Separability and Composite Commodities

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Separability and composite goods

Separable preferences

- Economists often study smaller sets of goods.
- When is it OK to study the demand for food, or entertainment, in isolation?
- When can you study the demand in year 2020, independently of choices made about consumption in other time periods?
- Answer: Two-stage budgeting.
- This requires separable preferences.

Separable preferences

- We've also been thinking about conventional consumer choices.
- But we often want to re-interpret consumption space.
- For example under uncertainty, $x \in \mathbf{R}^n_+$ could be state-contingent consumption: There is a set Ω of *states of the world*, with $S = |\Omega|$. And x is a contract that delivers a consumption vector $x_\omega \in \mathbf{R}^K_+$ if state ω occurs.
- When $K = 1$ these are called *monetary acts*.
- Then consumption space if $\mathbf{R}^n_+$ with $n = SK$.
- For decisions under uncertainty, separability is important (details later).

Separable preferences

- Another re-interpretation is intertemporal choice.
- Think of consumption over time.
- There are T time periods, and x is a *plan* to consume $x_t \in \mathbf{R}^K_+$ at time t.
- Then consumption space if $\mathbf{R}^n_+$ with $n = TK$.
- Again, separability will be crucial in standard models of intertemporal choice.

Example: intertemporal choice.

Suppose that $T = 2$ and $K = 1$. So: one good in each time period (the model used in macroeconomics).

Consider a consumer who solves:

$$\begin{array}{cc} \max_{x \in \mathbf{R}^2_+} & u(x_1) + \delta u(x_2) \\ \text{s.t} & p \cdot x \leq m \end{array}$$

Here $\delta \in (0, 1)$ is a subjective discount factor, capturing how utility in later periods translates into current utility.

Such a utility representation means that preferences are *additively separable.*

Example: intertemporal choice.

We may normalize $p_1 = 1$ and consider $p_2 = R$ to be the price of consumption in period 2 in terms of period-1 consumption: a market discount factor.

Suppose that u is C^1 and strictly monotone, and that we can focus on interior solutions. Then the consumer will solve:

$$\max u(x_1) + \delta u(w - Rc_1).$$

Example: intertemporal choice.

FOC:

$$u'(c_1) = R\delta u'(c_2);$$

a so-called *Euler equation* (the special name for equality of MRS and price in intertemporal models.).

If $R = \frac{1}{1+r}$, where r is a market interest rate. Then

$$\frac{\delta u'(c_2)}{u'(c_1)} = 1 + r.$$

Separable preferences: Notation

Let $X = \mathrm{R}^n_+$.

Let $A \subset \{1, \ldots, n\} = [n]$.

For $x \in \mathrm{R}^n$, write

$$x_A = (x_i)_{i \in A}$$
$$x_{-A} = (x_i)_{i \in [n] \setminus A}$$
$$x = (x_A, x_{-A})$$

Separable preferences

Idea:

$$u(x) = U(s(x_A), x_{-A})$$

- $s : \mathbb{R}_+^A \to \mathbb{R}$ is a "sub-utility."
- Consumer optimizes s subject to a budget constraint.
- Marginal rate of substitution:

$$MRS_{i,j} = \frac{D_i u(x)}{D_j u(x)} = \frac{D_i s(x_A)}{D_j s(x_A)},$$

  when $i, j \in A$.
- So MRS is independent of x_{-A}.
- Basically only prices p_A matter for the demand for x_A (of course expenditure on A goods is still affected by p_{-A}).

Separable preferences: Two-state budgeting

$$\begin{array}{cc} \max_{(x_A, x_{-A})} & U(s(x_A), x_{-A}) \\ \text{s.t} & p_A \cdot x_A + p_{-A} \cdot x_{-A} \leq m \end{array}$$

So solve

$$\begin{array}{cc} \max_{x_A} & s(x_A) \\ \text{s.t} & p_A \cdot x_A \leq m - p_{-A} \cdot x^*_{-A}(p, m) \end{array}$$

Obs: $x^*_{-A}(p, m)$ may depend on p_A (check that MRS is not independent of $i \in A$).

Separable preferences

A preference relation $\succeq$ on X is *A-separable* if, for all x, y

$$(x_A, x_{-A}) \succeq (y_A, x_{-A}) \text{ iff } (x_A, y_{-A}) \succeq (y_A, y_{-A})$$

Separable preferences

Let $X = \mathbb{R}^n_+$.

