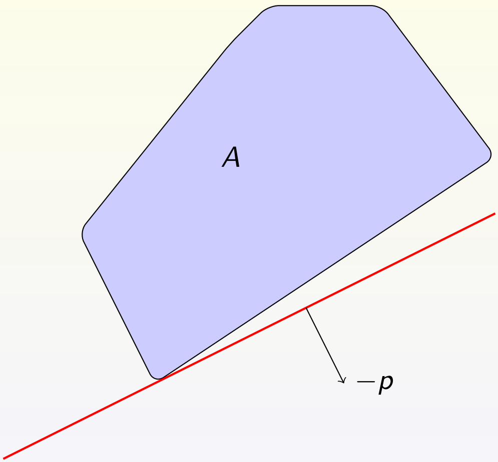
Support function



Support function



Support function



Support function

Lemma

The support function μ_A is concave and positively homogeneous of degree 1 (that is, $\mu_A(\lambda p) = \lambda \mu_A(p)$ for all p and all $\lambda \geq 0$).

Proof.

Each x defines a linear function

$$p \mapsto \ell_x(p) = p \cdot x$$

By the prev. exercise, $\mu_A = \inf_{x \in A} \ell_x$ is concave.

Homogeneity is obvious.

Support function

Theorem

Let C be a closed convex set. Then x is a supergradient of the support function μ_C at p iff if $x \in C$ and minimizes $p \cdot x$ over C.

In other words,

$$\partial \mu_C(p) = \{x \in C : p \cdot x = \mu_C(p)\} = \operatorname{argmin}\{p \cdot x : x \in C\}.$$

Support function: proof

By defn. of supergradient: $x \in \partial\mu_C(p)$ iff

$$\mu_C(p) + x \cdot (q - p) \geq \mu_C(q) \qquad \text{for all } q.$$

Support function: proof

($\Longrightarrow$) First, if $x \notin C$ then $x \notin \partial\mu_C(p)$.

Because if $x \notin C$ then by (a version of) the SHT there's q for which $q \cdot x < \mu_C(q)$; then by homogeneity there's M s.t $Mq \cdot x < \mu_C(Mq) + \big(p \cdot x - \mu_C(p)\big)$; which means that $x \notin \partial\mu_C(p)$.

So let $x \in \partial\mu_C(p)$. We know $x \in C$, and hence $p \cdot x \geq \mu_C(p)$. But $q = 0$ in the defn. of supergradient gives $\mu_C(p) \geq p \cdot x$.

Thus, $\partial\mu_C(p) \subseteq \{x \in C : p \cdot x = \mu_C(p)\}$

Support function: proof

($\Longleftarrow$) Suppose now $x \in \operatorname{argmin}\{p \cdot y : y \in C\}$. So $x \in C$ and $\mu_C(p) = p \cdot x$.

Then, for any q, $q \cdot x \geq \mu_C(q)$. Together with $\mu_C(p) = p \cdot x$ this implies

$$q \cdot x + \mu_C(p) - p \cdot x \geq \mu_C(q)$$

Thus $\{x \in C : p \cdot x = \mu_C(p)\} \subset \partial \mu_C(p)$, completing the proof.

Corollary

Let C be a closed convex set and suppose that x is the unique solution to

$$\begin{array}{ll} \min & p \cdot x \\ \text{s.t} & x \in C \end{array}$$

Then μ_C is differentiable at p and

$$D\mu_C(p) = x.$$

Expenditure minimization

Now apply these results to the expenditure function:

Minimize expenditure, subject to achieving a given target utility level:

$$\begin{array}{cc} \min & p \cdot x \\ \text{s.t} & u(x) \geq \theta \end{array}$$

The value of this problem is $e(p, \theta)$.

But note that

$$e(p, \theta) = \mu_{\{x : u(x) \geq \theta\}}(p).$$

The support function of the upper contour set at utility θ.

Expenditure minimization

Assume $u : \mathrm{R}^n_+$ is quasi-concave, cont. and LNS.

As a consequence of the prev. general theorem on support functions, we obtain that:

Corollary

$$\partial e(p, \theta) = x^h(p, \theta)$$

In particular, when $p \mapsto e(p, h)$ is differentiable we recover the Hicksian demand from expenditure by

$$x^h(p, \theta) = D_p e(p, \theta).$$

Duality – take II

The Lagrangean for utility maximization is

$$L(x, \lambda) = u(x) + \lambda(m - p \cdot x) = -\lambda p \cdot x + u(x) + \lambda m.$$

The Lagrangean for expenditure minimization is

$$L(x, \mu) = p \cdot x + \mu(\theta - u(x)) = -\mu u(x) + p \cdot x + \mu\theta.$$

To exploit this structure we assume differentiabilty.

Utility max

Now we develop demand theory under differentiability assumptions.

If preferences are "smooth" in a sense that we shall not go into, they admit not only a continuous utility representation, but also one that is differentiable.

