Representative consumer

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Representative consumer

Consider a collection of continuous preferences $(\succeq_i)_{i=1}^I$ on $\mathbf{R}^n_+$ such that each $\succeq_i$ gives rise to a continuous demand function x_i^*.

We want to know when there is a preference relation $\succeq^R$, giving rise to a demand function x^R with the property that:

$$x^R\Big(p, \sum_{i=1}^I m_i\Big) = \sum_{i=1}^I x_i^*(p, m_i).$$

In this case, we say that $(\succeq_i)_{i=1}^I$ *admits a representative consumer*.

Representative consumer

Obs.
If $(\succeq_i)$ admits a representative consumer then

$$\sum_i x_i^*(p, m_i) = \sum_i x_i^*(p, m_i')$$

whenever $\sum_i m_i = \sum_i m_i'$.

This is clearly a very restrictive property. It says that income distribution doesn't matter for aggregate demand.

Representative consumer

We shall see that homothetic preferences is central to the question of when an economy accepts a representative consumer.

Recall that a consumer has a homothetic preference relation if $x \succeq_i y$ implies $\alpha x \succeq_i \alpha y$ for all $x, y \in \mathbf{R}^n_+$ and $\alpha > 0$.

Representative consumer

Lemma

Let $\succeq$ be a monotone preference relation.

1. $\succeq$ is homothetic and cont. iff it can be represented by a continuous and homogeneous degree one utility function.
2. Suppose $\succeq$ generates a C^1 demand function x^*. Then there is a homothetic preference generating x^* iff $x^*(p, m)$ is homog. of degree one in income (i.e., $x^*(p, m) = mx^*(p, 1)$ for every $m \geq 0$).

The function $m \mapsto x_i^*(p, m)$ is the *income expansion path* (or the Engel curve) for consumer i.

So homotheticity means that income expansion paths are linear.

Representative consumer

Theorem (Antonelli's Theorem)

The preferences $(\succeq_i)_{i=1}^I$ admit a representative consumer iff there is a homothetic preference relation $\succeq$ on $\mathbf{R}^n_+$ such that each individual demand x_i^* is generated by $\succeq$.

Digression: Cauchy's equation.

Consider *Cauchy's equation*

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in R$.

This is an equation in which the unknown is the (real) function f: a so-called *functional equation*.

We may want to know if there are any solutions: there obviously are. For ex. $f(x) = x$.

More interestingly, we want to characterize all solutions.

Digression: The Cauchy equation.

Proposition

Let $f : \mathbf{R} \to \mathbf{R}$ be cont. and satisfy that $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1, x_2 \in \mathbf{R}$. Then there is $c \in \mathbf{R}$ (a constant) such that $f(x) = cx$. Note that $c = f(1)$.

Digression: The Cauchy equation – Proof.

First, let $n > 1$ be a positive integer and $r \in \mathrm{R}$. Note that

$$\begin{aligned} f(nr) &= f(r + (n-1)r) = f(r) + f((n-1)r) \\ &= f(r) + f(r) + f((n-2)r) = \ldots = nf(r). \end{aligned}$$

In second place, let $q = n/m \in \mathrm{Q}$ be a rational number, with $n, m \in \mathrm{Z}_+$ being positive integers. Then $f(n/m) = nf(1/m)$, which means that

$$mf(n/m) = mnf(1/m) = nf(m/m) = nf(1).$$

Hence $f(q) = qf(1)$. The same holds true when $q \leq 0$.

Now, since $f(q) = qf(1)$ for all rational numbers q, and f is continuous, $f(x) = xf(1)$ for all real numbers x.

Proof of Antonelli's theorem

($\Leftarrow$) Let $\succeq^R$ be a homothetic preference on $\mathbf{R}^n_+$ that generates every demand function x^*_i.

Let x^R be the demand generated by this preference relation. Note that $x^R(p,m) = mx^R(p,1)$ for all m as x^R is linear homogeneous (homog. of dg. one) in income.