Theorem

A continuous preference relation is A-separable iff there exists $s : \mathbb{R}^A_+ \to \mathbb{R}$ and a $U : \mathbb{R} \times \mathbb{R}^{n-|A|}_+ \to \mathbb{R}$ that is strictly increasing in its first component, s.t

$$u(x_A, x_{-A}) = U(s(x_A), x_{-A})$$

is a utility representation for $\succeq$.

Separable preferences: Idea

By cont. there exists a cont. utility representation u.

By separability, $x_A \mapsto u(x_A, x_{-A})$ represent the same (cont) preference for any x_{-A}.

Let $s : \mathrm{R}_+^A \to \mathrm{R}$ represent this preference.

In fact we can set $s(x_A) = u(x_A, x^0_{-A})$ for arbitrary fixed x^0_{-A}.

Additively separable preferences

Going back to the two re-interpretations of consumption space: Suppose $K = 1$. When there is uncertainty, it's common to assume a utility of the form

$$u(x) = \sum_{\omega \in \Omega} \pi(\omega) v(x_\omega)$$

For intertemporal consumption, it's common to assume

$$u(x) = \sum_{t=1}^{T} \delta^t v(x_t).$$

Both are of the general form:

$$u(x) = \sum_{i=1}^{n} v_i(x_i);$$

that is: *additively separable*.

Additively separable preferences

Let $X = \mathbb{R}^n_+$. A preference $\succeq$ is *additively separable* if there are functions v_i s.t

$$u(x) = \sum_{i=1}^{n} v_i(x_i)$$

is a utility representation of $\succeq$.

Clearly this implies separability of any set of goods from the rest. In fact, once we rule out certain degenerate cases, this condition is necessary and sufficient for additive separability.

Additively separable preferences: Debreu's thm

A preference $\succeq$ is non-trivial if there exists a set A of cardinality at least three with the property that for each $i \in A$ there is x and y with $x_{-i} = y_{-i}$ and $x \succ y$.

Theorem

Let $\succeq$ be non-trivial and cont. Then $\succeq$ is additively separable iff, for all $A \subset [n]$, $\succeq$ is A-separable.

Obs: $n = 2$ is not contemplated by this theorem.

Additively separable preferences: Debreu's thm

For $n = 2$, Debreu introduced the following *double cancellation* axiom:

$$(x_1, x_2) \succeq (y_1, y_2) \text{ and } (y_1, z_2) \succeq (z_1, x_2) \Longrightarrow (x_1, z_2) \succeq (z_1, y_2)$$

Theorem

Let $X = \mathbf{R}^2_+$ and $\succeq$ be cont. on X. Then $\succeq$ is additively separable iff it satisfies double cancellation.

Additively separable preferences

Exercise

Show that additively separable preference satisfy double cancellation.

Additively separable preferences: uniqueness

Let $X = \mathbf{R}^n_+$; $n \geq 2$.

Theorem

Let $\succeq$ be cont. and additively separable. Suppose that

$$u = \sum_i v_i \qquad \text{and} \qquad u' = \sum_i v'_i$$

are two additively separable representations of $\succeq$. If none of the v_i functions are constant, then there exists numbers $\alpha > 0$ and β_i s.t

$$v'_i = \alpha v_i + \beta_i$$

So additively separable utility representations are unique up to a "cardinal" (affine) transformation with a common scale.

Additively separable preferences: convexity

Let $X = \mathbf{R}^n_+$; $n \geq 2$.

Theorem

Let $\succeq$ be cont. and additively separable. Suppose that $u = \sum_i v_i$ is an additively separable representation of $\succeq$, and that none of the v_i functions are constant. If $\succeq$ is convex, then all, except at most one, of these functions v_i is concave.

Corollary

If $u = \sum_i \alpha_i v$, where $\alpha_i > 0$ are scalars, is a representation of a continuous and convex preference $\succeq$, then v is concave.

A curiosity: Spherical preferences

Let $X = \mathrm{R}^n_+$; write $x \perp y$ when $x \cdot y = 0$.

$\succeq$ satisfies *Origin independent orthogonal independence (OIOI)* if:

Whenever $z \perp x$ and $z \perp y$,

$$w + x \succeq w + y \qquad \text{iff} \qquad w + x + z \succeq w + y + z.$$

Spherical preferences: Chambers and Echenique

Theorem

Suppose that $n \geq 3$. Then a preference $\succeq$ is cont. and satisfies OIOI iff one of the following is true:

1. There is $u \in \mathrm{R}^n$ for which $x \succeq y$ iff $u \cdot x \geq u \cdot y$
2. There is $x^* \in \mathrm{R}^n$ for which $x \succeq y$ iff $\|x - x^*\| \leq \|y - x^*\|$
3. There is $x^* \in \mathrm{R}^n$ for which $x \succeq y$ iff $\|x - x^*\| \geq \|y - x^*\|$.

