Online Appendix

Approximate Expected Utility Rationalization

Federico Echenique Taisuke Imai Kota Saito

Contents

B Perturbed Subjective Expected Utility

We study the model of subjective expected utility (SEU), in which beliefs are not known. Instead, beliefs are subjective and unobservable. The analysis will be analogous to what we did for OEU, and therefore proceed at a faster pace. In particular, all the definitions and results parallel those of the section on OEU. The proof of the main result (the axiomatic characterization) is substantially more challenging here because both beliefs and utilities are unknown: there is a classical problem in disentangling beliefs from utility. The technique for solving this problem was introduced in Echenique and Saito (2015).

The following definition formalizes the concept of as-if choices.

Definition 10. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* Subjective Expected Utility (SEU) rational *if there exist* $\mu \in \Delta_{++}(S)$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* $k \in \mathcal{K}$,

$$y \in B(p^k, p^k \cdot x^k) \implies \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x_s^k).$$

Given a non-negative number e, we say that a dataset is e-belief-perturbed subjective expected utility (SEU) rational, if it can be rationalized using expected utility with perturbed beliefs for which the ratios of likelihood ratios do not differ by more than e.

Definition 11. *Let* $e \in \mathbf{R}_+$. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* e-belief-perturbed SEU rational *if there exist* $\mu^k \in \Delta_{++}(S)$ *for each* $k \in \mathcal{K}$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* $k \in \mathcal{K}$,

$$y \in B(p^k, p^k \cdot x^k) \implies \sum_{s \in S} \mu_s^k u(y_s) \leq \sum_{s \in S} \mu_s^k u(x_s^k)$$

and for each $k, l \in \mathcal{K}$ *and* $s, t \in S$

$$\frac{\mu_s^k / \mu_t^k}{\mu_s^l / \mu_t^l} \leq 1 + e. \tag{15}$$

Note that the definition of e-belief-perturbed SEU rationality differs from the definition of e-belief-perturbed OEU rationality, only in condition (15), establishing bounds on perturbations. Here there is no objective probability from which we can evaluate the deviation of the set $\{\mu^k\}$ of beliefs. Thus we evaluate perturbations *among* beliefs, as in (15).

Remark B.1. *The constraint on the perturbation applies for each* $k, l \in \mathcal{K}$ *and* $s, t \in S$, *so it implies for each* $k, l \in \mathcal{K}$ *and* $s, t \in S$

$$\frac{1}{1+e} \leq \frac{\mu_s^k / \mu_t^k}{\mu_s^l / \mu_t^l} \leq 1 + e.$$

Hence, when $e = 0$, *it must be that* $\mu_s^k / \mu_t^k = \mu_s^l / \mu_t^l$. *This implies that* $\mu^k = \mu^l$ *for a dataset that is* 0*-belief perturbed SEU rational.*

Next, we propose perturbed SEU rationality with respect to prices.

Definition 12. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-price-perturbed SEU rational *if there exist $\mu \in \Delta_{++}(S)$ and a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ and $\varepsilon^k \in \mathbf{R}_+^{|S|}$ for each $k \in \mathcal{K}$ such that, for all $k \in \mathcal{K}$,*

$$y \in B(\tilde{p}^k, \tilde{p}^k \cdot x^k) \implies \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x_s^k),$$

where for each $k \in \mathcal{K}$ and $s \in S$

$$\tilde{p}_s^k = p_s^k \varepsilon_s^k,$$

and for each $k, l \in \mathcal{K}$ and $s, t \in S$

$$\frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l} \leq 1 + e. \tag{16}$$

Again, the definition differs from the corresponding definition of price-perturbed OEU rationality only in condition (16), establishing bounds on perturbations. In condition (16), we measure the size of the perturbations by

$$\frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l},$$

not $\varepsilon_s^k / \varepsilon_t^k$ as in (5). This change is necessary to accommodate the existence of subjective beliefs. By choosing subjective beliefs appropriately, one can neutralize the perturbation in prices if $\varepsilon_s^k / \varepsilon_t^k = \varepsilon_s^l / \varepsilon_t^l$ for all $k, l \in \mathcal{K}$. That is, as long as $\varepsilon_s^k / \varepsilon_t^k = \varepsilon_s^l / \varepsilon_t^l$ for all $k, l \in \mathcal{K}$, if we can rationalize the dataset by introducing the noise with some subjective belief μ, then without using the noise, we can rationalize the dataset with another subjective belief μ' such that $\varepsilon_s^k \mu_s' / \varepsilon_t^k \mu_t' = \mu_s / \mu_t$.

Finally, we define utility-perturbed SEU rationality.

Definition 13. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-utility-perturbed SEU rational *if there exist $\mu \in \Delta_{++}(S)$, a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$, and $\varepsilon^k \in \mathbf{R}_+^{|S|}$ for each $k \in \mathcal{K}$ such that, for all $k \in \mathcal{K}$,*

$$y \in B(p^k, p^k \cdot x^k) \implies \sum_{s \in S} \mu_s \varepsilon_s^k u(y_s) \leq \sum_{s \in S} \mu_s \varepsilon_s^k u(x_s^k),$$

and for each $k, l \in \mathcal{K}$ and $s, t \in S$

$$\frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l} \leq 1 + e.$$

As in the previous section, given e, we can show that these three concepts of rationality are equivalent.

Theorem 3. *Let $e \in \mathbf{R}_+$ and D be a dataset. The following are equivalent:*

- *D is e-belief-perturbed SEU rational;*
- *D is e-price-perturbed SEU rational;*
- *D is e-utility-perturbed SEU rational.*

In light of Theorem 3, we shall speak simply of e-perturbed SEU rationality to refer to any of the above notions of perturbed SEU rationality.

Echenique and Saito (2015) prove that a dataset is SEU rational if and only if it satisfies a revealed-preference axiom termed the Strong Axiom for Revealed Subjective Expected Utility (SARSEU). SARSEU states that, for any test sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$, if each s appears as s_i (on the left of the pair) the same number of times it appears as s'_i (on the right), then

$$\prod_{i=1}^{n} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \leq 1.$$

SARSEU is no longer necessary for perturbed SEU rationality. This is easy to see, as we allow the decision maker to have a different belief μ^k for each choice k, and reason as in our discussion of SAROEU. Analogous to our analysis of OEU, we introduce a perturbed version of SARSEU to capture perturbed SEU rationality. Let $e \in \mathbf{R}_+$.

Axiom 2 (e-Perturbed SARSEU (e-PSARSEU))**.** *For any test sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1} \equiv \sigma$, if each s appears as s_i (on the left of the pair) the same number of times it appears as s'_i (on the right), then*

$$\prod_{i=1}^{n} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \leq (1+e)^{m(\sigma)}.$$

We can easily see the necessity of e-PSARSEU by reasoning from the first-order conditions, as in our discussion of e-PSAROEU. The main result of this section shows that e-PSARSEU is not only necessary for e-perturbed SEU rationality, but also sufficient.

Theorem 4. *Let $e \in \mathbf{R}_+$ and D be a dataset. The following are equivalent:*

- *D is e-perturbed SEU rational;*
- *D satisfies e-PSARSEU.*

It is easy to see that 0-PSARSEU is equivalent to SARSEU, and that by choosing e to be arbitrarily large it is possible to rationalize any dataset. As a consequence, we shall be interested in finding a minimal value of e that rationalizes a dataset. Echenique et al. (2019) apply the idea to datasets of choice under uncertainty collected in the laboratory as well as on the large-scale online survey of the general U.S. population.

B.1 Proof of Theorem 3

First, we prove a lemma that establishes Theorem 3 and proves useful for the sufficiency part of Theorem 4. This lemma provides "Afriat inequalities" for the problem at hand.

Lemma 6. *Given $e \in \mathbf{R}_+$, and let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent.*

(a) $(x^k, p^k)_{k=1}^K$ is e-belief-perturbed SEU rational.

(b) There are strictly positive numbers v_s^k, λ^k, μ_s^k, for $s \in S$ and $k \in \mathcal{K}$, such that

$$\mu_s^k v_s^k = \lambda^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies v_s^k \le v_{s'}^{k'}, \tag{17}$$

and for each $k, l \in \mathcal{K}$ and $s, t \in S$,

$$\frac{\mu_s^k / \mu_t^k}{\mu_s^l / \mu_t^l} \le 1 + e. \tag{18}$$

(c) $(x^k, p^k)_{k=1}^K$ is e-price-perturbed SEU rational.

(d) There are strictly positive numbers $\hat{v}_s^k$, $\hat{\lambda}^k$, μ_s, and ε_s^k for $s \in S$ and $k \in \mathcal{K}$, such that

$$\mu_s \hat{v}_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies \hat{v}_s^k \le \hat{v}_{s'}^{k'},$$

and for all $k, l \in \mathcal{K}$ and $s, t \in S$,

$$\frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l} \le 1 + e.$$

(e) $(x^k, p^k)_{k=1}^K$ is e-utility-perturbed SEU rational.

(f) There are strictly positive numbers $\hat{v}_s^k$, $\hat{\lambda}^k$, μ_s, and $\hat{\varepsilon}_s^k$ for $s \in S$ and $k \in \mathcal{K}$, such that

$$\mu_s \hat{\varepsilon}_s^k \hat{v}_s^k = \hat{\lambda}^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies \hat{v}_s^k \le \hat{v}_{s'}^{k'},$$

and for all $k, l \in \mathcal{K}$ and $s, t \in S$,

$$\frac{\hat{\varepsilon}_s^k / \hat{\varepsilon}_t^k}{\hat{\varepsilon}_s^l / \hat{\varepsilon}_t^l} \le 1 + e.$$

Proof. The equivalence between (a) and (b), the equivalence between (c) and (d), and the equivalence between (e) and (f) follow from standard arguments: see Echenique and Saito (2015) for details. Moreover, it is easy to see the equivalence between (d) and (f) with $\varepsilon_s^k = 1/\hat{\varepsilon}_s^k$ for each $k \in \mathcal{K}$ and $s \in S$. Hence, to prove the result, it suffices to show that (b) and (d) are equivalent.

To show that (d) implies (b), define $v = \hat{v}$ and

$$\mu_s^k = \frac{\mu_s}{\varepsilon_s^k} \Bigg/ \left(\sum_{s \in S} \frac{\mu_s}{\varepsilon_s^k} \right)$$

for each $k \in \mathcal{K}$ and $s \in S$ and

$$\lambda^k = \hat{\lambda}^k \Bigg/ \left(\sum_{s \in S} \frac{\mu_s}{\varepsilon_s^k} \right)$$

for each $k \in \mathcal{K}$. Then, $\mu^k \in \Delta_{++}(S)$. Since $\mu_s \hat{v}_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k$, we have $\mu_s^k v_s^k = \lambda^k p_s^k$. Moreover, for each $k, l \in \mathcal{K}$ and $s, t \in S$,

$$\frac{\mu_s^k / \mu_t^k}{\mu_s^l / \mu_t^l} = \frac{\varepsilon_t^k / \varepsilon_s^k}{\varepsilon_t^l / \varepsilon_s^l} \leq 1 + e.$$

To show (b) implies (d), for all $s \in S$ define $\hat{v} = v$ and

$$\mu_s = \sum_{k \in \mathcal{K}} \frac{\mu_s^k}{K}.$$

Then, $\mu \in \Delta_{++}(S)$. For all $k \in \mathcal{K}$, $\hat{\lambda}^k = \lambda^k$. For all $k \in \mathcal{K}$ and $s \in S$, define

$$\varepsilon_s^k = \frac{\mu_s}{\mu_s^k}.$$

For each $k \in \mathcal{K}$ and $s \in S$, since $\mu_s^k v_s^k = \lambda^k p_s^k$, we have $\mu_s v_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k$. Finally, for each $k, l \in \mathcal{K}$ and $s, t \in S$,

$$\frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l} = \frac{\mu_t^k / \mu_s^k}{\mu_t^l / \mu_s^l} \leq 1 + e.$$

□

B.2 Proof of the Necessity Direction of Theorem 4

Lemma 7. *Given $e \in \mathbf{R}_+$, if a dataset is e-belief-perturbed SEU rational then the dataset satisfies e-PSARSEU.*

Proof. Fix any sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n \equiv \sigma$ of pairs that satisfies conditions (i) and (ii) in Definition 7 and another condition that each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right), which we refer to as condition (iii) throughout this section. By the standard argument using the concavity of u, for each i, there exist $v_{s_i}^{k_i}, v_{s_i'}^{k_i'}, \lambda^{k_i}, \lambda^{k_i'}, \mu_{s_i}^{k_i}, \mu_{s_i'}^{k_i'}$ such that $v_{s_i'}^{k_i'} \geq v_{s_i}^{k_i}$ and $v_{s_i}^{k_i} = \frac{\lambda^{k_i} p_{s_i}^{k_i}}{\mu_{s_i}^{k_i}}$, and $v_{s_i'}^{k_i'} = \frac{\lambda^{k_i'} p_{s_i'}^{k_i'}}{\mu_{s_i'}^{k_i'}}$. Thus, we have

$$1 \geq \prod_{i=1}^n \frac{\lambda^{k_i} \mu_{s_i'}^{k_i'} p_{s_i}^{k_i}}{\lambda^{k_i'} \mu_{s_i}^{k_i} p_{s_i'}^{k_i'}} = \prod_{i=1}^n \frac{\mu_{s_i'}^{k_i'}}{\mu_{s_i}^{k_i}} \prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}},$$

where the second equality holds by condition (ii). See the proof of Lemma 10 of Echenique and Saito (2015) for detail. Thus,

$$\prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq \prod_{i=1}^{n} \frac{\mu_{s_i}^{k_i}}{\mu_{s_i'}^{k_i'}}.$$

In the following, we evaluate the right-hand side. For each (k, s), we first cancel out the same μ_s^k as much as possible both from the denominator and the numerator. Then, the number of μ_s^k remained in the numerator is $d(\sigma, k, s)$ as defined in Definition 8. Since the number of terms in the numerator and the denominator must be the same, the number of the remaining fractions is $m(\sigma) \equiv \sum_{s\in S} \sum_{k\in\mathcal{K}:d(\sigma,k,s)>0} d(\sigma, k, s)$. So by relabeling the index i to j if necessary, we obtain

$$\prod_{i=1}^{n} \frac{\mu_{s_i}^{k_i}}{\mu_{s_i'}^{k_i'}} = \prod_{j=1}^{m(\sigma)} \frac{\mu_{s_j}^{k_j}}{\mu_{s_j'}^{k_j'}}.$$

Consider the corresponding sequence $(x_{s_j}^{k_j}, x_{s_j'}^{k_j'})_{j=1}^{m(\sigma)}$. Since the sequence is obtained by canceling out x_s^k from the first element and the second element of the pairs the same number of times; and since the original sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n}$ satisfies conditions (ii) and (iii), it follows that $(x_{s_j}^{k_j}, x_{s_j'}^{k_j'})_{j=1}^{m(\sigma)}$ satisfies conditions (ii) and (iii).

By condition (iii), we can assume without loss of generality that $s_j = s_j'$ for each j. Fix $s^* \in S$. Then by condition (15) of e-belief perturbed SEU, for each $j \in \{1, \ldots, m(\sigma)\}$,

$$\frac{\mu_{s_j}^{k_j}}{\mu_{s_j'}^{k_j'}} = \frac{\mu_{s_j}^{k_j}}{\mu_{s_j}^{k_j'}} \leq (1+e)\frac{\mu_{s^*}^{k_j'}}{\mu_{s^*}^{k_j}}.$$

Moreover by condition (ii),

$$\prod_{j=1}^{m(\sigma)} \frac{\mu_{s^*}^{k_j'}}{\mu_{s^*}^{k_j}} = 1.$$

Therefore,

$$\prod_{i=1}^{n} \frac{\mu_{s_i}^{k_i}}{\mu_{s_i'}^{k_i'}} = \prod_{j=1}^{m(\sigma)} \frac{\mu_{s_i}^{k_j}}{\mu_{s_j'}^{k_j'}} \leq (1+e)^{m(\sigma)} \prod_{j=1}^{n} \frac{\mu_{s^*}^{k_j'}}{\mu_{s^*}^{k_j}} = (1+e)^{m(\sigma)},$$

and hence,

$$\prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq (1+e)^{m(\sigma)}.$$

□

B.3 Proof of the Sufficiency Direction of Theorem 4

The outline of the argument is the same as the proof of Theorem 2 and Echenique and Saito (2015). As in the proof of Theorem 2, we need three lemmas to prove the sufficiency direction.

We know from Lemma 6 that it suffices to find a solution to the Afriat inequalities (actually first-order conditions). So we set up the problem to find a solution to a system of linear inequalities obtained from using logarithms to linearize the Afriat inequalities in Lemma 6.

The first lemma, Lemma 8, establishes that e-PSARSEU is sufficient for e-belief-perturbed SEU rationality when the logarithms of the prices are rational numbers.

The second lemma, Lemma 9, establishes that we can approximate any dataset satisfying e-PSARSEU with a dataset for which the logarithms of prices are rational, and for which e-PSARSEU is satisfied.

Finally, Lemma 10 establishes the result by using another version of the theorem of the alternative, stated as Lemma 11 above.

The statements of the lemmas follow. The rest of the section is devoted to the proof of these lemmas.