In fact, if they are smooth enough, the utility function is differentiable as many times as we wish.

Utility max

- u is C^1 with $Du \gg 0$
- u is strictly quasi-concave
- Restrict attention to $p \gg 0$ and $m > 0$
- Utility max. has interior solutions: this is often achieved by an "Inada" assumption. Can assume that $u(x) > 0$ only if $x \gg 0$.

Utility max

Under these assumptions,

$$\begin{array}{cc} \max & u(x) \\ \text{s.t} & p \cdot x \leq m \end{array}$$

has a unique, interior, continuous solution $x^*(p, m) \gg 0$.
The Lagrangean for this problem is

$$L(x, \lambda; p, m) = u(x) + \lambda(m - p \cdot x).$$

The gradient of the constraint is $-p \neq 0$, so the Lagrange Multiplier Theorem applies.

Utility max

Let $x^*(p, m)$ be the solution with Lagrange multiplier $\lambda^*(p, m)$. The first-order conditions are

$$D_i u(x^*) - \lambda^* p_i = 0 \quad i = 1, \dots, n.$$

Since $p \gg 0$ and each $D_i u > 0$ we have

$$\lambda^* > 0.$$

The *indirect utility function* v is the value function for this problem:

$$v(p, m) = u\big(x^*(p, m)\big).$$

Roy's identity

Then by the Envelope Theorem,

$$D_m v(p,m) = D_m L = \lambda^*(p,m)$$
$$D_{p_j} v(p,m) = D_{p_j} L = -\lambda^*(p,m) x_j^*(p,m).$$

Together these imply *Roy's Identity*, namely:

$$x_j^*(p,m) = -\frac{D_{p_j} v(p,m)}{D_m v(p,m)}.$$

Expenditure minimization

$$\begin{array}{cc} \min & p \cdot x \\ \text{s.t} & u(x) \geq \theta \end{array}$$

The Lagrangean for this problem is:

$$p \cdot x + \mu\big(\theta - u(x)\big).$$

Expenditure minimization

Let $x^h(p, \theta)$ solve the problem and let $\hat{\mu}(p, \theta)$ be the Lagrange multiplier. The *expenditure function* e is the value function for this problem:

$$e(p, \theta) = p \cdot x^h(p, \theta).$$

Then by the Envelope Theorem,

$$D_\theta e(p, \theta) = \hat{\mu}(p, \theta) \qquad \text{and} \qquad D_{p_j} e(p, \theta) = x_j^h(p, \theta).$$

Expenditure minimization

Moreover, by our prev. observation, e is concave in p. Thus e is twice differentiable in p almost everywhere, and where it is differentiable:

$$\begin{bmatrix} \frac{\partial^2 e(p,\upsilon)}{\partial^2 p_1} & \dots & \frac{\partial^2 e(p,\upsilon)}{\partial p_n \partial p_1} \\ \vdots & & \vdots \\ \frac{\partial^2 e(p,\upsilon)}{\partial p_1 \partial p_n} & \dots & \frac{\partial e(p,\upsilon)}{\partial^2 p_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1^h}{\partial p_1} & \dots & \frac{\partial x_1^h}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial x_n^h}{\partial p_1} & \dots & \frac{\partial x_n^h}{\partial p_n} \end{bmatrix}$$

is symmetric and negative semidefinite.

Expenditure minimization

In particular then,

$$D_{p_j} x_j^h \leq 0$$

and

$$D_{p_i} x_j^h = D_{p_j} x_i^h$$

for $i, j = 1, \ldots, n$,

Price effects for compensated demand are symmetric.

Back to Marshallian demand

What is the relationship between Hicksian and ordinary demand? From the equivalence of expenditure minimization and utility maximization we have

$$x^*{}_i\big(p, e(p,\theta)\big) = x_i^h(p,\theta),$$

which implies

$$\frac{\partial x^*{}_i}{\partial p_j} + \frac{\partial x^*{}_i}{\partial m}\frac{\partial e}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j}.$$

Rearranging,

$$\frac{\partial x^*{}_i}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j} - \frac{\partial x^*{}_i}{m}\frac{\partial e}{\partial p_j}.$$

Slutsky equation

But $\frac{\partial e}{\partial p_j} = x_j^h$ and $x_j^h(p,\theta) = x^{*j}(p,m)$ where $m = e(p,\theta)$.
Thus

$$\boxed{\frac{\partial x^*{}_i(p,m)}{\partial p_j} = \frac{\partial x_i^h\big(p, v(p,m)\big)}{\partial p_j} - x^*{}_j(p,m)\frac{\partial x^*{}_i(p,m)}{\partial m}.}$$

This is the famous *Slutsky equation*.

Slutsky equation: own price effect

$$\frac{\partial {x^*}_i(p,m)}{\partial p_i} = \underbrace{\frac{\partial x_i^h\big(p, v(p,m)\big)}{\partial p_i}}_{\leq 0} - {x^*}_i(p,m)\underbrace{\frac{\partial {x^*}_i(p,m)}{\partial m}}_{\geq 0}.$$

If i is normal.