Note that (1) is a linear preference (only weak. mon. SP); and (2) are "Euclidean" preferences, of the type used in Political Economy (x^* is an *ideal point*). These are only st. convex SP. The preferences in (3) are "anti-Euclidean."

Composite commodities

Table 1: Trend in product variety (number of models) for some products in the USA[6]

Product	1970	1998	2012
Automobile models	140	260	684
Newspapers	339	790	>5,000
TV screens (size)	5	15	43
Movies (at the cinema)	267	458	1,410
Breakfast cereals	160	340	4,945
Types of milk	4	19	>50
Mouthwash	15	66	113
Sports shoes	5	285	3,371
Brands of mineral water	16	50	195
Types of tights	5	90	594

Theory can handle 4900 different kinds of cereal, but empirical works needs to aggregate.

Composite commodity theorem (Hicks/Leontief)

Partition the set of $[n]$ commodities into K groups, group k having n_k elements.

Assume the commodities are numbered s.t $x = (x_1, \ldots, x_K)$ where each $x_k \in \mathbf{R}^{n_k}_{+}$.

We want to know when we can treat the groups as "composite commodities." For ex. "food," instead of milk, bread, etc. or "entertainment" instead of books, movies, etc.

Composite commodity theorem

Example:

- Scanner data
- Many different kinds of cereal, bottled water, etc. – just intractable.
- Aggregate into a single “cereal category.”
- How? By calculating expenditure on cereal. When is this OK?

Composite commodity theorem

Fix a price vector $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_K)$, where each $\bar{p}_k \in \mathrm{R}^{\mathrm{n}_k}$.

Idea:

- Keep relative prices between any two goods that belong to the same group.
- These are kept fixed as defined in $\bar{p}$.
- Vary relative prices and consider the resulting demand.

Composite commodity theorem

Define $\varphi\colon \mathrm{R}^K \to \mathrm{R}^n$ by

$$\varphi(\pi) = (\pi_1 \bar{p}_1, \ldots, \pi_K \bar{p}_K),$$

and note that φ is linear and monotonic. Define $\xi\colon \mathrm{R}^n_+ \to \mathrm{R}^K_+$ by

$$\xi(x) = (\bar{p}_1 \cdot x_1, \ldots, \bar{p}_K \cdot x_K),$$

and observe that ξ is linear and maps R^n onto R^K.

Composite commodity theorem

Obs. For all $x \in \mathrm{R}^n_+$, and $\pi \in \mathrm{R}^K_{++}$,

$$\pi \cdot \xi(x) = \varphi(\pi) \cdot x.$$

Let $\hat{u}\colon \mathrm{R}^n_+ \to \mathrm{R}_+$ be a utility with indirect utility function v that satisfies properties N, P, M, H, Q, S, Z.

Let u be the utility defined from v by the inversion formula, and let $x^*\colon \mathrm{R}^n_{++} \times \mathrm{R}_+ \to \mathrm{R}^n_+$ be its demand function. (For simplicity of notation, assume there is a demand function, not a correspondence.)

Composite commodity theorem

Define $X^*\colon \mathrm{R}^K_{++} \times \mathrm{R} \to \mathrm{R}^K_+$ by

$$X^*\big(\pi, m\big) = \xi\Big(x^*\big(\varphi(\pi), m\big)\Big).$$

Interpret X^* as a demand function for the composite commodities $\xi_1, \ldots, \xi_K$.

Composite commodity theorem

Theorem (Composite Commodity Theorem)

Under the conditions above, there is an upper semicontinuous quasiconcave monotone utility function

$$U\colon \mathrm{R}_+^K \to \mathrm{R}_+$$

that generates the demand X^*.

Punchline

If we want to analyze the market for breakfasts,

- milk
- cereal
- eggs
- bread.

We need:

1. Separability so as to ignore the rest of the goods that the agents' buy (housing, transportation, entertainment, etc.)
2. Composite commodity theorem to aggregate all 4900 kinds of cereal into a composite commodity. Same with bread, milk, etc.