Lemma 8. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)_{k=1}^k$ satisfy e-PSARSEU. Suppose that $\log(p^k_s) \in \mathbf{Q}$ for all k and s and $\log(1+e) \in \mathbf{Q}$. Then there are numbers v^k_s, λ^k, μ^k_s for $s \in S$ and $k \in \mathcal{K}$ satisfying* (17) *and* (18) *in Lemma 6.*

Lemma 9. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)_{k=1}^k$ satisfy e-PSARSEU. Then for all positive numbers $\overline{\varepsilon}$, there exist a positive real number $e' \in [e, e+\overline{\varepsilon}]$ and $q^k_s \in [p^k_s - \overline{\varepsilon}, p^k_s]$ for all $s \in S$ and $k \in \mathcal{K}$ such that $\log q^k_s \in \mathbf{Q}$ and the dataset $(x^k, q^k)_{k=1}^k$ satisfy e'-PSARSEU.*

Lemma 10. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)_{k=1}^k$ satisfy e-PSARSEU. Then there are numbers v^k_s, λ^k, μ^k_s for $s \in S$ and $k \in \mathcal{K}$ satisfying* (17) *and* (18) *in Lemma 6.*

B.3.1 Proof of Lemma 8

The proof is similar to the proof of Echenique and Saito (2015), which corresponds to the case with $e = 0$. By log-linearizing system (17), and inequality (18) in Lemma 6, we have for all $s \in S$ and $k \in \mathcal{K}$, such that

$$\log \mu^k_s + \log v^k_s = \log \lambda^k + \log p^k_s, \tag{19}$$

$$x^k_s > x^{k'}_{s'} \implies \log v^k_s \leq \log v^{k'}_{s'}, \tag{20}$$

and for all $k, l \in \mathcal{K}$ and $s, t \in S$,

$$\log \mu^k_s - \log \mu^k_t - \log \mu^l_s + \log \mu^l_t \leq \log(1+e). \tag{21}$$

We are going to write the system of inequalities (19)-(21) in matrix form. The formulation follows Echenique and Saito (2015), with some modifications.

Let A be a matrix with $K \times |S|$ rows and $2(K \times |S|) + K + 1$ columns, defined as follows: We have one row for every pair (k, s), two columns for every pair (k, s), one column for each k, and one last column. In the row corresponding to (k, s), the matrix has zeroes everywhere with the following exceptions: it has 1's in columns for (k, s); it has a -1 in the column for k; it has $-\log p_s^k$ in the very last column. The matrix A looks as follows:

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccccccc} \cdots & v_s^k & v_t^k & v_s^l & v_t^l & \cdots & \cdots & \mu_s^k & \mu_t^k & \mu_s^l & \mu_t^l & \cdots & \cdots & \lambda^k & \lambda^l & \cdots & p \end{array} \\
\begin{array}{c} \\ (k,s) \\ (k,t) \\ (l,s) \\ (l,t) \\ \\ \end{array} &
\left[\begin{array}{cccccc|cccccc|cccc|c}
 & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\
\cdots & 1 & 0 & 0 & 0 & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & \cdots & -1 & 0 & \cdots & -\log p_s^k \\
\cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & -1 & 0 & \cdots & -\log p_s^k \\
\cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & -1 & \cdots & -\log p_s^l \\
\cdots & 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & -1 & \cdots & -\log p_s^l \\
 & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots
\end{array}\right]
\end{array}.
$$

Next, we write the system of inequalities (20) and (21) in matrix form. There is one row in matrix B for each pair (k, s) and (k', s') for which $x_s^k > x_{s'}^{k'}$. In the row corresponding to $x_s^k > x_{s'}^{k'}$, we have zeroes everywhere with the exception of a -1 in the column for (k, s) and a 1 in the column for (k', s'). Matrix B has additional rows, that capture the system of inequalities (21): We do not need a constraint for each quadruple (k, l, s, t), as some of them would be redundant. Specifically, we need the constraints $\frac{\mu_s^k/\mu_t^k}{\mu_s^l/\mu_t^l} \le 1 + e$, and $\frac{\mu_s^l/\mu_t^l}{\mu_s^k/\mu_t^k} \le 1 + e$, which is equivalent to $\frac{\mu_s^k/\mu_t^k}{\mu_s^l/\mu_t^l} \ge 1/(1 + e)$. But note that $\frac{\mu_t^l/\mu_s^l}{\mu_t^k/\mu_s^k} \le 1 + e$ is redundant, as $\frac{\mu_t^l/\mu_s^l}{\mu_t^k/\mu_s^k} = \frac{\mu_s^k/\mu_t^k}{\mu_s^l/\mu_t^l}$. So for each (s, t) with $s < t$, and each $k \ne l$ we are going to have the constraint (k, l, s, t).[1] For each such (k, l, s, t) we have two rows. One of these rows has a 1 in the column for μ_s^k and μ_t^l, a -1 in the column for μ_t^k and μ_s^l, and $\log(1 + e)$ in the very last column; one of these rows has a 1 in the column for μ_t^k and μ_s^l, a -1 in the column for μ_s^k and μ_t^l, and $\log(1 + e)$ in the very last column. So this part of matrix B is as follows:

$$
\begin{array}{c}
\begin{array}{ccccccccccccccccc} \cdots & v_s^k & v_t^k & v_s^l & v_t^l & \cdots & \cdots & \mu_s^k & \mu_t^k & \mu_s^l & \mu_t^l & \cdots & \cdots & \lambda^k & \lambda^l & \cdots & p \end{array} \\
\left[\begin{array}{cccccc|cccccc|cccc|c}
 & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & -1 & 1 & 1 & -1 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) \\
\cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & -1 & -1 & 1 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) \\
 & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots
\end{array}\right]
\end{array}.
$$

[1] The inequality $s < t$ is simply a device to ensure that we choose only one of the two ordered pairs of s and t.

Finally, we have a matrix E which has a single row and has zeroes everywhere except for 1 in the last column.

To sum up, there is a solution to the system (19)-(21) if and only if there is a vector $u \in \mathbf{R}^{2(K\times|S|)+K+1}$ that solves the system of equations and linear inequalities

$$S1: \begin{cases} Au = 0, \\ Bu \geq 0, \\ Eu > 0. \end{cases}$$

The entries of A, B, and E are either 0, 1 or -1, with the exception of the last column of A and B. Under the hypotheses of the lemma we are proving, the last column consists of rational numbers. By Motzkin's theorem, then, there is such a solution u to $S1$ if and only if there is no rational vector (θ, η, π) that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

In the following, we shall prove that the non-existence of a solution u implies that the dataset must violate e-PSARSEU. Suppose then that there is no solution u and let (θ, η, π) be a rational vector as above, solving system $S2$.

The outline of the rest of the proof is similar to the proof of Theorem 2. Since (θ, η, π) are rational vectors, by multiplying all of their entries by a large enough integer, we can without loss of generality assume that (θ, η, π) are integer vectors.

Then we transform the matrices A and B using θ and η. (i) If $\theta_r > 0$, then create θ_r copies of the rth row; (ii) omitting row r when $\theta_r = 0$; and (iii) if $\theta_r < 0$, then θ_r copies of the rth row multiplied by -1.

Similarly, we create a new matrix by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

By using the transformed matrices and the fact that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$ and $\eta \geq 0$, we can prove the following claims:

Claim. There exists a sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ of pairs that satisfies conditions (i) and (ii) in Definition 7.

Proof. The proof is the same as in the proof of Lemma 11 in Echenique and Saito (2015). □

Claim. In the sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*} \equiv \sigma^*$, each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right).

Proof. Recall our construction of the matrix B. We have a constraint for each quadruple (k, l, s, t) with $s < t$. Denote the weight on the rows capturing $\frac{\mu_s^k/\mu_t^k}{\mu_s^l/\mu_t^l} \le 1 + e$ by $\eta(k, l, s, t)$. Let $n(x_s^k) \equiv \#\{i \mid x_s^k = x_{s_i}^{k_i}\}$ and $n'(x_s^k) \equiv \#\{i \mid x_s^k = x_{s_i'}^{k_i'}\}$. For notational convenience, define $\eta(k, l, s, t) = 0$ for all quadruples (k, l, s, t) with $t < s$.

For each $k \in \mathcal{K}$ and $s \in S$, in the column corresponding to μ_s^k in matrix A, remember that we have 1 if we have $x_s^k = x_{s_i}^{k_i}$ for some i and -1 if we have $x_s^k = x_{s_i'}^{k_i'}$ for some i. This is because a row in A must have 1 (-1) in the column corresponding to v_s^k if and only if it has 1 (-1, respectively) in the column corresponding to μ_s^k. By summing over the column corresponding to μ_s^k, we have $n(x_s^k) - n'(x_s^k)$.

Now we consider matrix B. In the column corresponding to μ_s^k and $s < t$, we have -1 in the row multiplied by $\eta(k, l, s, t)$ and 1 in the row multiplied by $\eta(l, k, s, t)$. By summing over the column corresponding to μ_s^k, we also have $-\sum_{l \ne k} \sum_{t \ne s} \eta(k, l, s, t) + \sum_{l \ne k} \sum_{t \ne s} \eta(l, k, s, t)$.

For each $k \in \mathcal{K}$ and $s \in S$, the column corresponding to μ_s^k of matrices A and B must sum up to zero; so we have

$$n(x_s^k) - n'(x_s^k) - \sum_{l \ne k} \sum_{t \ne s} \eta(k, l, s, t) + \sum_{l \ne k} \sum_{t \ne s} \eta(l, k, s, t) = 0.$$

Therefore, for each s,

$$\begin{aligned}
\sum_{k \in \mathcal{K}} \left(n(x_s^k) - n'(x_s^k) \right) &= \sum_{k \in \mathcal{K}} \left[\sum_{l \ne k} \sum_{t \ne s} \eta(k, l, s, t) - \sum_{l \ne k} \sum_{t \ne s} \eta(l, k, s, t) \right] \\
&= \sum_{t \ne s} \left[\sum_{k \in \mathcal{K}} \sum_{l \ne k} \eta(k, l, s, t) - \sum_{k \in \mathcal{K}} \sum_{l \ne k} \eta(l, k, s, t) \right] \\
&= 0.
\end{aligned}$$

This means that each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right). □

Claim. $\prod_{i=1}^{n^*} (p_{s_i}^{k_i} / p_{s_i'}^{k_i'}) > (1 + e)^{m(\sigma^*)}$.

Proof. By the fact that the last column must sum up to zero and E has one at the last column, we have

$$\sum_{i=1}^{n^*} \log \frac{p_{s_i'}^{k_i'}}{p_{s_i}^{k_i}} + \left(\sum_{k \in \mathcal{K}} \sum_{l \ne k} \sum_{s \in S} \sum_{t \ne s} \eta(k, l, s, t) \right) \log(1 + e) = -\pi < 0.$$

Hence, by multiplying -1, we have

$$\sum_{i=1}^{n^*} \log \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} - \left(\sum_{k \in \mathcal{K}} \sum_{l \ne k} \sum_{s \in S} \sum_{t \ne s} \eta(k, l, s, t) \right) \log(1 + e) > 0.$$

Remember that for all $k \in \mathcal{K}$ and $s \in S$,

$$n(x_s^k) - n'(x_s^k) = +\sum_{l \neq k} \sum_{t \neq s} \eta(k,l,s,t) - \sum_{l \neq k} \sum_{t \neq s} \eta(l,k,s,t) \leq \sum_{l \neq k} \sum_{t \neq s} \eta(k,l,s,t).$$

Since $d(\sigma^*, k, s) = n(x_s^k) - n'(x_s^k)$, we have

$$\begin{aligned} m(\sigma^*) \equiv \sum_{s \in S} \sum_{k \in \mathcal{K}: d(\sigma^*,k,s)>0} d(\sigma^*, k, s) &= \sum_{s \in S} \sum_{k \in \mathcal{K}} \max\{n(x_s^k) - n'(x_s^k), 0\} \\ &\leq \sum_{s \in S} \sum_{k \in \mathcal{K}} \sum_{l \neq k} \sum_{t \neq s} \eta(k,l,s,t). \end{aligned}$$

Therefore,

$$\sum_{i=1}^{n^*} \log \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} > \left(\sum_{k \in \mathcal{K}} \sum_{l \neq k} \sum_{s \in S} \sum_{t \neq s} \eta(k,l,s,t) \right) \log(1+e) \geq m(\sigma^*) \log(1+e).$$

This is a contradiction. □

B.3.2 Proof of Lemma 9

Let $\mathcal{X} = \{x_s^k \mid k \in \mathcal{K}, s \in S\}$. Consider the set of sequences that satisfy conditions (i) and (ii) in Definition 7, and (iii) in e-PSARSEU:

$$\Sigma = \left\{ (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n \subset \mathcal{X}^2 \;\middle|\; \begin{array}{l} (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n \text{ satisfies conditions (i) and (ii)} \\ \text{in Definition 7 and (iii) for some } n \end{array} \right\}.$$

For each sequence $\sigma \in \Sigma$, we define a vector $t_\sigma \in \mathbf{N}^{K^2|S|^2}$. For each pair $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})$, we shall identify the pair with $((k_i, s_i), (k_i', s_i'))$. Let $t_\sigma((k,s),(k',s'))$ be the number of times that the pair $(x_s^k, x_{s'}^{k'})$ appears in the sequence σ. One can then describe the satisfaction of e-PSARSEU by means of the vectors t_σ. Observe that t depends only on $(x^k)_{k=1}^K$ in the dataset $(x^k, p^k)_{k=1}^K$. It does not depend on prices.

For each $((k,s),(k',s'))$ such that $x_s^k > x_{s'}^{k'}$, define $\delta((k,s),(k',s')) = \log(p_s^k / p_{s'}^{k'})$. And define $\delta((k,s),(k',s')) = 0$ when $x_s^k \leq x_{s'}^{k'}$. Then, δ is a $K^2|S|^2$-dimensional real-valued vector. If $\sigma = (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$, then

$$\delta \cdot t_\sigma = \sum_{((k,s),(k',s')) \in (\mathcal{K} \times S)^2} \delta((k,s),(k',s')) t_\sigma((k,s),(k',s')) = \log \left(\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \right).$$

So the dataset satisfies e-PSARSEU if and only if $\delta \cdot t_\sigma \leq m(\sigma) \log(1+e)$ for all $\sigma \in \Sigma$.

Enumerate the elements in $\mathcal{X}$ in increasing order: $y_1 < y_2 < \cdots < y_N$, and fix an arbitrary $\underline{\xi} \in (0,1)$. We shall construct by induction a sequence $\{(\varepsilon_s^k(n))\}_{n=1}^N$, where $\varepsilon_s^k(n)$ is defined for all (k,s) with $x_s^k = y_n$.

By the denseness of the rational numbers, and the continuity of the exponential function, for each (k,s) such that $x_s^k = y_1$, there exists a positive number $\varepsilon_s^k(1)$ such that $\log(p_s^k \varepsilon_s^k(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon_s^k(1) < 1$. Let $\varepsilon(1) = \min\{\varepsilon_s^k(1) \mid x_s^k = y_1\}$.

In second place, for each (k,s) such that $x_s^k = y_2$, there exists a positive $\varepsilon_s^k(2)$ such that $\log(p_s^k \varepsilon_s^k(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon_s^k(2) < \varepsilon(1)$. Let $\varepsilon(2) = \min\{\varepsilon_s^k(2) \mid x_s^k = y_2\}$.

In third place, and reasoning by induction, suppose that $\varepsilon(n)$ has been defined and that $\underline{\xi} < \varepsilon(n)$. For each (k,s) such that $x_s^k = y_{n+1}$, let $\varepsilon_s^k(n+1) > 0$ be such that $\log(p_s^k \varepsilon_s^k(n+1)) \in \mathbf{Q}$, and $\underline{\xi} < \varepsilon_s^k(n+1) < \varepsilon(n)$. Let $\varepsilon(n+1) = \min\{\varepsilon_s^k(n+1) \mid x_s^k = y_n\}$.

This defines the sequence $(\varepsilon_s^k(n))$ by induction. Note that $\varepsilon_s^k(n+1)/\varepsilon(n) < 1$ for all n. Let $\bar{\xi} < 1$ be such that $\varepsilon_s^k(n+1)/\varepsilon(n) < \bar{\xi}$.

For each $k \in \mathcal{K}$ and $s \in S$, let $q_s^k = p_s^k \varepsilon_s^k(n)$, where n is such that $x_s^k = y_n$. We claim that the dataset $(x^k, q^k)_{k=1}^K$ satisfies e-PSARSEU. Let δ^* be defined from $(q^k)_{k=1}^K$ in the same manner as δ was defined from $(p^k)_{k=1}^K$.

For each pair $((k,s),(k',s'))$ with $x_s^k > x_{s'}^{k'}$, if n and m are such that $x_s^k = y_n$ and $x_{s'}^{k'} = y_m$, then $n > m$. By definition of ε,

$$\frac{\varepsilon_s^k(n)}{\varepsilon_{s'}^{k'}(m)} < \frac{\varepsilon_s^k(n)}{\varepsilon(m)} < \bar{\xi} < 1.$$

Hence,

$$\delta^*((k,s),(k',s')) = \log \frac{p_s^k \varepsilon_s^k(n)}{p_{s'}^{k'} \varepsilon_{s'}^{k'}(m)} < \log \frac{p_s^k}{p_{s'}^{k'}} + \log \bar{\xi} < \log \frac{p_s^k}{p_{s'}^{k'}} = \delta((k,s),(k',s')).$$

Now we choose e' such that $e' \geq e$ and $\log(1+e') \in \mathbf{Q}$.

Thus, for all $\sigma \in \Sigma$, $\delta^* \cdot t_\sigma \leq \delta \cdot t_\sigma \leq m(\sigma)\log(1+e) \leq m(\sigma)\log(1+e')$ as $t_. \geq 0$ and the dataset $(x^k, p^k)_{k=1}^K$ satisfies e-PSARSEU.

Therefore, the dataset $(x^k, q^k)_{k=1}^K$ satisfies e'-PSARSEU. Finally, note that $\underline{\xi} < \varepsilon_s^k(n) < 1$ for all n and each $k \in \mathcal{K}, s \in S$. So that by choosing $\underline{\xi}$ close enough to 1 we can take $(q^k)_{k=1}^K$ to be as close to $(p^k)_{k=1}^K$ as desired. We also can take e' to be as close to e as desired.

B.3.3 Proof of Lemma 10

Consider the system comprised by (19), (20), and (21) in the proof of Lemma 8. Let A, B, and E be constructed from the dataset as in the proof of Lemma 8. The difference with respect to Lemma 8 is that now the entries of A_4 and B_4 may not be rational. Note that the entries of E, B_i, and A_i, for $i = 1,2,3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (19), (20), and (21). Then, by the argument in the proof of Lemma 8 there is no solution to system $S1$. Lemma 11 (in Appendix B.4) with $\mathbf{F} = \mathbf{R}$ implies that there is a real vector (θ, η, π) such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$ and $\eta \geq 0, \pi > 0$. Recall that $E_4 = 1$, so we obtain that $\theta \cdot A_4 + \eta \cdot B_4 + \pi = 0$.