Then,

$$\begin{aligned}\sum_{i=1}^{I} x^*_i(p, m_i) &= \sum_{i=1}^{I} x^R(p, m_i) \\ &= x^R(p,1)\sum_{i=1}^{I} m_i \\ &= x^R\Big(p, \sum_{i=1}^{I} m_i\Big)\end{aligned}$$

Proof of Antonelli's theorem

($\Rightarrow$) Let $p \in \mathbf{R}^n_{++}$ and $m \in \mathbf{R}_+$. Set $m_i = m$ and $m_j = 0$ for all $j \neq i$. Then

$$\begin{aligned} x^R(p,m) &= x^R(p, \sum_i m_i) = \sum_j x^*_j(p,m_j) \\ &= x^*_i(p,m_i) + \sum_{j\neq i} x^*_j(p,m_j) \\ &= x^*_i(p,m). \end{aligned}$$

Hence $x^R = x^*_i$.

Proof of Antonelli's theorem

In second place, we show that for each p the function $m \mapsto x^R(p, m)$ satisfies additivity (the condition in the Cauchy eq.).

Let $m = m_1 + m_2$. Let now $m_i = 0$ for $i \neq 1, 2$. Then

$$\begin{aligned} x^R(p, m_1 + m_2) = x^R(p, \sum_i m_i) &= \sum_j x_j^*(p, m_j) \\ &= x_1^*(p, m_1) + x_2^*(p, m_2) \\ &= x^R(p, m_1) + x^R(p, m_2) \end{aligned}$$

Now we know from the solution of the Cauchy equation that $x^R(p, m) = m x^R(p, 1)$.

Take stock

- Antonelli's thm. says a representative consumer isn't going to exist unless we have very restrictive assumptions.
- These rule out consumer heterogeneity.
- *And* require homotheticity.
- But the property of rep. consumer is global. What if we only require it for a smaller set of possible income distributions?
- This is tied to the existence of a normative rep. consumer.

Take stock

- So far we've looked at a positive rep. consumer. One that reproduces aggregate demand.
- We often want one for which we can use the utility to draw welfare conclusions about the underlying economy.
- A so-called normative rep. consumer.
- Turns out one can construct examples of positive representative consumer who isn't normative.
- In fact makes Pareto dominated choices . . .
- This motivates an approach that (1) starts from a welfare criterion, and (2) restricts endogenously the possible income distributions.

Samuelsonian aggregation

Let $(\succeq)_{i=1}^{I}$ be a collection of cont. preferences where each $\succeq_i$ is represented by a cont. utility function $u_i : \mathrm{R}_+^n \to \mathrm{R}$.
Let $W : \mathrm{R}^I \to \mathrm{R}$ be a strictly monotone increasing function. We refer to W as a *social welfare function*.

Consider the problem:

$$\max_{(x_1,\ldots,x_I)\in\mathbb{R}_+^{nI}} \quad W\left(u_1(x_1),\ldots,u_I(x_I)\right) \tag{P1}$$

$$\text{subject to} \quad p\cdot\sum_{i=1}^{I} x_i \leq m.$$

Samuelsonian aggregation

The problem above implies choosing an aggregate consumption bundle $z = \sum_{i=1}^{I} x_i \in \mathbf{R}^{nI}$ for the economy.

The objective is to maximize the social welfare in the economy, as given by W.

The constraint reflects an aggregate budget constraint.

Samuelsonian aggregation

Consider the alternative problem:

$$\max_{z \in \mathbf{R}_+^{nI}} \left\{ \max_{(x_1,\ldots,x_I) \in \mathbf{R}_+^{nI}} W\left(u_1(x_1), \ldots, u_I(x_I)\right) \quad \text{s.t.} \quad \sum_{i=1}^{I} x_i = z \right\} \qquad \text{(P2)}$$

$$\text{subject to} \quad p \cdot z \leq m.$$

Samuelsonian aggregation

Define the value function of the inner max above as

$$U(z) = \sup\{W(u_1(x_1), \dots, u_I(x_I)) : (x_1, \dots, x_I) \in \mathrm{R}_+^{nI} \text{ and } \sum_{i=1}^{I} x_i = z\}.$$

Rewrite (P2) as:

$$\max_{\mathrm{R}_+^{nI}} \quad U(z) \quad \text{subject to} \quad p \cdot z \leq m. \tag{P2'}$$

Samuelsonian aggregation

Let $x_i^*(p, m_i)$ be the demand function generated by $\succeq_i$.
Consider the following problem:

$$\max_{\{m_i\}_{i=1}^{I}} \; W\left(u_1(x_1^*(p, m_1)), ..., u_I(x_I^*(p, m_I))\right) \tag{P4}$$

$$\text{subject to} \quad \sum_{i=1}^{I} m_i \leq m.$$

Samuelsonian aggregation

Interpret these problems. Problem 1 maximizes social welfare function by choosing an individual bundle for each consumer subject to the aggregate budget constraint.