Slutsky equation: own price effect

$$\frac{\partial x^*{}_i(p,m)}{\partial p_i} = \underbrace{\frac{\partial x_i^h\big(p, v(p,m)\big)}{\partial p_i}}_{\leq 0} - x^*{}_i(p,m)\underbrace{\frac{\partial x^*{}_i(p,m)}{\partial m}}_{\leq 0}.$$

If i is inferior.

Slutsky equation

Rearranging:

$$\begin{bmatrix} \frac{\partial^2 e}{\partial^2 p_1} & \cdots & \frac{\partial^2 e}{\partial p_n \partial p_1} \\ \vdots & & \vdots \\ \frac{\partial^2 e}{\partial p_1 \partial p_n} & \cdots & \frac{\partial e}{\partial^2 p_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1^h}{\partial p_1} & \cdots & \frac{\partial x_1^h}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial x_n^h}{\partial p_1} & \cdots & \frac{\partial x_n^h}{\partial p_n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x^*{}_1}{\partial p_1} + x^*{}_1 \frac{\partial x^*{}_1}{\partial m} & \cdots & \frac{\partial x^*{}_1}{\partial p_n} + x^*{}_n \frac{\partial x^*{}_1}{\partial m} \\ \vdots & & \vdots \\ \frac{\partial x^*{}_n}{\partial p_1} + x^*{}_1 \frac{\partial x^*{}_n}{\partial m} & \cdots & \frac{\partial x^*{}_n}{\partial p_j} + x^*{}_n \frac{\partial x^*{}_n}{\partial m} \end{bmatrix}$$

is symmetric and negative semidefinite.

From indirect utility to utility

Consider the following *inversion formula:*

$$u(x) = \inf_p v(p, p \cdot x)$$

When is it valid? Meaning, the u recovered gives rise to v.

From indirect utility to utility

- N (Nonnegativity): $v(p, m) \geq 0$ for all (p, m).
- P (Monotonicity in p): $(m > 0 \ \& \ p' \gg p) \implies v(p', m) < v(p, m)$.
- M (Monotonicity in m): $m' > m \implies v(p, m') > v(p, m)$.
- H (Homogeneity): $v(\lambda p, \lambda m) = v(p, m)$ for all $\lambda > 0$.
- Q (Quasiconvexity in p): $v(p, m)$ is quasiconvex in p.
- S (Upper semicontinuity): v is upper semicontinuous on $\mathbf{R}^n_{++} \times \mathbf{R}_+$.
- Z (The zero property): For all p, p', $v(p, 0) = v(p', 0) = \min_{(p,m) \in \mathbf{R}^n_{++} \times \mathbf{R}_+} v(p, m)$.

From indirect utility to utility

Theorem

Let $v\colon \mathbf{R}^n_{++} \times \mathbf{R}_+ \to \mathbf{R}$ satisfy properties N, P, M, H, Q, S, and Z. Then the function $u\colon X \to \mathbf{R}$ defined by the inversion formula is upper semicontinuous, monotone and quasiconcave utility; and

$$v(p, m) = \max\big\{u(x) : x \in \mathbf{R}^n_+ \text{ and } p \cdot x \le m\big\}.$$

This result appears in Krishna and Sonnenschein (1990). I learned it from Kim Border.

From expenditure to utility

You may also consider:

$$u(x) = \max\{\theta : \forall p \in \mathrm{R}^N_{++},\ e(p,\theta) \leq p \cdot x\}$$

If e was derived using another utility representation $\tilde{u}$ with $\theta = \tilde{u}(x)$ and $\theta' = \tilde{u}(y)$.

Then it's easy to see that $\theta \leq \theta'$ iff $u(x) \leq u(y)$. Hence u represents the same preferences as $\tilde{u}$.

Money metric utility

A related idea is the *money metric utility function:*

$$m(p, x) = e(p, u(x))$$

For fixed p, $x \mapsto m(p, x)$ represents the same preferences as u. Utility is thus expressed in monetary terms.

Note it can be obtained directly from preferences by

$$m(p, x) = \inf\{p \cdot y : y \succeq x\}.$$

So that m is independent of u.

Money metric utility

Let $X = \mathbb{R}^n_+$.

The next theorem is due to Blackorby and Donaldson:

Theorem

Let $\succeq$ on X be cont., LNS and convex. Then $x \mapsto m(p, x)$ is a concave utility representation of $\succeq$ (for all $p \in \mathbb{R}^n_{++}$) iff $\succeq$ is homothetic.

So: concavity, which is useful, comes at a significant cost for the MMU.

Assignment

Exercise

Complete diagram 3.G.3 in MWG. Add the inversion formula, and the connection between utility, expenditure and money-metric utility.