Let $(q^k)_{k=1}^K$ vectors of prices and a positive real number e' be such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies e'-PSARSEU and $\log q_s^k \in \mathbf{Q}$ for all k and s and $\log(1 + e') \in \mathbf{Q}$. (Such $(q^k)_{k=1}^K$ and e' exist by Lemma 9.) Construct matrices A', B', and E' from this dataset in the same way as A, B, and E is constructed in the proof of Lemma 8. Since only prices q^k and the bound e' are different in this dataset, only A'_4 and B'_4 may be different from A_4 and B_4, respectively. So $E' = E$, $B'_i = B_i$ and $A'_i = A_i$ for $i = 1, 2, 3$.

By Lemma 9, we can choose prices q^k and e' such that $|(\theta \cdot A'_4 + \eta \cdot B'_4) - (\theta \cdot A_4 + \eta \cdot B_4)| < \pi/2$. We have shown that $\theta \cdot A_4 + \eta \cdot B_4 = -\pi$, so the choice of prices q^k and e' guarantees that $\theta \cdot A'_4 + \eta \cdot B'_4 < 0$. Let $\pi' = -\theta \cdot A'_4 - \eta \cdot B'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A'_i = A_i$, $B'_i = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $\theta \cdot A'_4 + \eta \cdot B'_4 + \pi' E_4 = \theta \cdot A'_4 + \eta \cdot B'_4 + \pi' = 0$. We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution to $S2$ for matrices A', B', and E'.

Lemma 11 then implies that there is no solution to system $S1$ for matrices A', B', and E'. So there is no solution to the system comprised by (19), (20), and (21) in the proof of Lemma 8. However, this contradicts Lemma 8 because the dataset (x^k, q^k) satisfies e'-PSARSEU, $\log(1+e') \in \mathbf{Q}$, and $\log q_s^k \in \mathbf{Q}$ for all $k \in \mathcal{K}$ and $s \in S$.

B.4 Theorem of the Alternative

We shall use the following lemma, which is a version of the Theorem of the Alternative. This is Theorem 1.6.1 in Stoer and Witzgall (1970). We shall use it here in the cases where F is either the real or the rational number field.

Lemma 11. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong to a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{F}^n$ such that $Au = 0$, $Bu \geq 0$, $Eu \gg 0$.*

2. *There is $\theta \in \mathbf{F}^r$, $\eta \in \mathbf{F}^l$, and $\pi \in \mathbf{F}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

The next lemma is a direct consequence of Lemma 11. See Lemma 12 in Chambers and Echenique (2014) for proof.

Lemma 12. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E are rational numbers. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{R}^n$ such that $Au = 0$, $Bu \geq 0$, and $Eu \gg 0$.*
2. *There is $\theta \in \mathbf{Q}^r$, $\eta \in \mathbf{Q}^l$, and $\pi \in \mathbf{Q}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

C Computing e_*

We demonstrate how to calculate e_* given a dataset of choice under risk. To calculate the value, it is easier to use price-perturbed OEU rationality, rather than belief-perturbed OEU rationality. Formally, for a given dataset $(x^k, p^k)_{k=1}^K$, we want to compute e_* such that the dataset is price perturbed OEU rational given the number e. We can transform this problem into an easier problem with the following remark.

Remark C.1. *Given $e \in \mathbf{R}_+$, a dataset $(x^k, p^k)_{k=1}^K$ is e-price-perturbed OEU rational if and only if there are strictly positive numbers v_s^k, λ^k, μ_s, and ε_s^k for $s \in S$ and $k \in \mathcal{K}$, such that*

$$\mu_s^* v_s^k = \lambda^k \varepsilon_s^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies v_s^k \leq v_{s'}^{k'},$$

and for all $k \in \mathcal{K}$ and $s, t \in S$

$$\frac{1}{1+e} \leq \frac{\varepsilon_s^k}{\varepsilon_t^k} \leq 1 + e.$$

By the remark, e_* can be obtained by solving the following problem:

$$\begin{aligned} \min_{(\mu_s, v_s^k, \lambda^k, \varepsilon_s^k)_{k,s}} \ \max_{k \in \mathcal{K}, s, t \in S} \ & \frac{\varepsilon_s^k}{\varepsilon_t^k} \\ \text{s.t.} \quad & \mu_s^* v_s^k = \lambda^k \varepsilon_s^k p_s^k, \\ & x_s^k > x_{s'}^{k'} \implies v_s^k \leq v_{s'}^{k'}. \end{aligned}$$

We replace ε_s^k in the objective function using the equality constraint $\mu_s^* v_s^k = \lambda^k \varepsilon_s^k p_s^k$. By canceling out λ^k and log-linearizing, we obtain the following:

$$\begin{aligned} \min_{(v_s^k)_{k,s}} \ \max_{k \in \mathcal{K}, s, t \in S} \ & (\log \mu_s^* + \log v_s^k - \log p_s^k) - (\log \mu_t^* + \log v_t^k - \log p_t^k) \\ \text{s.t.} \quad & x_s^k > x_{s'}^{k'} \implies \log v_s^k \leq \log v_{s'}^{k'}. \end{aligned} \tag{$\star$}$$

We have the following result:

Remark C.2. *For any dataset $(x^k, p^k)_{k=1}^K$, e_* is the solution of the problem ($\star$), which always exists.*

Implementation. In the empirical applications, we solve the problem ($\star$) using `Matlab` (MathWorks).

For each subject, the decision in every trial is characterized by a tuple (a_1, a_2, x_1, x_2) where a_i represents the intercept of the budget line on each axis (here we call the x-axis "account 1" and the

y-axis "account 2"), and x_i represents the subject's allocation to account i. In order to rewrite the choice data in a price-consumption format as in the theory, we set prices $p_1 = 1$ (normalization) and $p_2 = a_1/a_2$. This gives us a dataset $(x^k, p^k)_{k=1}^K$.

Remember that the problem we are going to solve is:

$$\begin{aligned}
&\min_{(v_s^k)_{k,s}} \max_{k\in\mathcal{K},s,t\in S} (\log \mu_s^* + \log v_s^k - \log p_s^k) - (\log \mu_t^* + \log v_t^k - \log p_t^k) \\
&\text{s.t.} \quad x_s^k > x_{s'}^{k'} \implies \log v_s^k \le \log v_{s'}^{k'}.
\end{aligned} \tag{$\star$}$$

Our main task is to express this problem in matrix form.

Let z be a column vector of length $K \times |S| + K \times |S| + |S|$, whose first $K \times |S|$ entries correspond to each of $\log v_s^k$ and the last $K \times |S| + |S|$ entries are all 1.

$$z' = \Big[\underbrace{\cdots \quad \log v_s^k \quad \cdots}_{K\times|S|} \quad \underbrace{1 \quad \cdots \quad 1}_{K\times|S|} \quad \underbrace{1 \quad \cdots \quad 1}_{|S|} \Big].$$

This vector contains the control variables of the problem, $(v_s^k)_{k,s}$. The reason why we have $K \times |S| + |S|$ additional rows of 1 in the vector will become clear shortly.

We construct two matrices A and B. The first matrix A has $K \times |S| \times (|S| - 1)$ rows and $K \times |S| + K \times |S| + |S|$ columns, and looks as follows:

$$\begin{array}{cc}
 & \begin{array}{cccccc|cccccc|cccc} \cdots & v_s^k & v_t^k & v_s^l & v_t^l & \cdots & \cdots & p_s^k & p_t^k & p_s^l & p_t^l & \cdots & \cdots & \mu_s^* & \mu_t^* & \cdots \end{array} \\
\begin{array}{c} \vdots \\ (k,s,t) \\ (k,t,s) \\ (l,s,t) \\ (l,t,s) \\ \vdots \end{array} &
\left[\begin{array}{cccccc|cccccc|cccc}
 & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \\
\cdots & 1 & -1 & 0 & 0 & \cdots & \cdots & -\log p_s^k & \log p_t^k & 0 & 0 & \cdots & \cdots & 1 & -1 & \cdots \\
\cdots & -1 & 1 & 0 & 0 & \cdots & \cdots & \log p_s^k & -\log p_t^k & 0 & 0 & \cdots & \cdots & -1 & 1 & \cdots \\
\cdots & 0 & 0 & 1 & -1 & \cdots & \cdots & 0 & 0 & -\log p_s^l & \log p_t^l & \cdots & \cdots & 1 & -1 & \cdots \\
\cdots & 0 & 0 & -1 & 1 & \cdots & \cdots & 0 & 0 & \log p_s^l & -\log p_t^l & \cdots & \cdots & -1 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots &
\end{array} \right]
\end{array}.$$

Similarly, the second matrix B has $K \times |S| + K \times |S| + |S|$ columns. There is one row for every pair (k, s) and (k', s') with $x_s^k > x_{s'}^{k'}$. In the row corresponding to (k, s) and (k', s') we have zeroes everywhere with the exception of a -1 in the column for v_s^k and a 1 in the column for $v_{s'}^{k'}$.

Note that Az is a vector in which each of the $K \times |S| \times (|S| - 1)$ elements corresponds to $(\log \mu_s^* + \log v_s^k - \log p_s^k) - (\log \mu_t^* + \log v_t^k - \log p_t^k)$ for some combination of $k \in \mathcal{K}$ and $s, t \in S$. Hence, the objective function of the problem ($\star$) can be written as

$$\max_{k\in\mathcal{K},s,t\in S} (\log \mu_s^* + \log v_s^k - \log p_s^k) - (\log \mu_t^* + \log v_t^k - \log p_t^k) = \max_i\, (Az)_i\,,$$

where $(Az)_i$ denotes the ith element of vector Az. Similarly, each element of the vector Bz is $-\log v_s^k + \log v_{s'}^{k'}$ where (k, s) and (k', s') are such that $x_s^k > x_{s'}^{k'}$. Hence, $Bz \ge 0$ captures the constraint of the problem ($\star$).

Taken together, we can express the problem ($\star$) in matrix form:

$$\begin{aligned} &\min_z \ \max_i \, (Az)_i \\ &\text{s.t.} \ \ Bz \geq 0 \end{aligned}$$

We use the `Matlab` function `fmincon` to find a solution z^* of this convex programming problem. Finally, we obtain e_* from the optimized value of the problem:

$$e_* = \exp\left(\log(1 + e_*)\right) - 1 = \exp\left(\max_i \, (Az^*)_i\right) - 1.$$

D Illustration with Two-Budget Examples

D.1 Perturbed OEU Rationalization and e_*

We present simple examples of pairs of observations, in order to gain some insights about (perturbed) OEU rationalization and minimal e. For simplicity, we assume that there are two equally likely states ($\mu_1^* = \mu_2^* = 0.5$). Consider two budget sets $B^k(p^k, I^k)$ with $(p^1, I^1) = ((1, 2/3), 32)$ and $(p^2, I^2) = ((1, 1/2), 18)$, which are shown in Figure D.1, panel A. Let $\bar{x}_s^k = I^k/p_s^k$ denote the maximum amount of x_s one can choose in budget k (i.e., the x_s-intercept of the budget line).

We generate synthetic choice data on these budgets. For each budget B^k, we first take 31 equally-spaced points $\{0, \bar{x}_1^k/30, 2\bar{x}_1^k/30, \ldots, \bar{x}_1^k\}$ on the set $[0, \bar{x}_1^k]$ of all possible x_1^k. Each of these x_1^k specifies a point $x^k = (x_1^k, x_2^k)$ on the frontier of budget B^k, given by $(\eta\bar{x}_1^k/30, (30-\eta)\bar{x}_2^k/30)$, $\eta = 0, 1, \ldots, 30$. We now have a set of 31 equally-spaced points on the frontier of budget B^k and, by taking all possible combinations of x^1 and x^2 from these sets, we generate 961 synthetic choice data. We then calculate e_* for each of these synthetic datasets.

We find that there are five possible values of e_* (0, 0.155, 0.5, 0.732, 1) when choices are made on these two budget sets. Figure D.1B shows that we can partition the space $[0, \bar{x}_1^1] \times [0, \bar{x}_1^2]$ into five regions, depending on the value of e_*.[2]

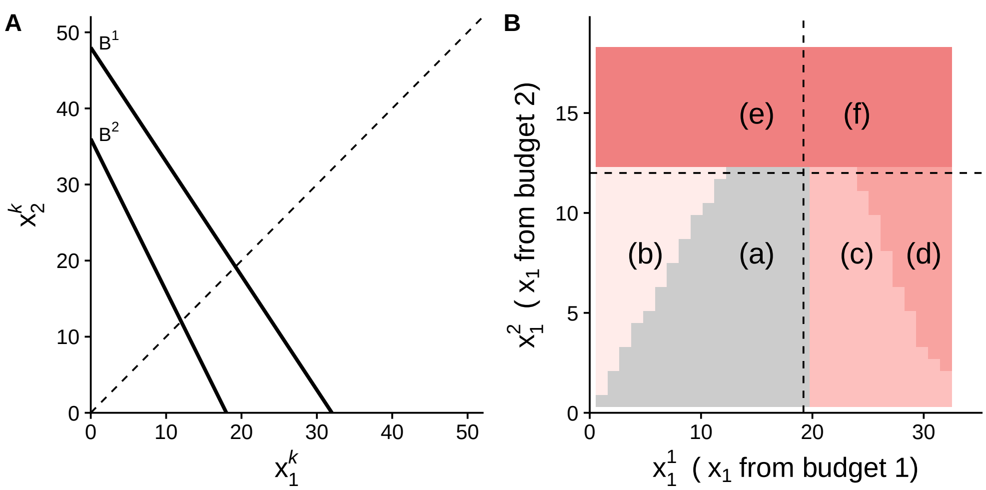


Figure D.1: Example. (A) Two budgets $B^k(p^k, I^k)$, $k = 1, 2$. (B) e_* for all combinations of x^1 and x^2 from two budgets. *Notes*: In panel B, darker colors correspond to larger e_* and the gray area corresponds to $e_* = 0$. The vertical (horizontal) dashed line indicates the value of x_1^1 (x_1^2) at which $x_1^1 = x_2^1$ ($x_1^2 = x_2^2$) holds.

[2] Choices in areas (e) and (f) have the same value of e_*. We treat them separately for later discussion.

Table D.1: Two-budget examples for illustration of e_*.

		Intercept		Income	Price		Allocation		Perturbed price		
Example	k	$\bar{x}_1^k$	$\bar{x}_2^k$	I^k	p_1^k	p_2^k	x_1^k	x_2^k	$\tilde{p}_1^k$	$\tilde{p}_2^k$	e_*
(a)	1	32	48	32	1	2/3	15.0	25.5	–	–	0.000
	2	18	36	18	1	1/2	8.0	20.0	–	–	
(b)	1	32	48	32	1	2/3	5.0	40.5	1.127	0.651	0.155
	2	18	36	18	1	1/2	8.0	20.0	0.921	0.532	
(c)	1	32	48	32	1	2/3	24.0	12.0	0.889	0.889	0.500
	2	18	36	18	1	1/2	8.0	20.0	0.788	0.585	
(d)	1	32	48	32	1	2/3	30.0	3.0	0.956	1.104	0.732
	2	18	36	18	1	1/2	8.0	20.0	0.711	0.616	
(e)	1	32	48	32	1	2/3	15.0	25.5	0.790	0.790	1.000
	2	18	36	18	1	1/2	15.0	6.0	0.857	0.857	
(f)	1	32	48	32	1	2/3	24.0	12.0	0.889	0.889	1.000
	2	18	36	18	1	1/2	15.0	6.0	0.857	0.857	

Now, to dig deeper, we choose six examples of choice data, one from each area (a)-(f). See Table D.1 for the corresponding list of six datasets and their associated e_*. The dataset in example (a) is rationalized by OEU and hence $e_* = 0$. Let us investigate the other five examples which are not OEU rationalizable.

Example (b). The dataset in example (b) is not OEU rationalizable. To see this, consider a sequence consisting of two pairs $\sigma = ((x_1^2, x_1^1), (x_2^1, x_2^2))$. It satisfies the requirement of a test sequence (Definition 7 in Section 3) since we have $x_1^2 > x_1^1$ and $x_2^1 > x_2^2$ and each $k \in \{1, 2\}$ appears once on the left of the pair and once on the right of the pair. However, it does not satisfy the conclusion of SAROEU since

$$\frac{p_1^2}{p_1^1} \cdot \frac{p_2^1}{p_2^2} = \frac{1}{1} \cdot \frac{2/3}{1/2} = \frac{4}{3} > 1.$$

Following Definition 8, we obtain the number $m(\sigma) = 2$. Then the sequence σ satisfies the conclusion of e-PSAROEU with $e = 0.155$:

$$\frac{p_1^2}{p_1^1} \cdot \frac{p_2^1}{p_2^2} = \frac{4}{3} \leq (1 + 0.155)^2.$$

Remark D.1. *Figure D.2 presents a geometrical argument for why this dataset is not OEU rational. Suppose, toward a contradiction, that the dataset is OEU rational. In panel B of Figure D.2, we include a budget set B^3 that has the same relative prices as the budget set B^2 but with a larger income so that the budget line passes through x^1 (i.e., B^3 is a parallel shift of B^2). Since the demand function of a risk-averse OEU agent is normal, the agent's choice x^3 from budget B^3 must be larger than the choice x^2 from budget B^2, which is indicated by the dash-dotted lines. The choice X^3 must lie in the line segment on B^3 that consists of bundles larger than x^2. However, such a choice would violate WARP. Hence, the (counterfactual) choice implied by risk-averse OEU at budget B^3 would be inconsistent with utility maximization, contradicting the assumption of OEU rationality. See Echenique and Saito (2015) for a similar discussion.*

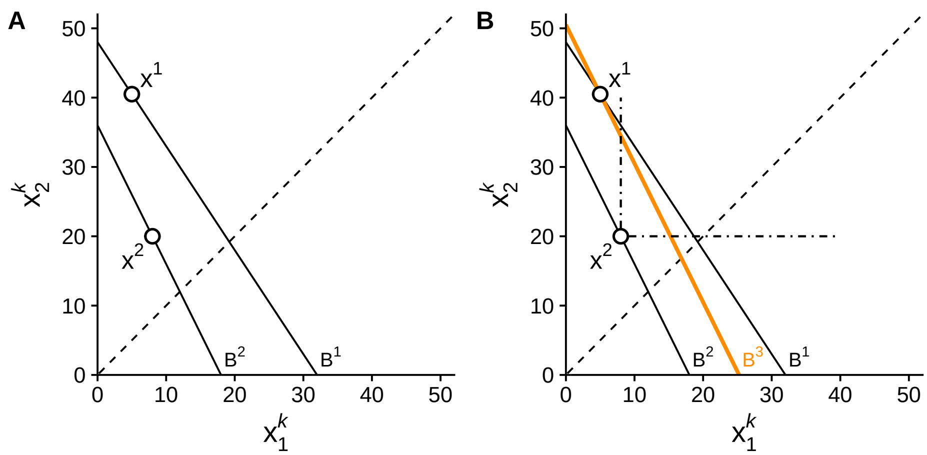


FIGURE D.2: Example (b) in Table D.1 is not OEU rationalizable. (A) The original dataset. (B) New budget set B^3 is added.