This induces an optimal aggregate bundle $z \in \mathrm{R}^n$. The second problem is a maximization in two steps. First, given an arbitrary aggregate bundle of consumption $z \in \mathrm{R}^n$, the social planner maximizes the social welfare function, i.e., decides on the optimal allocation of the aggregate bundle among the consumers.

Samuelsonian aggregation

Then, given the income restriction of the economy, the social planner decides the optimal aggregate consumption bundle.

Problem P2' makes explicit that we can write the second problem as that of a representative consumer whose utility function is the value function resulting from the first of the two nested problems above.

Therefore, the utility function of our representative consumer represents the optimal division of an aggregate consumption bundle among the consumers.

Another way of addressing this problem is in terms of the distribution of income. A planner allocates aggregate income among consumers, who then go and purchase according to their indiv. demand functions.

Gorman polar form

Consider a collection $(\succeq_i)$ of preferences on $\mathbf{R}^n_+$, each with C^1 utilities u_i, demands x_i^*, and indirect utilities v_i.

The function $m \mapsto x_i^*(p, m)$ is the *income expansion path* (or the Engel curve) for consumer i.

Recall that if $(\succeq_i)$ admits a representative consumer then

$$\sum_i x_i^*(p, m_i) = \sum_i x_i^*(p, m_i')$$

whenever $\sum_i m_i = \sum_i m_i'$.

So that aggregate demand is independent of the distribution of income.

Gorman polar form

Let $w \in \mathrm{R}^I$ s.t $\sum_i w_i = 0$ and consider incomes $m_i + \varepsilon w_i$.
These modified incomes add up to the same. So

$$\sum_i x_i^*(p, m_i) = \sum_i x_i^*(p, m_i + \varepsilon w_i)$$

Consider the derivative with respect to ε and evaluate at $\varepsilon = 0$.
Then we see that

$$\sum_i D_m x_i^*(p, m_i) w_i = 0$$

So, for example, if we consider $w_i = 1$, $w_j = -1$ and $w_h = 0$ for all other h then we conclude that

$$D_m x_i^*(p, m_i) = D_m x_i^*(p, m_j)$$

The income effect on any good, at any (p, m) must be the same for any two agents.

Gorman polar form

When does demand have the property that income effects are linear and the same for all consumers?

Individual indirect utilities have the Gorman form when:

$$v_i(p, m_i) = \alpha_i(p) + \beta_i(p)m_i,$$

with α_i and β_i being such that the properties that we have seen characterize indirect utility are satisfied.

Gorman polar form

Then by Roy's law

$$x_{il}^*(p,m) = \frac{-D_{p_l} v_i(p,m)}{D_m v_i(p,m)}$$
$$= \frac{-D_{p_l}\alpha_i(p)}{\beta_i(p)} - \frac{D_{p,l}\beta_i(p)}{\beta_i(p)} m$$

Now, we need income effects to be the same for any two consumers. So we have to set $\beta_i = \beta$.

Gorman polar form

Conclude that when consumers indirect utilities are of the Gorman polar form

$$v_i(p, m_i) = \alpha_i(p) + \beta(p)m_i,$$

then demand is of the form

$$x_i^*(p, m_i) = -a_i(p) - b(p)m_i,$$

so that aggregate demand is

$$\sum_i x_i^*(p, m_i) = -\sum_i a_i(p) - b(p)\sum_i m_i = x^R(p, \sum_i m_i),$$

where x^R is derived from indirect utility

$$v_R(p, m) = \sum_i \alpha_i(p) + \beta(p)m,$$

which also has the Gorman form.

Gorman polar form

We have shown that the Gorman form is sufficient for the existence of a representative consumer "locally," in a nbd of p and $m_1, \ldots, m_n$.

Note that if we want this to hold globally then by taking $m_i \to 0$ we'd need demand to $\to 0$. Hence $\alpha_i(p) = 0$.

The Gorman form is also necessary for linear and parallel income expansion paths, but we'll skip a discussion of this fact.