Examples (c)-(f). Next, we consider choices in regions (c)-(f), which are not OEU rationalizable because they involve violations of FOSD-monotonicity under the assumption of $\mu_1^* = \mu_2^* = 0.5$. When there are two equally likely states, choosing an option (x_1, x_2) at prices (p_1, p_2) violates monotonicity with respect to first-order stochastic dominance (FOSD-monotonicity) when either (i) $p_1 > p_2$ and $x_1 > x_2$ or (ii) $p_2 > p_1$ and $x_2 > x_1$ holds. Since the two states have the same objective probability in our datasets, choosing a larger payoff in the more expensive state violates FOSD-monotonicity. In Figure D.1B above, any allocation that appears on the right or above the dashed lines violates FOSD-monotonicity.

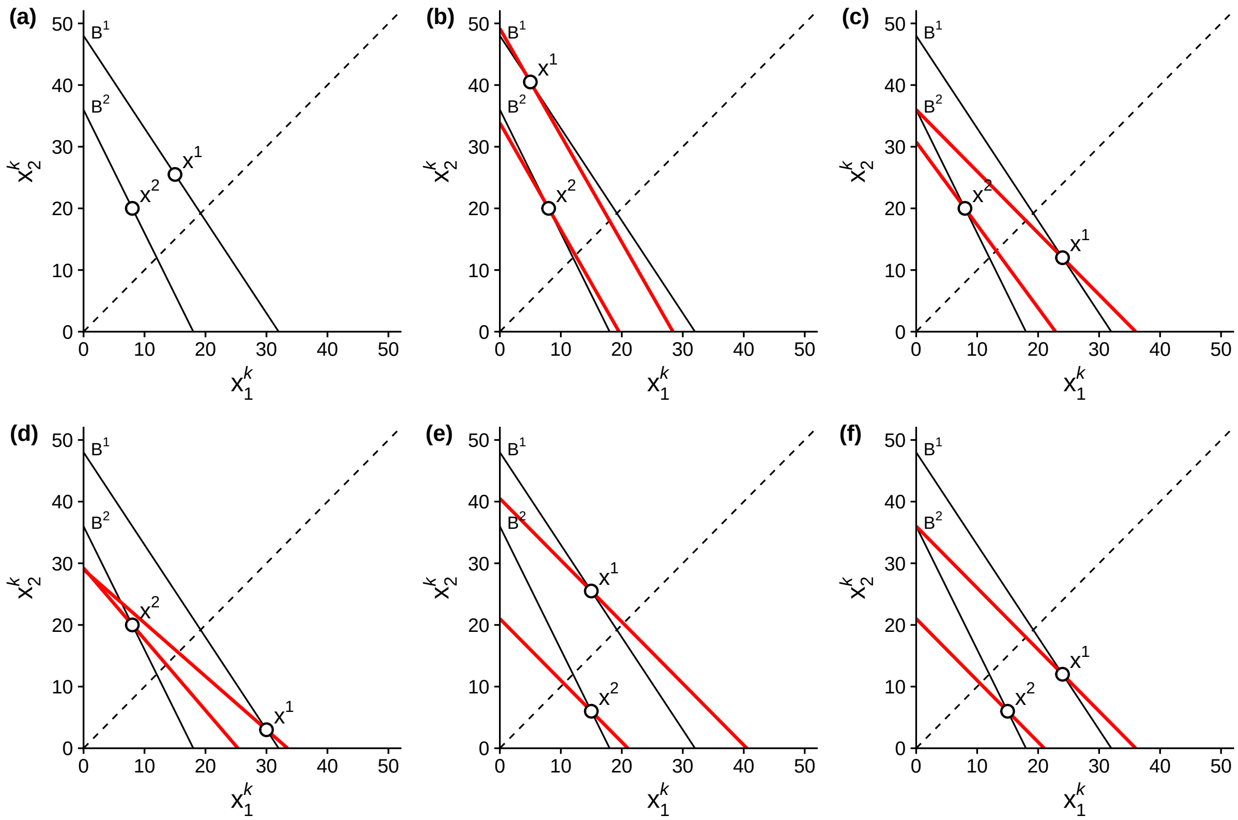


FIGURE D.3: Minimal price perturbation. *Notes*: Black lines represent the original, "true", budget lines, and red lines represent the minimally perturbed budget lines. Choices in example (a) are OEU rationalizable.

Minimal price perturbation. Let us consider how we rationalize choices in examples (b)-(f) with perturbed prices. Solving the constrained minimization problem described in Section C gives us a collection of ratios of perturbations $\{\varepsilon_1^k/\varepsilon_2^k\}_{k\in\mathcal{K}}$ that corresponds to e_* (these are simply part of the output of the minimization program). We can compute perturbed relative prices

$$\frac{\tilde{p}_1^k}{\tilde{p}_2^k} = \frac{p_1^k}{p_2^k}\frac{\varepsilon_1^k}{\varepsilon_2^k}.$$

Note that perturbed budgets must pass through the chosen bundles. Assuming that the income I^k is unchanged, we obtain perturbed prices $\tilde{p}_1^k$ and $\tilde{p}_2^k$ (see Table D.1). Figure D.3 illustrates these "minimally-perturbed" budget lines under which observed choices are e_*-perturbed OEU rationalizable.

Consider again the sequence $\sigma = ((x_1^2, x_1^1), (x_2^1, x_2^2))$ in example (b). We established above that it does not satisfy SAROEU under the original prices, but it does satisfy the conclusion of

SAROEU under the perturbed prices:

$$\frac{\tilde{p}_1^2}{\tilde{p}_1^1} \cdot \frac{\tilde{p}_2^1}{\tilde{p}_2^2} = \frac{0.921}{1.127} \cdot \frac{0.651}{0.532} = 1.$$

Note that the argument in Remark D.1 does not work in the perturbed dataset since perturbed budget lines are parallel to each other:

$$\frac{\tilde{p}_2^1}{\tilde{p}_1^1} = \frac{0.651}{1.127} = 0.578 = \frac{0.532}{0.921} = \frac{\tilde{p}_2^2}{\tilde{p}_1^2}.$$

Let us now move on to examples involving violations of FOSD-monotonicity. Consider example (c), in which allocation x^1 violates FOSD-monotonicity. A price perturbation eliminates this violation of FOSD-monotonicity by rotating budget B^1 so that $\tilde{p}_1^1 = \tilde{p}_2^1$. Example (d) is similar to example (c), but allocation x^1 is located further away from the 45-degree line while the other allocation x^2 is fixed. In this case, unlike example (c), rotating budget B^1 just to make $\tilde{p}_1^1 = \tilde{p}_2^1$ is not enough— we need to rotate it more and make $\tilde{p}_1^1 < \tilde{p}_2^1$.

In examples (e) and (f), violation of FOSD-monotonicity occurs on the "most extreme" budget in the dataset, which is B^2.[3] In this case, the size of the minimal perturbation necessary to eliminate the FOSD-monotonicity violation (rotating it so that perturbed prices become $\tilde{p}_1^2 = \tilde{p}_2^2$) dominates and the location of x^1 on budget B^1 does not matter (see Figure D.1B). In other words, e_* for this case is determined by the relative price $\max\{p_1^2/p_2^2, p_2^2/p_1^2\} = 2$.

[3]Budget B^2 is more extreme than budget B^1 in the sense that $\max\{p_1^2/p_2^2, p_2^2/p_1^2\} = 2 > 3/2 = \max\{p_1^1/p_2^1, p_2^1/p_1^1\}$.

D.2 Comparing e_* and the GRID Method

Polisson et al. (2020) develop a general method called the Generalized Restriction of Infinite Domain (GRID) for testing consistency with models of choice under risk and uncertainty. Using GRID, they provide a way to calculate CCEI for departures from OEU (called EU-CCEI) and risk-averse OEU (called cEU-CCEI).

In Section 4.2, we discuss the relationship between e_* and EU-CCEI as well as cEU-CCEI using real datasets from three experiments. To have a better understanding about the similarities and differences between our approach and the GRID method, we look at simple examples with two equally likely states and two budgets, as in Section D.1.

We consider seven examples listed in Table D.2, which cover different configurations of budget lines, exhibiting different properties (such as the point at which they cross, and relative steepness). We generated synthetic choice data following the same procedure as in Section D.1. Three measures, e_*, cEU-CCEI, and EU-CCEI, were calculated for each synthetic dataset.

Table D.2: Two-budget examples for comparing measures of deviation from OEU.

		Intercept		Price		Allocation		Measure		
Example	k	$\bar{x}_1^k$	$\bar{x}_2^k$	p_1^k	p_2^k	x_1^k	x_2^k	e_*	cEU-CCEI	EU-CCEI
(a)	1	32	48	1	2/3	5.0	40.5	0.155	0.985	1.000
	2	18	36	1	1/2	8.0	20.0			
(b)	1	32	48	1	2/3	5.0	40.5	0.080	0.993	1.000
	2	28	16	1	7/4	15.0	7.4			
(c)	1	32	48	1	2/3	15.0	25.5	0.162	0.969	1.000
	2	36	40	1	9/10	5.0	34.4			
(d)	1	32	48	1	2/3	15.0	25.5	0.061	0.995	1.000
	2	40	30	1	4/3	30.0	7.5			
(e)	1	32	48	1	2/3	8.0.0	36.0	0.155	0.986	1.000
	2	40	20	1	2	20.0	10.0			
(f)	1	32	36	1	8/9	6.0	29.25	0.333	0.959	1.000
	2	40	20	1	2	20.0	10.0			
(g)	1	32	33	1	32/33	6.0	26.8	0.393	0.952	0.980
	2	40	20	1	2	20.0	10.0			

In Figures D.4 and D.5, we plot two budgets in each example (first column), e_* under each pair (x_1^1, x_1^2) of payoffs in state 1 (second column), cEU-CCEI (third column), and EU-CCEI (fourth column). Observations in the gray regions are rationalizable by risk-averse OEU (second and third columns) or general OEU (fourth column). Otherwise, the darker the region is, the further the observation is from risk-averse OEU (second and third columns) or general OEU (fourth column). For ease of interpretation, we show a sample pair of choices (x^1, x^2) represented by hollow circles in the panels in the first column. The values of e_*, cEU-CCEI, and EU-CCEI associated with these sample choices are shown in the last three columns of Table D.2.

We observe that properties of e_* discussed in Section D.1 are general and not specific to the budget lines used in that particular example: The space $[0, \bar{x}_1^1] \times [0, \bar{x}_1^2]$ is partitioned into five areas depending on the value of e_*, and a violation of FOSD-monotonicity is penalized more if it occurs on the most extreme budget line.

In Figures D.4 and D.5, the comparison between the second column (e_*) and the third column (cEU-CCEI) highlights some differences between our perturbed OEU and the GRID method. The comparison between these two columns and the fourth column (EU-CCEI) shows an implication of assuming risk aversion.

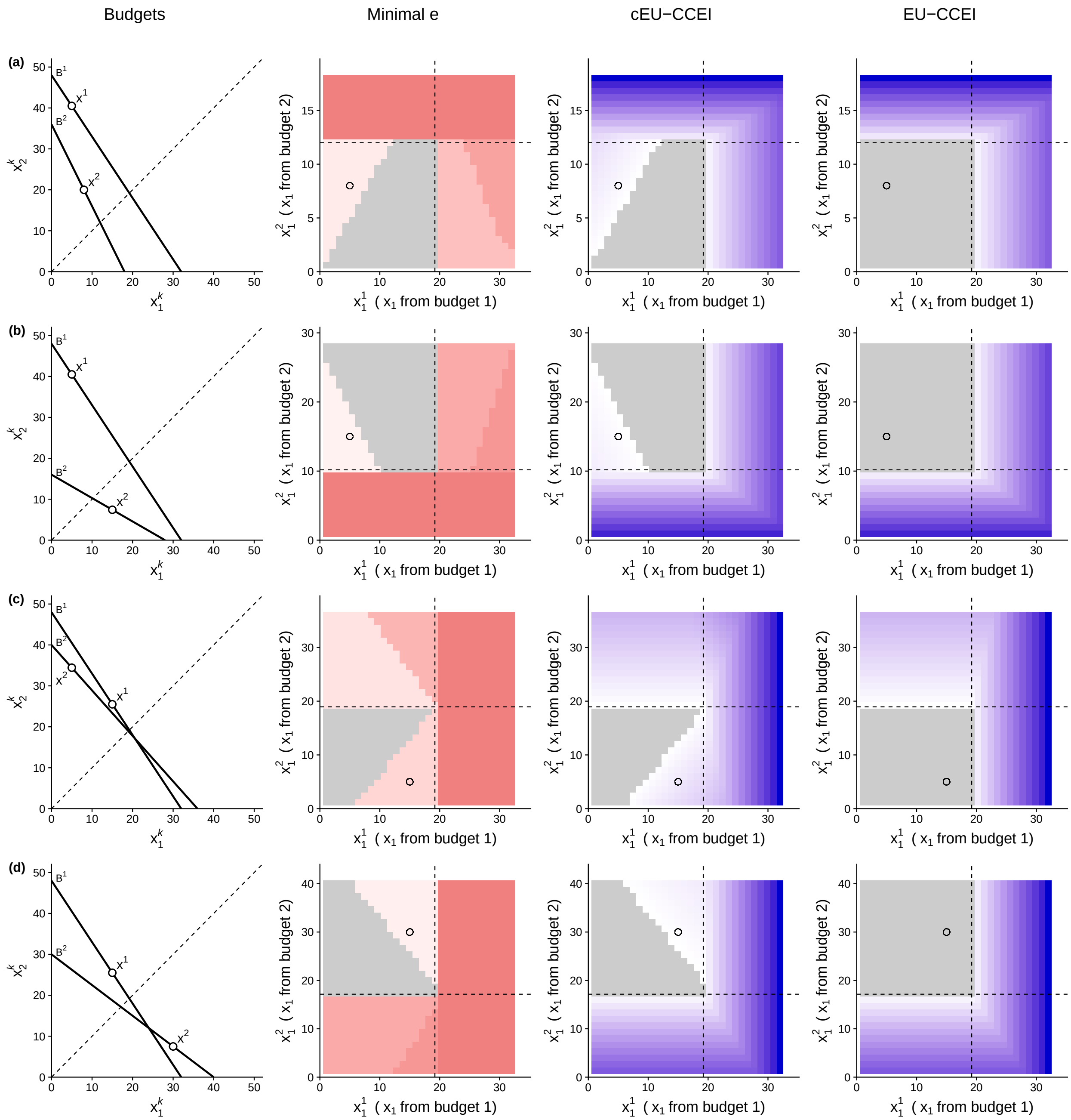


FIGURE D.4: Examples (a)-(d). *Notes*: In the second to fourth columns, darker colors correspond to a larger distance from (risk-averse or general) OEU. The vertical (horizontal) dashed line indicates the value of x_1^1 (x_1^2) at which $x_1^1 = x_2^1$ ($x_1^2 = x_2^2$) holds.

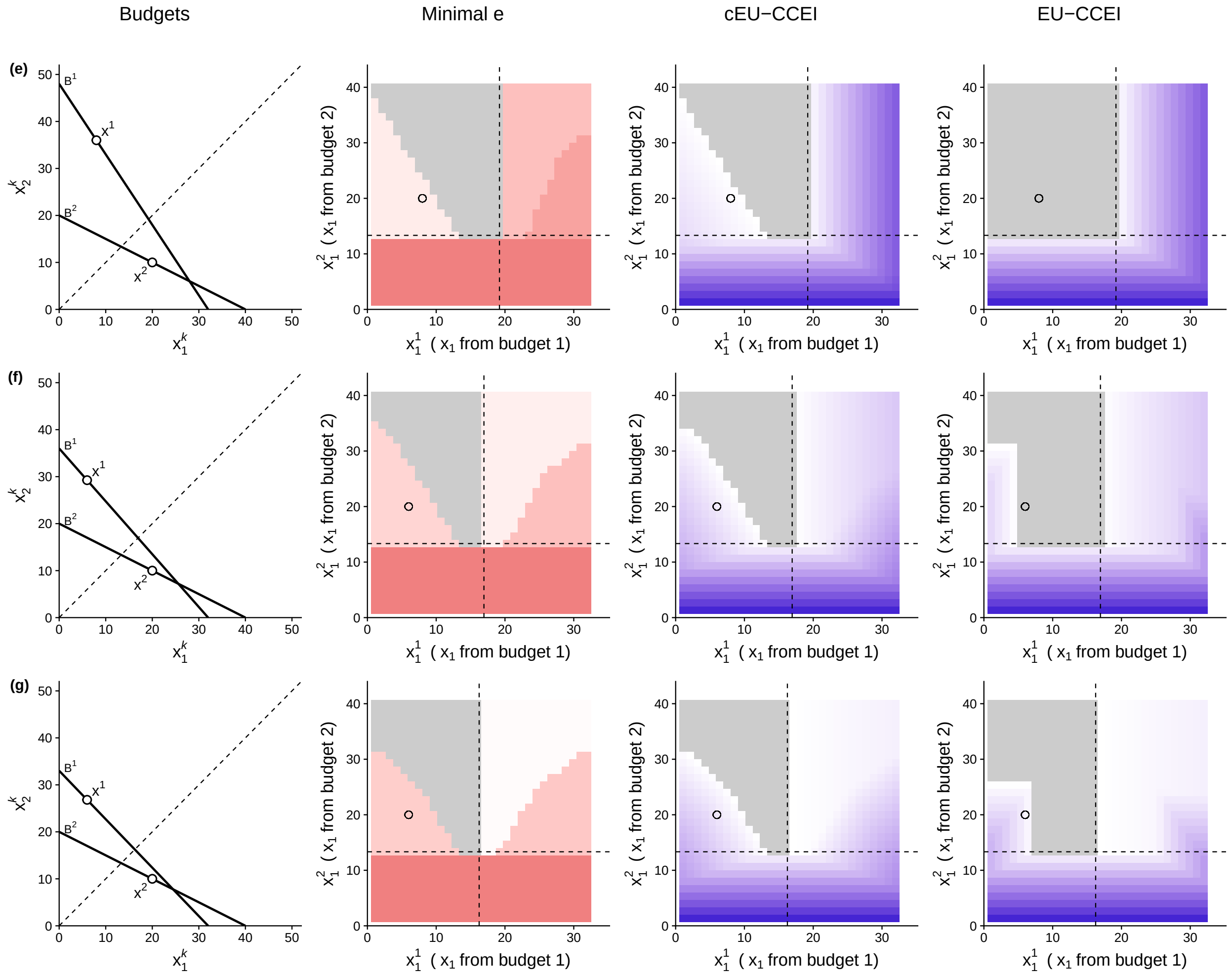


FIGURE D.5: Examples (e)-(g). *Notes*: In the second to fourth columns, darker colors correspond to a larger distance from (risk-averse or general) OEU. The vertical (horizontal) dashed line indicates the value of x_1^1 (x_1^2) at which $x_1^1 = x_2^1$ ($x_1^2 = x_2^2$) holds.

Treatment of violations of FOSD-monotonicity. Perturbed OEU and the GRID treat violations of FOSD-monotonicity differently and the difference is most visible when the violation occurs on the most extreme budget line. Suppose that budget B^1 is more extreme than budget B^2 (as in examples (c)-(d) in Figure D.4) and x^1 violates FOSD-monotonicity. In our perturbed OEU framework, it does not matter how far away x^1 is from the 45-degree line. It also does not matter the location of the other choice x^2 on budget B^2. In the GRID method, the distance from risk-averse OEU depends on both of these factors. The value of cEU-CCEI decreases as x^1 moves away from the 45-degree line, and it is also influenced by the location of x^2.

The following Figure D.6 illustrates the first point clearly.

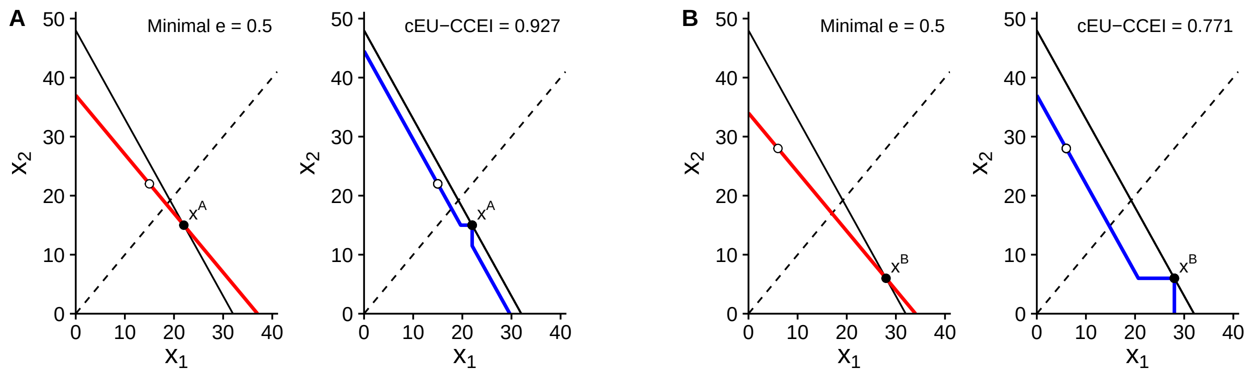


FIGURE D.6: Correcting violation of FOSD-monotonicity using price perturbation and the GRID method.

Consider a budget line characterized by prices $p^A = p^B = (1, 2/3)$ and income $I^A = I^B = 32$, and two choices from this budget which violate FOSD-monotonicity: $x^A = (22, 15)$ in panel A and $x^B = (28, 6)$ in panel B. Both approaches, perturbed OEU and the GRID method, eliminate the violation of FOSD-monotonicity by modifying the budget line so that it passes through the original choice and its "mirror image" (which is shown as a hollow circle in Figure D.6). Perturbed OEU rotates the budget line to achieve this while the GRID method shifts it down.[4] This explains why the value of e_* is the same in both cases while the value of cEU-CCEI is smaller for x^B than for x^A.

Consider another budget $p^C = (1, 2)$ and income $I^C = 40$ and a choice on the budget $x^C = (10, 15)$, which violates FOSD-monotonicity. The value of cEU-CCEI associated with this dataset is 0.875 (Figure D.7C). Now, let us add this observation (p^C, x^C) to previous single-budget examples (Figure D.6). Since budget (p^C, I^C) is more extreme than budgets (p^A, I^A) and (p^B, I^B), the size of

[4]Following the terminology in Polisson et al. (2020), a shrunken version of budget B^k for a given number $e \in [0, 1]$ is defined by $B^k(e) = \{z \in \mathbf{R}^2_+ : p^k \cdot z \leq ep^k \cdot x^k\} \cup \{x^k\}$. The downward extension of a shrunken budget is given by $\{z \in \mathbf{R}^2_+ : p^k \cdot z \leq ep^k \cdot x^k\} \cup \{z \in \mathbf{R}^2_+ : z \leq x^k\}$. Blue lines in Figure D.6 show the frontiers of the downward extension of $B^k(e^k)$, where e^k is set at the value of cEU-CCEI.

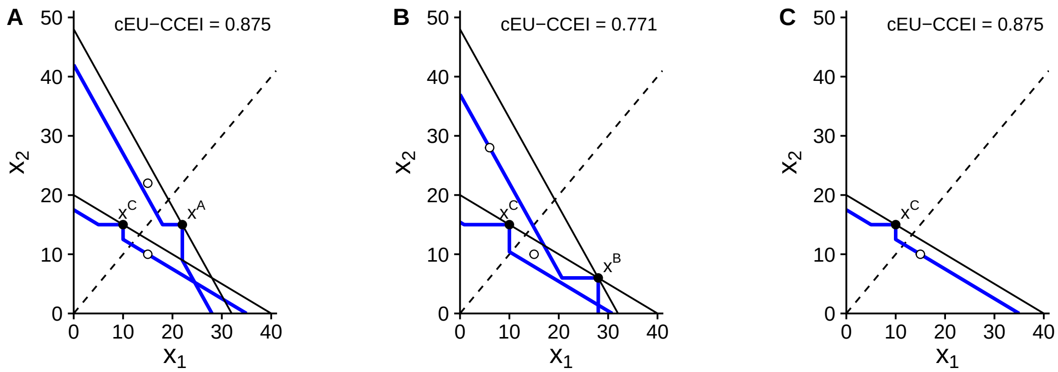


FIGURE D.7: Correcting violations of FOSD-monotonicity using the GRID method.

minimal perturbation is fully determined by the slope of budget C and the location of the other choice (x^A or x^B) does not matter (as discussed in Section D.1). The value of cEU-CCEI, however, depends on the location of the other choice x^A or x^B, fixing x^C the same.

Figure D.7 illustrates the point. In panel A, x^C is a more severe violation of FOSD-monotonicity ($0.875 < 0.927$). In panel B, x^B is a more severe violation ($0.875 > 0.771$).

E Minimum Perturbation Test

We provide a detailed exposition of how we formulate our statistical test. Our approach is inspired by the methodology laid out in Varian (1985), Echenique et al. (2011), and Echenique et al. (2016).

Let $D_{\text{obs}} = (p^k, x^k)_{k=1}^K$ denote an observed dataset and $D_{\text{true}} = (\tilde{p}^k, x^k)_{k=1}^K$ denote the "true" dataset, the one the agent had in mind. Let us suppose that observed prices and the "true" prices are related in the following way: $\tilde{p}_s^k = p_s^k \tilde{\varepsilon}_s^k$, where $\tilde{\varepsilon}_s^k > 0$ for all $s \in S$ and $k \in \mathcal{K}$.

The null hypothesis we consider is

$$\text{H}_0: \text{ The "true" dataset } D_{\text{true}} = (\tilde{p}^k, x^k)_{k=1}^K \text{ is OEU rational.}$$

If we could (somehow) observe the "true" dataset D_{true}, we could compute the "test statistic"

$$\mathcal{E} = \max_{k \in \mathcal{K}, s, t \in S} \frac{\tilde{\varepsilon}_s^k}{\tilde{\varepsilon}_t^k}$$

and we would reject the null hypothesis if $\mathcal{E}$ is "too large" in the sense that it exceeds the critical value.

There are two major challenges in this approach. First, we do not observe the "true" dataset D_{true} and hence we cannot compute the test statistic $\mathcal{E}$ itself. Second, we need to impose some assumptions to derive the distribution of $\mathcal{E}$.

We address how we tackle each of these issues below.

Lower bound on $\mathcal{E}$. Under the null hypothesis that the "true" dataset is OEU rational, a slight modification of Lemma 7 in Echenique and Saito (2015) implies that there exist strictly positive numbers $\tilde{v}_s^k$ and $\tilde{\lambda}^k$ for all $s \in S$ and $k \in \mathcal{K}$ such that

$$\log \mu_s^* + \log \tilde{v}_s^k - \log \tilde{\lambda}^k - \log \tilde{p}_s^k = 0 \quad \text{and} \quad x_s^k > x_{s'}^{k'} \Longrightarrow \log \tilde{v}_s^k \leq \log \tilde{v}_{s'}^{k'}.$$

Substituting the relationship $\tilde{p}_s^k = p_s^k \tilde{\varepsilon}_s^k$ for all $s \in S$ and $k \in \mathcal{K}$ yields

$$\log \mu_s^* + \log \tilde{v}_s^k - \log \tilde{\lambda}^k - \log p_s^k - \log \tilde{\varepsilon}_s^k = 0 \quad \text{and} \quad x_s^k > x_{s'}^{k'} \Longrightarrow \log \tilde{v}_s^k \leq \log \tilde{v}_{s'}^{k'}.$$

This implies that the tuple $(\tilde{v}_s^k, \tilde{\lambda}^k, \tilde{\varepsilon}_s^k)_{s \in S, k \in \mathcal{K}}$ satisfies the constraints (but is not necessarily a solution) of the optimization problem:

$$\begin{aligned} \min_{(v_s^k, \lambda^k, \varepsilon_s^k)_{s,k}} \quad & \max_{k \in \mathcal{K}, s, t \in S} \frac{\varepsilon_s^k}{\varepsilon_t^k} \\ \text{s.t.} \quad & \log \mu_s^* + \log v_s^k - \log \lambda^k - \log p_s^k - \log \varepsilon_s^k = 0 \\ & x_s^k > x_{s'}^{k'} \Longrightarrow \log v_s^k \leq \log v_{s'}^{k'}. \end{aligned}$$

Note that this optimization problem is the one we use to calculate e_* given the observed dataset D_{obs} (see Section C above). Therefore, under the null hypothesis, e_* gives an observable lower bound on the test statistic $\mathcal{E}$:

$$e_*((p^k, x^k)_{k=1}^K) \leq \max_{k \in \mathcal{K}, s,t \in S} \frac{\tilde{\varepsilon}_s^k}{\tilde{\varepsilon}_t^k}.$$

(By writing $e_*((p^k, x^k)_{k=1}^K)$, we make it explicit that e_* is calculated from the observed dataset.)

With this observation in mind, we construct a test, using $e_*((p^k, x^k)_{k=1}^K)$ instead of $\mathcal{E}$, as follows:

$$\text{Reject the null hypothesis if } \int_{e_*((p^k,x^k)_{k=1}^K)}^{\infty} f_{\mathcal{E}}(z)dz < \alpha.$$

where α is the size of the test and $f_{\mathcal{E}}$ is the density function of the distribution of $\mathcal{E} = \max_{k,s,t} \tilde{\varepsilon}_s^k / \tilde{\varepsilon}_t^k$. Given a nominal size α, we can find a critical value C_α satisfying $\Pr[\mathcal{E} > C_\alpha] = \alpha$. So we reject the null hypothesis if $e_*((p^k, x^k)_{k=1}^K) > C_\alpha$, and we are certain that $\mathcal{E} > C_\alpha$ is indeed the case (since $e_*((p^k, x^k)_{k=1}^K) \leq \mathcal{E}$). This also means that our test is conservative since the true size of the test is smaller than α: $\Pr[e_*((p^k, x^k)_{k=1}^K) > C_\alpha \mid H_0 \text{ is true}] \leq \Pr[\mathcal{E} > C_\alpha \mid H_0 \text{ is true}] = \alpha$).

Parameter tuning for the distribution of noise. In order to perform the test, we need to know the distribution of $\mathcal{E}$ and the critical value C_α given a significance level α. We obtain the distribution of $\mathcal{E}$ by assuming that the noise term ε follows a log-normal distribution, $\varepsilon \sim \Lambda(\nu, \xi^2)$. [5] In other words, noise terms $\tilde{\varepsilon}_s^k$ in the "true" dataset are i.i.d. draws from $\Lambda(\nu, \xi^2)$.

The crucial step in our approach is the selection of parameters (ν, ξ^2). Once we know (ν, ξ^2), we can simulate the distribution of $\mathcal{E} = \max_{k,s,t} \tilde{\varepsilon}_s^k / \tilde{\varepsilon}_t^k$. It is natural to choose these parameters so that there is no price perturbation on average (i.e., $\mathbf{E}[\varepsilon] = 1$). However, there is no objective guide to choosing an appropriate level of $\text{Var}(\varepsilon)$. Therefore, we use variation in (relative) prices observed in the data.

Let $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ denote random variables of observed prices and "true" prices, respectively. Assuming that the noise term ε is independent of the random selection of prices $\mathfrak{p}$ in the experiment,

[5] Note that parameters (ν, ξ^2) correspond to the mean and the variance of the random variable in the log-scale. In other words, $\log \varepsilon \sim N(\nu, \xi^2)$. The moments of the log-normal distribution $\varepsilon \sim \Lambda(\nu, \xi^2)$ are then calculated by $\mathbf{E}[\varepsilon] = \exp(\nu + \xi^2/2)$ and $\text{Var}(\varepsilon) = \exp(2\nu + \xi^2)(\exp(\xi^2) - 1)$.

we have

$$
\begin{aligned}
& \operatorname{Var}(\tilde{\mathfrak{p}}) = \operatorname{Var}(\mathfrak{p}) \cdot \operatorname{Var}(\varepsilon) + \operatorname{Var}(\mathfrak{p}) \cdot \mathbf{E}[\varepsilon]^2 + \mathbf{E}[\mathfrak{p}]^2 \cdot \operatorname{Var}(\varepsilon) \\
\iff \quad & \frac{\operatorname{Var}(\tilde{\mathfrak{p}})}{\operatorname{Var}(\mathfrak{p})} = \mathbf{E}[\varepsilon]^2 + \left(1 + \frac{\mathbf{E}[\mathfrak{p}]^2}{\operatorname{Var}(\mathfrak{p})}\right) \operatorname{Var}(\varepsilon) \\
\iff \quad & \operatorname{Var}(\varepsilon) = \left(\frac{\operatorname{Var}(\tilde{\mathfrak{p}})}{\operatorname{Var}(\mathfrak{p})} - 1\right) \left(1 + \frac{\mathbf{E}[\mathfrak{p}]^2}{\operatorname{Var}(\mathfrak{p})}\right)^{-1}. \qquad (22)
\end{aligned}
$$

We use the variation in prices observed in the data, $(p_s^k)_{s \in S, k \in \mathcal{K}}$, as proxies for $\mathbf{E}[\mathfrak{p}]$ and $\operatorname{Var}(\tilde{\mathfrak{p}})$. In this way, we transform the question of selecting the variance of the noise term, $\operatorname{Var}(\varepsilon)$, into a question of selecting "reasonable" variance of perturbed prices, $\operatorname{Var}(\tilde{\mathfrak{p}})$.

Price misperception as a hypothesis test. Let us consider an agent who has trouble telling the distributions of prices, $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$, apart (that is why the agent misperceives prices). In particular, we assume that the agent has trouble telling the two variances apart.

Let us consider an agent who has trouble telling the two variances apart. More generally, the agent has trouble telling the distributions of prices apart, which is why she is confusing actual and perceived prices, but the distribution depends only on the variance; so we focus on variance. Consider a hypothesis test for the null hypothesis that the variance of a normal random variable with known mean has variance σ_0^2 against the alternative that $\sigma^2 \geq \sigma_0^2$. Let $\hat{\sigma}_n^2$ be the sample variance.

The agent performs an upper-tailed chi-squared test defined as

$$
\begin{aligned}
\mathrm{H}_0^A &: \ \sigma^2 = \sigma_0^2, \\
\mathrm{H}_1^A &: \ \sigma^2 > \sigma_0^2.
\end{aligned}
$$

The test statistic is:

$$
T_n = \frac{(n-1)\hat{\sigma}_n^2}{\sigma_0^2}
$$

where n is the sample size (i.e., the number of budget sets). The sampling distribution of the test statistic T_n under the null hypothesis follows a chi-squared distribution with $n-1$ degrees of freedom.[6]

We consider the probability η^I of rejecting the null hypothesis when it is true, a type I error; and the probability η^{II} of failing to reject the null hypothesis when the alternative $\sigma^2 = \sigma_1^2 > \sigma_0^2$ is true, a type II error. The test rejects the null hypothesis that the variance is σ_0^2 if

$$
T_n > \chi^2_{1-\alpha, n-1}
$$

[6] The superscript A in H_0^A and H_1^A is to distinguish the hypothesis test, which the agent is assumed to perform, from the one we (researchers) perform to interpret the magnitude of e_* discussed above.

where $\chi^2_{1-\alpha,n-1}$ is the critical value of a chi-squared distribution with $n-1$ degree of freedom at the significance level α, defined by $\Pr[\chi^2 < \chi^2_{1-\alpha,n-1}] = 1 - \eta^I$.[7]

Under the alternative hypothesis that $\sigma^2 = \sigma_1^2 > \sigma_0^2$, the statistic $(\sigma_0^2/\sigma_1^2) \cdot T_n$ follows a chi-squared distribution (with $n-1$ degrees of freedom). Then, the probability η^{II} of making a type II error is given by

$$\begin{aligned} \eta^{II} = \Pr[T_n < \chi^2_{1-\alpha,n-1} \mid \mathrm{H}_1 : \sigma_1^2 > \sigma_0^2 \text{ is true}] &= \Pr\left[\frac{\sigma_0^2}{\sigma_1^2} \cdot T_n < \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1}\right] \\ &= \Pr\left[\chi^2 < \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1}\right]. \end{aligned}$$

Let $\chi^2_{\beta,n-1}$ be the value that satisfies $\Pr[\chi^2 < \chi^2_{\beta,n-1}] = \eta^{II}$. Then, given η^I and η^{II}, we obtain

$$\begin{aligned} \Pr\left[\chi^2 < \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1}\right] = \eta^{II} \iff & \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1} = \chi^2_{\beta,n-1} \\ \iff & \frac{\sigma_1^2}{\sigma_0^2} = \frac{\chi^2_{1-\alpha,n-1}}{\chi^2_{\beta,n-1}}. \end{aligned}$$

As a consequence, given a measured variance σ_0^2, calculated from observed prices, and assumed values for η^I and η^{II}, we can back out the minimum "detectable" value of the variance σ_1^2. From this variance of prices, we obtain $\mathrm{Var}(\varepsilon)$ using equation (22).

[7] An alternative approach, without assuming that a distribution for T_n, and based on a large sample approximation to the distribution of T_n, yields very similar results. Calculations and empirical findings are available from the authors upon request.

F Supplementary Empirical Analysis

F.1 Summary Statistics of e_*

TABLE F.1: Summary Statistics of e_*

	N	Mean	SD	Q1	Median	Q3	Min	Max
Panel A: All data								
CKMS	1182	3.034	1.816	1.563	2.729	4.184	0.035	8.772
CMW	1119	2.480	1.126	1.659	2.533	3.592	0.000	4.387
CS	1423	2.490	1.707	1.157	2.081	3.370	0.000	10.021
Panel B: CCEI = 1								
CKMS	270	3.058	2.176	1.154	2.662	4.868	0.035	8.637
CMW	210	2.534	1.505	0.786	3.087	3.592	0.000	4.387
CS	315	2.103	1.971	0.693	1.156	3.044	0.000	8.858

Notes: Q1 and Q3 indicate the first and the third quartile, respectively.

F.2 First-Order Stochastic Dominance

In the portfolio allocation environment studied in the three studies we looked at, choosing an allocation (x_1, x_2) from a budget line defined by prices (p_1, p_2) violates *monotonicity with respect to first-order stochastic dominance (FOSD-monotonicity)* when either (i) $p_1 > p_2$ and $x_1 > x_2$ or (ii) $p_2 > p_1$ and $x_2 > x_1$ (i.e., the choice involves more allocation toward more-expensive security).

Table F.2 presents the average fraction (out of 25) of choices violating FOSD-monotonicity and the number of subjects without FOSD-monotonicity violations. On average, subjects made 24-34% violations of FOSD-monotonicity. The number of subjects who made no FOSD-violating choices is less than 10% for all datasets. As discussed in Choi et al. (2014), choices can be consistent with GARP even with violations of FOSD-monotonicity. The average fraction of FOSD-violating choices calculated from the subsample of GARP-compliant (CCEI = 1) subjects is close to the one we obtain from the whole sample. The entire distributions are presented in Figure F.1.

TABLE F.2: FOSD violation.

	All subjects			CCEI = 1		
	CKMS	CMW	CS	CKMS	CMW	CS
Number of subjects	1,182	1,119	1,423	270	210	315
Average fraction of FOSD-mon. violations	0.335	0.320	0.239	0.364	0.308	0.220
Fraction of subjects without FOSD-mon. violations	0.025	0.047	0.067	0.067	0.176	0.159

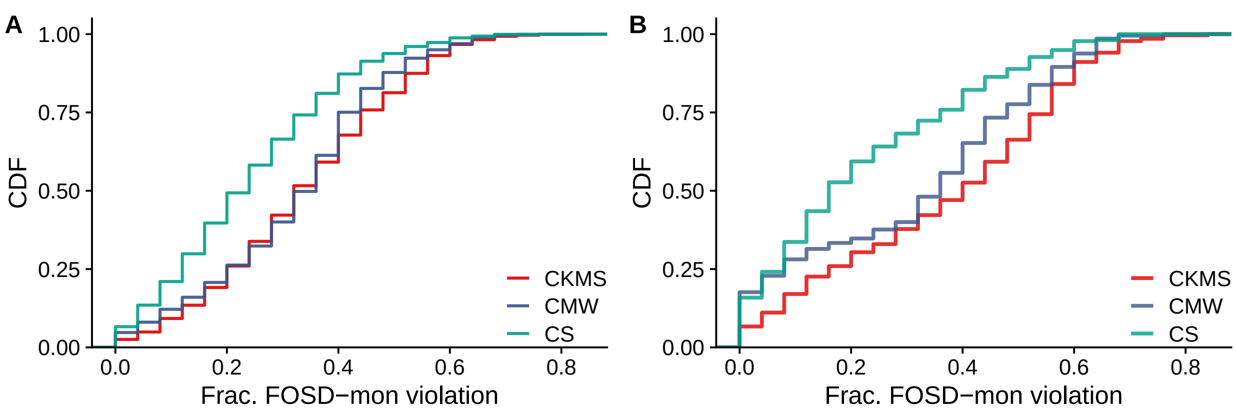


FIGURE F.1: Empirical CDFs of the fraction of choices that violate FOSD-monotonicity. (A) All subjects. (B) Subjects with CCEI = 1.

F.3 Choices on the 45-Degree Line

In the experiments, subjects made choices of allocations (x_1, x_2) by clicking on the budget line graphically presented on the screen. Note that points on the 45-degree line correspond to equal allocations between the two accounts ($x_1 = x_2$) and therefore involve no risk (i.e., the 45-degree line is the "full insurance" line). If a subject's all choices are on the 45-degree line (call such pattern *diagonal allocations*), we can rationalize the data with EU and hence $e_* = 0$.

It is, however, extremely difficult (or almost impossible) to choose the point "exacctly" on the 45-degree line in practice. Actual choices subjects made may be slightly off from the 45-degree line, and it can generate large e_* (through violations of the downward-sloping demand) while CCEI and EU-CCEI stay close to 1 (see Figure 8, panel D). In this section, we examine how much of the disagreement between e_* and CCEI or EU-CCEI is driven by small deviations from the diagonal allocations.

To this end, we first redefine diagonal allocations. Instead of requiring all choices to be exactly on the 45-degree line, we call a data *almost diagonal allocations* if all choices are inside small balls (with fixed radius r) drawn around the intersections of budget lines and the 45-degree line. We can control the size of acceptable deviations by changing the radius r of the ball. The idea is shown in Figure F.2. In this example, chosen allocations (black dots) are not exactly on the 45-degree line, but they are inside the balls around the diagonal allocations (red circles).[8]

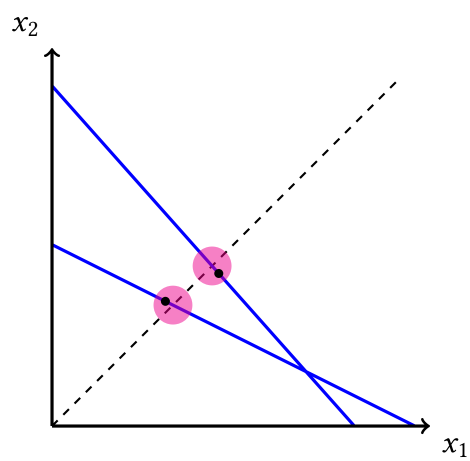


FIGURE F.2: Almost diagonal allocations.

[8]These choices also violate FOSD-monotonicity. We would expect relatively large e_* from this choice pattern, but its CCEI is 1 because it satisfies GARP.

Table F.3: Fraction of subjects who made almost diagonal allocations.

		Radius of the ball (r)			
Study	N	0.05	0.20	0.50	1.00
CKMS	1182	0.000	0.000	0.035	0.083
CMW	1119	0.008	0.040	0.097	0.120
CS	1423	0.005	0.022	0.048	0.060

Table F.3 shows the fraction of subjects who made almost diagonal allocations (in all 25 questions) under different sizes of r. Between 6% and 12% of subjects made such a choice pattern when the radius is set to $r = 1$.

Figures F.3 and F.4 below show the relationship between e_* and CCEI as well as EU-CCEI, as in Figure 9 (Section 4.2). The bottom panels in each figure focus on subjects who made almost diagonal allocations (the radius of the ball is set to $r = 1$) in all 25 questions, and the top panels present the rest of the subjects.

The bottom panels in each figure confirm that almost diagonal allocations yield values of CCEI and EU-CCEI that are close to 1. The same subjects have dispersed values of e_*, including the highest value in each experiment.

It does not mean that the disagreement between e_* and CCEI-based measures comes mainly from slight deviations from the diagonal allocations. The top panels in each figure show that there are choice patterns, other than almost diagonal allocations, that have CCEI/EU-CCEI ≈ 1 and large e_*.

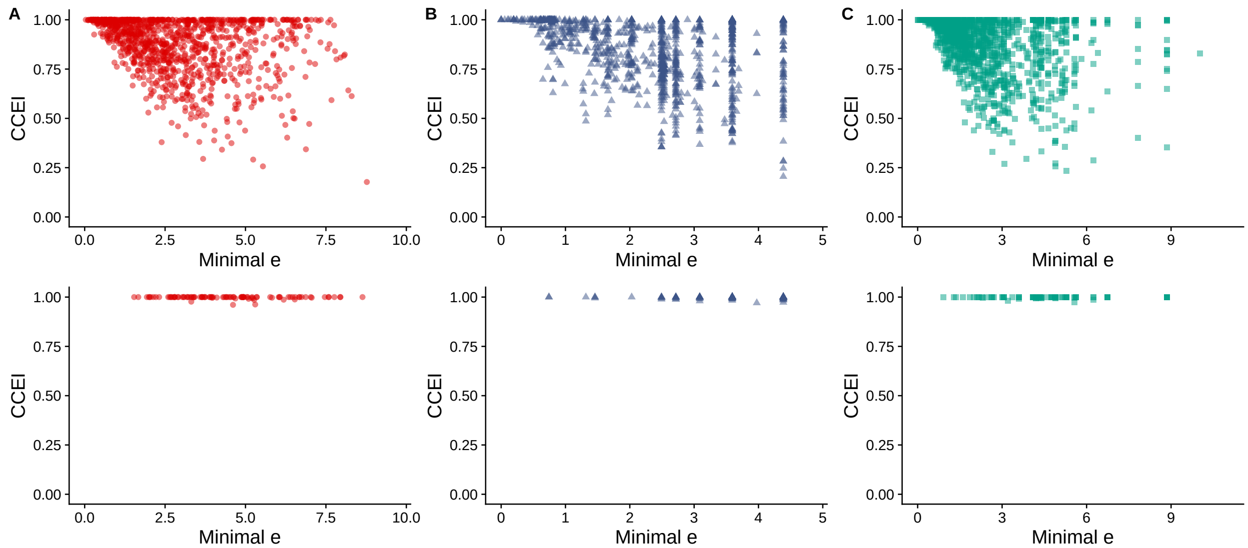


FIGURE F.3: Correlation between e_* and CCEI. The top panels show subjects who did not choose almost diagonal allocations and the bottom panels show those who selected almost diagonal allocations. Panels: (A) CKMS, (B) CMW, (C) CS. *Notes*: The radius of the ball is set to $r = 1$.

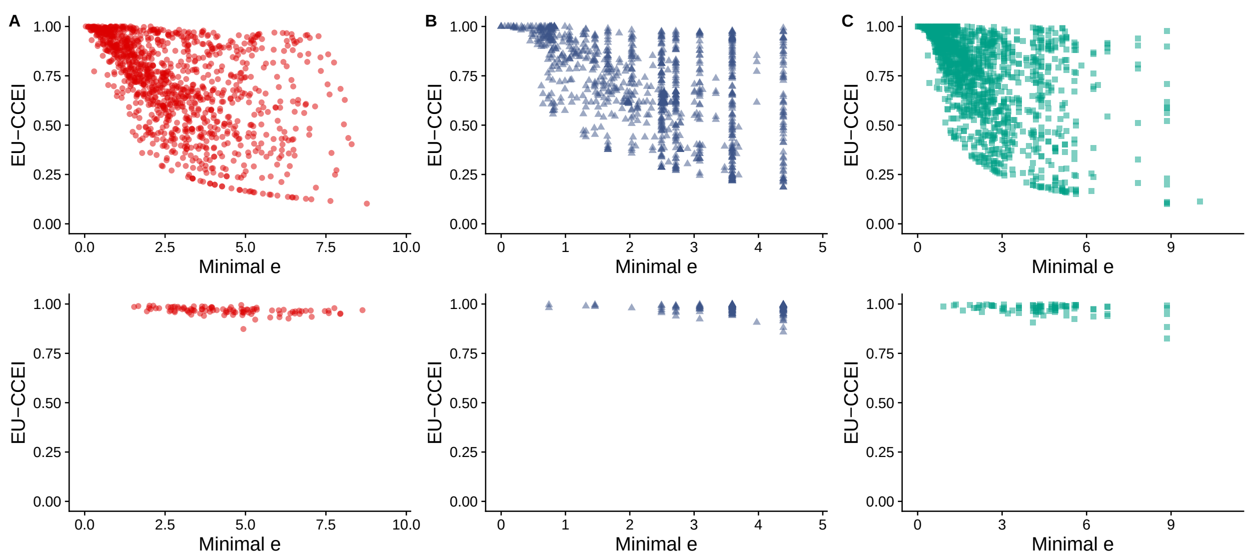


FIGURE F.4: Correlation between e_* and EU-CCEI. The top panels show subjects who did not choose almost diagonal allocations and the bottom panels show those who selected almost diagonal allocations. Panels: (A) CKMS, (B) CMW, (C) CS. *Notes*: The radius of the ball is set to $r = 1$.

F.4 Sensitivity

As is clear from the definition, our measure e_* is a bound that has to hold across all observations and states (see conditions (4), (5), and (6) in the definitions of e-perturbed OEU in Section 3). It is possible that a couple of "bad" choices significantly influence the measure. This section presents several robustness checks for the main empirical result.

Dropping critical mistakes. In this robustness check, we recalculate e_* using subsets of observed choices that exclude outliers. More precisely, for each subject, we calculate e_* for all combinations of $25 - m$ choices and pick the smallest e_*. We do this for $m = 1, 2$.

By construction, dropping critical mistakes shifts the distribution of the measure (Figure F.5). However, it does not dramatically change the correlational patterns between e_* and CCEI (Figure F.6) nor between e_* and demographic characteristics (Figures F.7 and F.8). In this sense, the main empirical results are robust to the presence of a small number of bad choices.

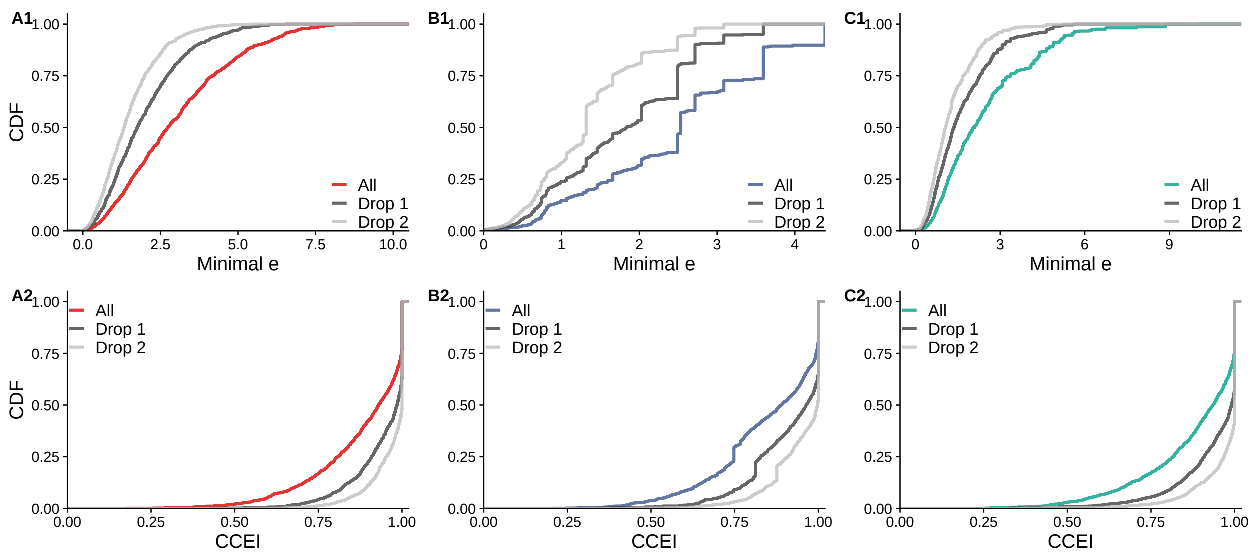


FIGURE F.5: Empirical CDFs of e_* and CCEI, using all observations or subsets of observations dropping one or two critical mistakes. Panels: (A) CKMS, (B) CMW, (C) CS.

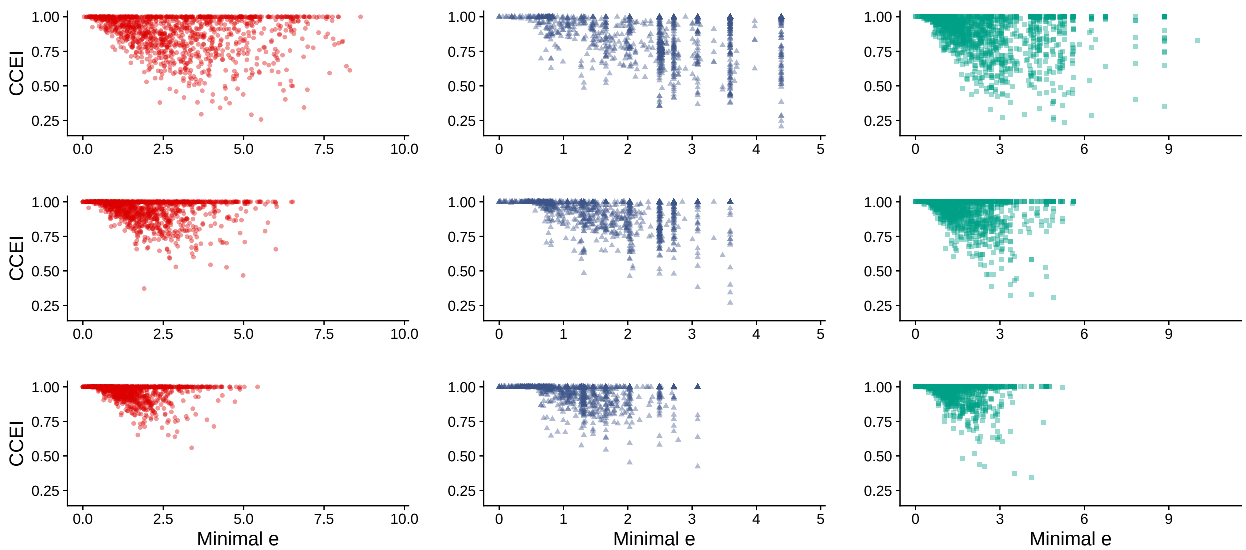


FIGURE F.6: Correlation between e_* and CCEI. (Top panels) All 25 observations. (Middle panels) Drop one critical mistake. (Bottom panels) Drop two critical mistakes.

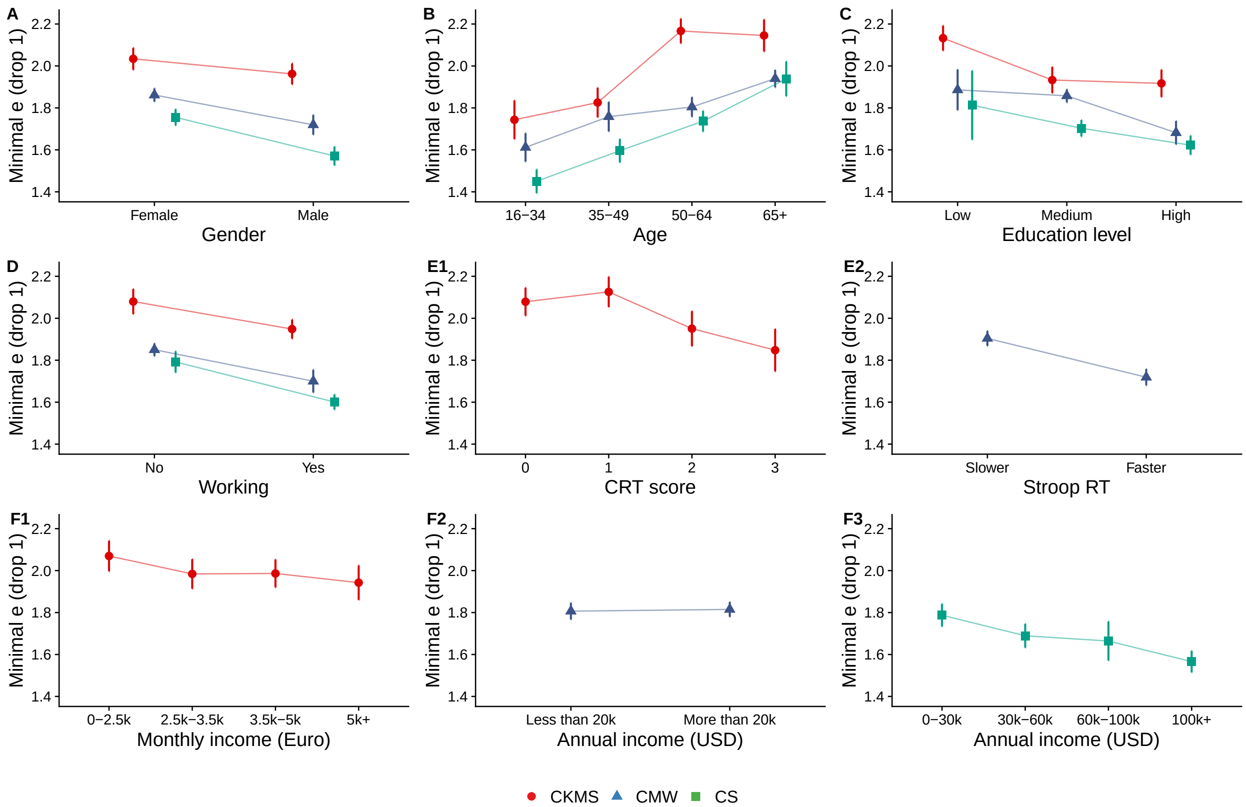


FIGURE F.7: Robustness of demographic correlations in Figure 10. For each subject, e_* is recalculated after dropping one critical mistake. *Notes*: Dots represent mean e_* and bars represent standard errors of means.

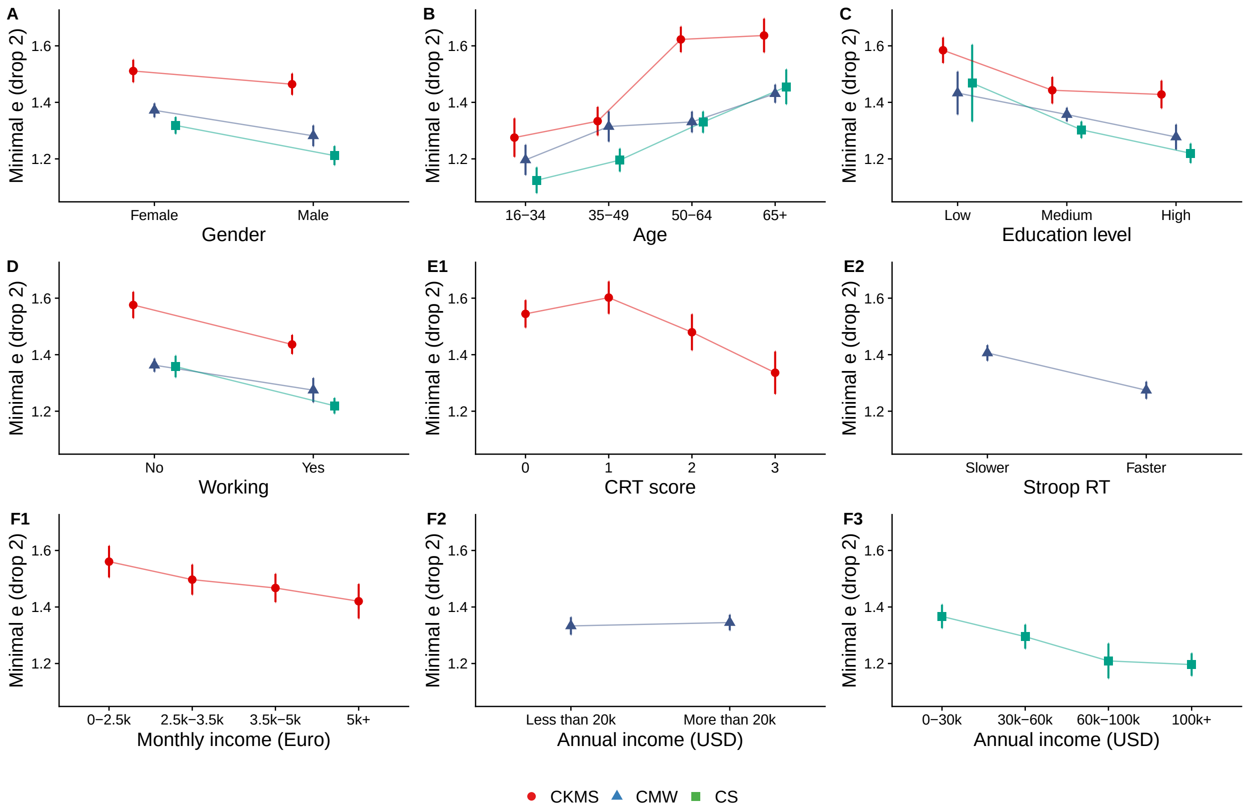


FIGURE F.8: Robustness of demographic correlations in Figure 10. For each subject, e_* is recalculated after dropping two critical mistakes. *Notes*: Dots represent mean e_* and bars represent standard errors of means.

"Average" perturbation. Let $\bar{e}$ be the solution to the following minimization problem:

$$\begin{aligned} \min_{(\varepsilon_s^k)_{k,s}} \quad & \sum_{k \in \mathcal{K}} \sum_{s \in S} \frac{\left|\log \varepsilon_s^k\right|}{KS} \\ \text{s.t.} \quad & (x^k, q^k)_{k=1}^K \text{ is OEU rational} \\ & q_s^k = p_s^k \varepsilon_s^k \text{ for each } s \in S, k \in \mathcal{K} \end{aligned}$$

The idea behind this alternative measure is simple. As in the case of e-price-perturbed utility, we search for sets of multiplicative noises $(\varepsilon_s^k)_{k,s}$ which could rationalize the observed data. Instead of looking at the uniform bound $\max_{s,t,k}(\log \varepsilon_s^k - \log \varepsilon_t^k)$ and minimizing it, we take the average of these perturbations and minimize it. A similar idea was applied to quantify the distance from several models of time preferences in Echenique et al. (2016).

Figure F.9 presents the relationship between $\bar{e}$, e_*, and CCEI. Figure F.10 shows the correlation between $\bar{e}$ and demographic variables. These figures do not show correlational patterns that are markedly different from those presented in the main empirical results (Figures 9 and 10 in Section 4.2).

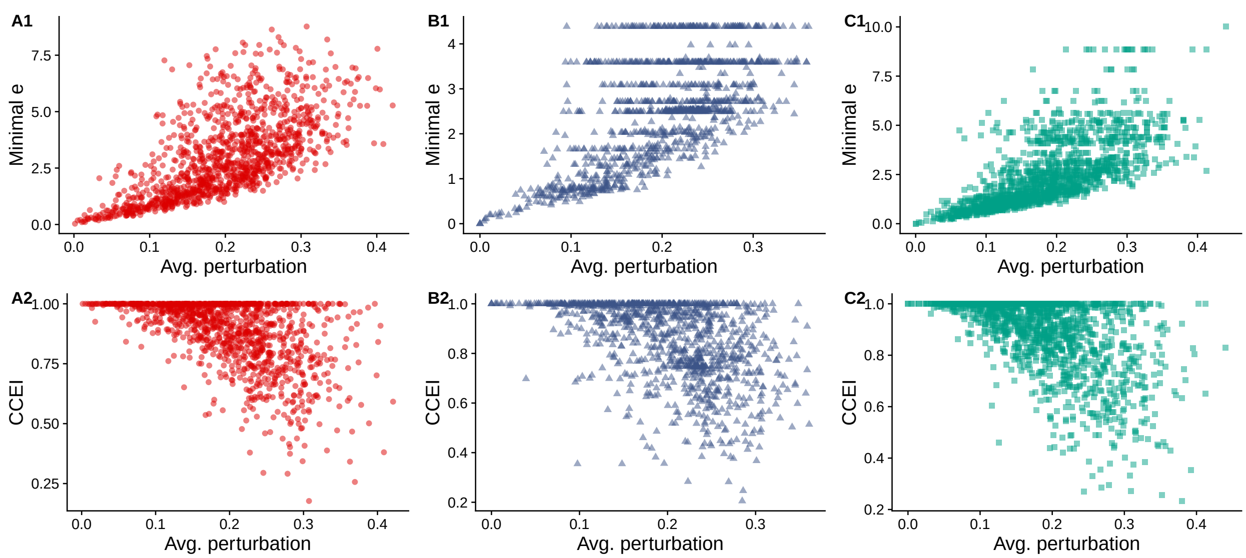


FIGURE F.9: Correlation between $\bar{e}$ and e_* (top panels) and $\bar{e}$ and CCEI (bottom panels). Panels: (A) CKMS, (B) CMW, (C) CS.

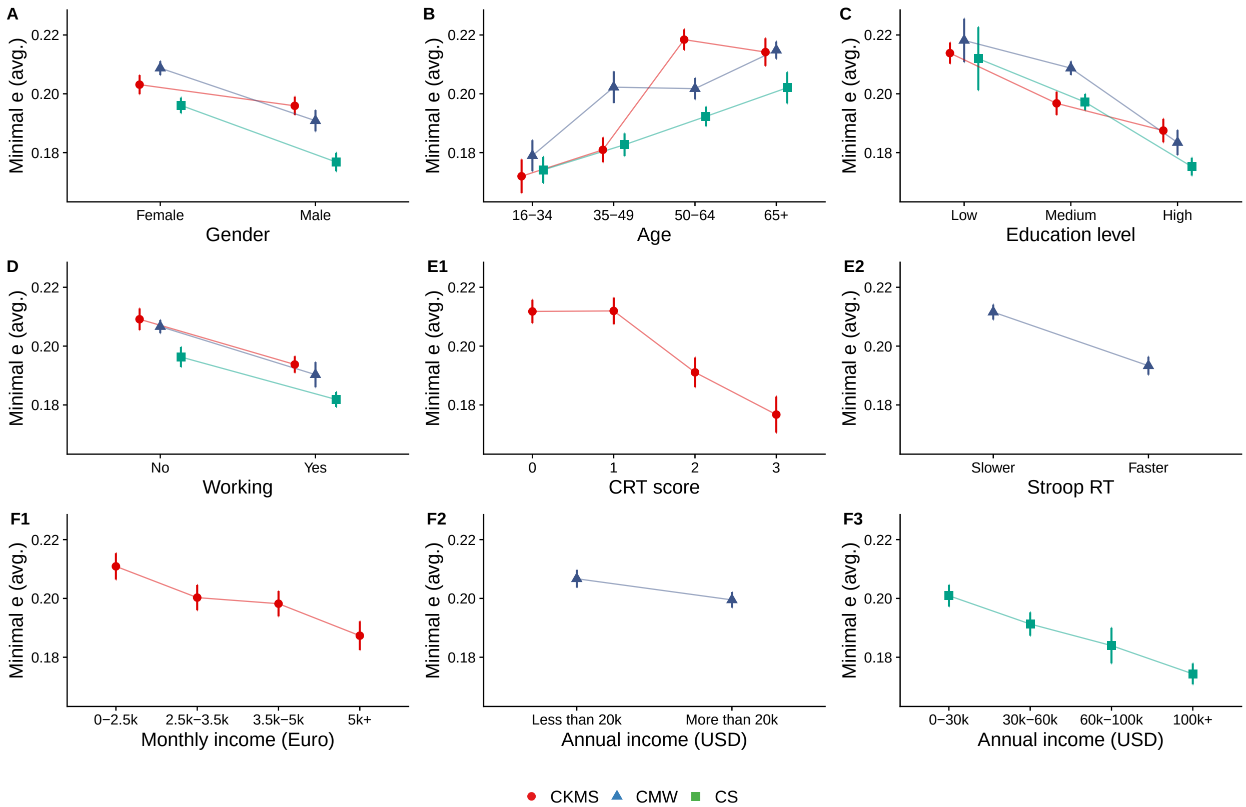


FIGURE F.10: Correlation between $\bar{e}$ and demographic variables. *Notes*: Dots represent mean $\bar{e}$ and bars represent standard errors of means.

F.5 Properties of e_*

e_* from observed and simulated choices. The statistical approach described in Section 4.3 is one way to assess "how big" the observed e_*'s are. Another way is to simulate choice data assuming some behavioral model and calculate e_* on the simulated dataset.

Following Bronars (1987), we simulate synthetic subjects who choose an allocation uniformly randomly from each budget line. Since subjects in CKMS and CS faced a randomly selected set of budgets, we first randomly select one set of budgets (from the observed sets of budgets) and then randomly choose allocations on these budgets. We calculate e_*, and CCEI, using these simulated choices. We repeat this 10,000 times for each of the three datasets.

Figure F.11 compares the observed and simulated e_*. The distribution of observed e_* locates left of simulated e_* in all three datasets (all differences are statistically significant, according to the two-sample Kolmogorov-Smirnov test). The actual subjects' behavior is thus closer to OEU rationality compared to random behavior (even though the uniformly random choice is unrestrictive and may not be the best benchmark).

In the second simulation, we generate random choices that respect FOSD-monotonicity. The distributions of e_* in this simulation, shown in dark gray lines in Figure F.11, exhibit a stark difference from those from real subjects: they have smaller median values and are distributed on narrower ranges.

Figure F.12 looks at the correlation between e_* and CCEI and compares the pattern in observed and simulated datasets (panels A-C in the top row are the same as Figure 9).

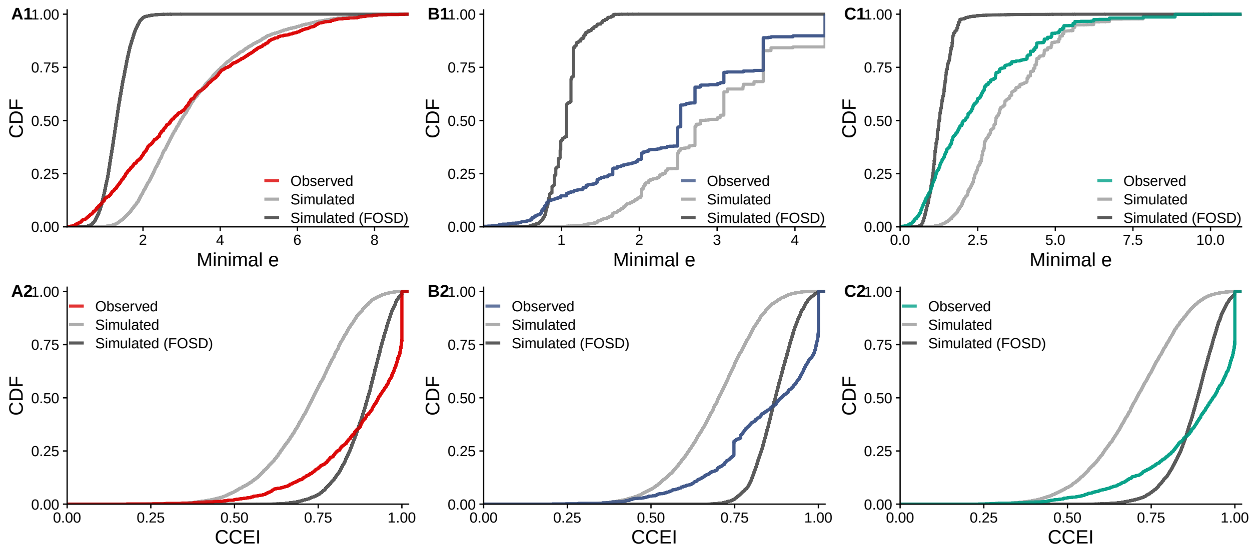


FIGURE F.11: Comparison between observed and simulated e_* (top panels) and CCEI (bottom panels). Panels: (A) CKMS, (B) CMW, (C) CS.

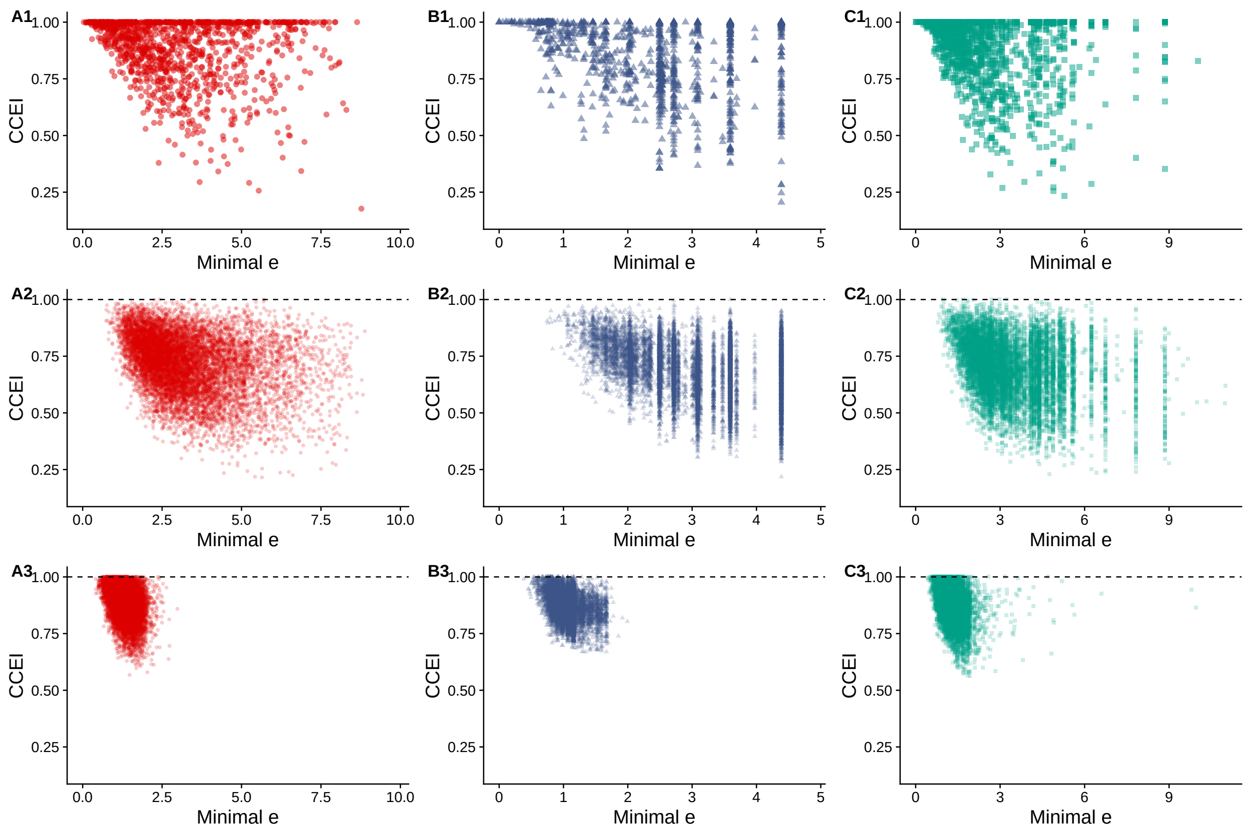


FIGURE F.12: Correlation between e_* and CCEI. Panels: (A) CKMS, (B) CMW, (C) CS. *Notes*: Measures are calculated using observed choices (top row), uniformly random choices (middle row), and uniformly random choices satisfying FOSD-monotonicity (bottom row). Panels A1-C1 are identical to those in Figure 9.

Upper bound of e_*. The value of e_* depends on the structure of the budgets an agent faces. In particular, it is clear from e-PSAROEU that $1+e_*$ is bounded by the maximum ratio of risk-neutral prices:

$$1 + e_* \leq \max_{k \in \mathcal{K}, s, t \in S} \frac{\rho_s^k}{\rho_t^k}.$$

The right-hand side captures the slope of the "most extreme" budget line.

Since CKMS, CMW, and CS experiments all used two equally likely states, the ratio of risk-neutral prices is equal to the ratio of prices (i.e., $\rho_s^k/\rho_t^k = p_s^k/p_t^k$). Figure F.13 shows the relationship between the observed e_* and the participant-specific upper bound. (Since all subjects faced the same set of budgets in the CMW study, there is only one vertical line in panel B.)

About 13% of the subjects (475/3724 in the merged data; 221/1182 in CKMS; 114/1119 in CMW; 140/1423 in CS) have their e_* exactly at the upper bound. The reason is that these subjects made a violation of FOSD-monotonicity (choosing a larger payoff in a more expensive state) in the most extreme budget among the set of budgets they faced during the experiment. See discussion in Section D.1 above.

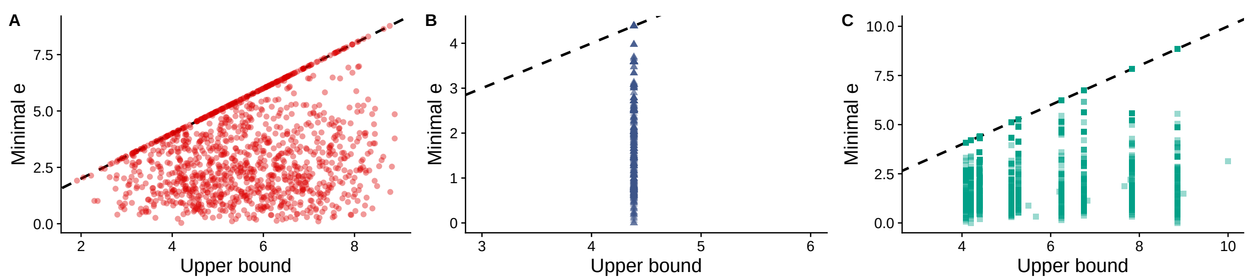


FIGURE F.13: Bound of e_*. The x-axis in each plot is the upper bound of e_*, given by $\max_{k,s,t} p_s^k/p_t^k - 1$. *Notes*: There is no variation in bounds in the CMW data (panel B) since all subjects faced the same set of budgets. In the CMW data (panel B), all points line up at 4.387, which is given by the most extreme budget in that study with prices $(p_1, p_2) = (1, 5.387)$. In the CS data (panel C), the x-axis is cut at 10 for better visualization. There are 22 additional observations in the data with the bounds ranging from 11 to 48.

F.6 Illustration of e_*-Perturbed OEU

In Figure 8, we present choice patterns of six selected subjects with CCEI $= 1$ and varying degrees of e_*. Panels A-F plot observed choices and panels a-f plot the relationship between $\log(x_2/x_1)$ and $\log(p_2/p_1)$, which illustrates how much the dataset conforms to the downward-sloping demand. The measure e_*, roughly speaking, captures the degree of deviation from the downward-sloping demand.

Consider an observed dataset $(x^k, p^k)_{k=1}^K$ and a perturbed dataset $(x^k, \tilde{p}^k)_{k=1}^K$, where $\tilde{p}_s^k = p_s^k \varepsilon_s^k$ and $\varepsilon_s^k \geq 0$ for all $s \in S$ and $k \in \mathcal{K}$. Since we fix the chosen bundle $(x^k)_{k=1}^K$ and rotate the budget lines around them, price perturbation "moves" points in panels a-f horizontally.

To make the dataset e-price-perturbed OEU rational (Definition 4), we need to move the points horizontally until they satisfy the downward-sloping demand. Note that the horizontal distance for each observation k, before and after e-price perturbation, is given by

$$\log\left(\frac{\tilde{p}_2^k}{\tilde{p}_1^k}\right) - \log\left(\frac{p_2^k}{p_1^k}\right) = \log\left(\frac{\tilde{p}_2^k/p_2^k}{\tilde{p}_1^k/p_1^k}\right) = \log\left(\frac{\varepsilon_2^k}{\varepsilon_1^k}\right).$$

We thus need to look at the maximal horizontal adjustment among observations, and the measure e_* is obtained by minimizing it.

Figure F.14 shows the idea behind the calculation of e_* using price perturbation. It plots the same six subjects as in Figure 8. In panels A-F, red dotted lines represent the original budgets and blue solid lines represent perturbed budgets. In panels a-f, green circles represent the original dataset and blue triangles represent the perturbed dataset. Red arrows connect points that correspond to the maximal adjustment. The figure shows that e_*-perturbed datasets are closer to the downward-sloping demand in the sense of less dispersion.

We can draw several observations about the practical aspect of e_*. First, observe that the "cheapest" way for correcting choices violating FOSD-monotonicity is to perturb budgets corresponding to these observations so that $\tilde{p}_1^k = \tilde{p}_2^k$. Second, the figure provides an intuitive explanation of why e_* can be large for choice patterns like panel D. Since clicking on the point exactly on the 45-degree line is a challenging task, choices would scatter around the 45-degree line, occasionally falling in the region of FOSD-monotonicity. No matter how small these deviations from the 45-degree line are, e-price perturbation requires horizontal adjustments to achieve the downward-sloping demand. If the necessary adjustment is applied on a relatively extreme budget line, e_* for such a subject can be very high.

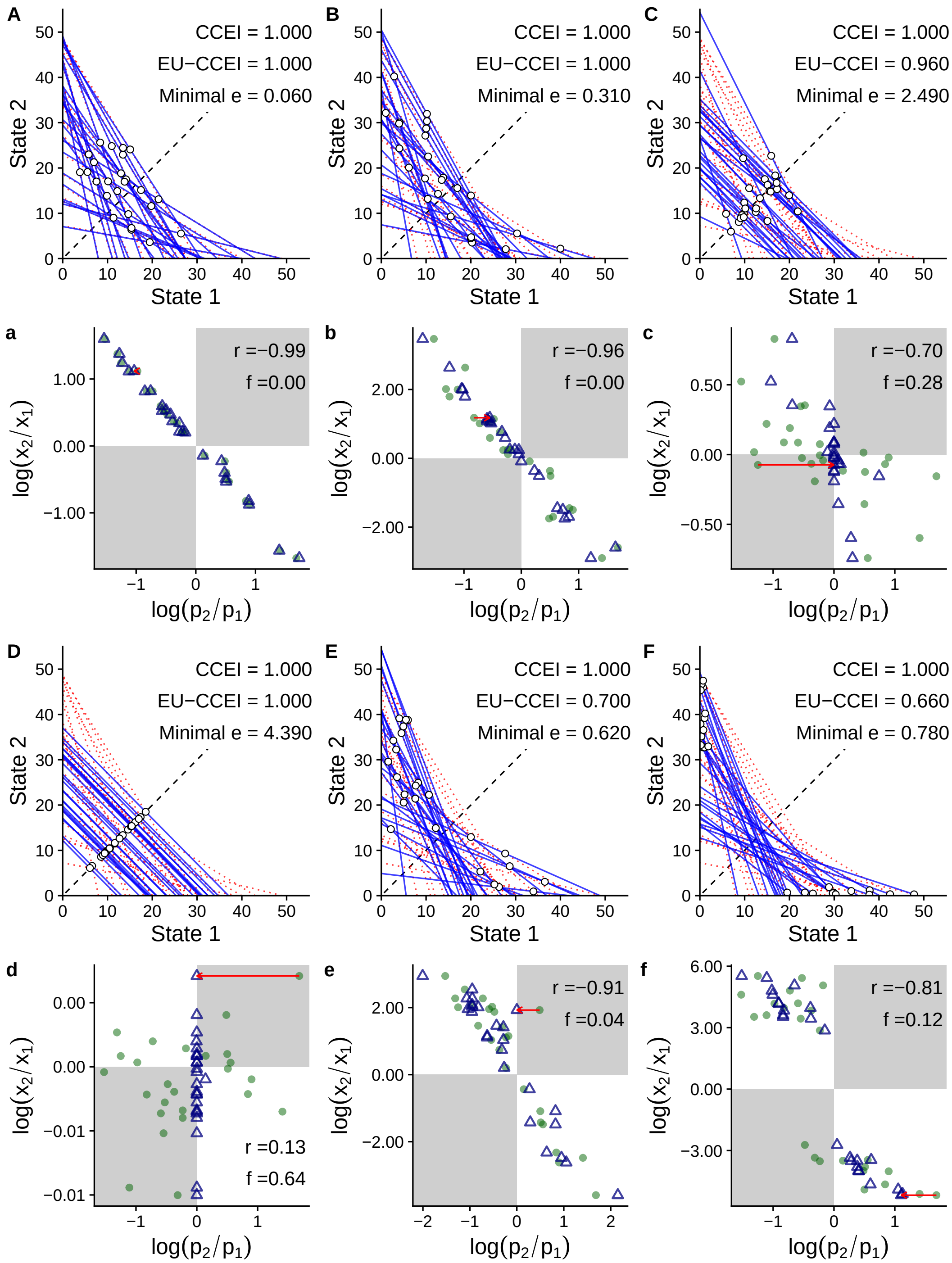


FIGURE F.14: Illustration of e-price-perturbed OEU rationalization. (A-F) Perturbed budgets (blue solid lines) and the original budgets (red dotted lines). (a-f) The relation between $\log(x_2/x_1)$ and $\log(p_2/p_1)$ (green circles), and $\log(x_2/x_1)$ and $\log(\tilde{p}_2/\tilde{p}_1)$ (blue triangles). Red arrows indicate observations requiring the largest adjustment.

F.7 Comparing Measures

We calculate CCEI at which a subject is consistent with a given model, stochastically monotone utility maximization (Nishimura et al., 2017), EU, and concave EU, using the GRID method developed in Polisson et al. (2020).[9] We call these measures F-GARP, EU-CCEI, and cEU-CCEI. For a given dataset, the measures are ordered as

$$\text{cEU-CCEI} \leq \text{EU-CCEI} \leq \text{F-GARP} \leq \text{CCEI},$$

since the models we look at are nested in this order. Note that Polisson et al. (2020) calculated and reported CCEI, F-GARP, EU-CCEI, and cEU-CCEI for the CKMS dataset but not for the CMW and the CS datasets.

Figures F.15-F.17 compare e_*, CCEI, and these three additional measures.[10] Panels on the diagonal show the distribution of each measure. Pairwise scatter plots are presented below the diagonal, and Spearman's correlation coefficients are shown above the diagonal (all $p < 0.001$; uncorrected for multiple comparisons).

The first column in each figure shows the relationship between e_* and other measures. The second and the fourth panels in this column (e_* vs. CCEI and e_* vs. EU-CCEI) are identical to those presented in Figure 9. As we discussed in Section 4.2 of the paper, we see that there are a significant number of subjects whose CCEI and EU-CCEI are close to one but their e_*'s are widely dispersed and further away from zero.

This observation is not specific to CCEI and EU-CCEI. In the third and the fifth panels of the same column, we can see a similar pattern between e_* and F-GARP as well as e_* and EU-CCEI. The pattern is a general feature that distinguishes the idea behind the measures: e_* is based on rotating budget lines while the other measures, which are all variants of CCEI, are based on shrinking budget sets.

[9] A stochastically monotone utility function gives strictly higher utility to bundle x compared to another bundle y if x first-order stochastically dominates y and gives them the same utility if two bundles are stochastically equivalent. In the environment we consider (two states with equally likely objective probabilities), a utility function is stochastically monotone if and only if it is symmetric and strictly increasing.

Choi et al. (2014) also discuss a similar idea. They propose an additional measure, which jointly captures the extent of GARP violations and violations of stochastic dominance, by combining the observed data and its "mirror-image". More precisely, they assume that if an allocation (x_1, x_2) is chosen under the budget constraint $p_1x_1 + p_2x_2 = 1$, then (x_2, x_1) would have been chosen under the mirror-image budget constraint $p_2x_1 + p_1x_2 = 1$. They then re-calculate CCEI for the "combined" data consisting of 50 (25 budgets × 2) choices.

[10] We did not compute cEU-CCEI for 23 subjects (8 in CMW, and 15 in CS) since the code spent a significantly long computation time. (Polisson et al. (2020) used a high-performance computing facility.) We also treated cEU-CCEI for six subjects in CS as missing values, since the code incorrectly returned cEU-CCEI = 0. Note that F-GARP and EU-CCEI for these 29 subjects are included in Figures F.15-F.17.

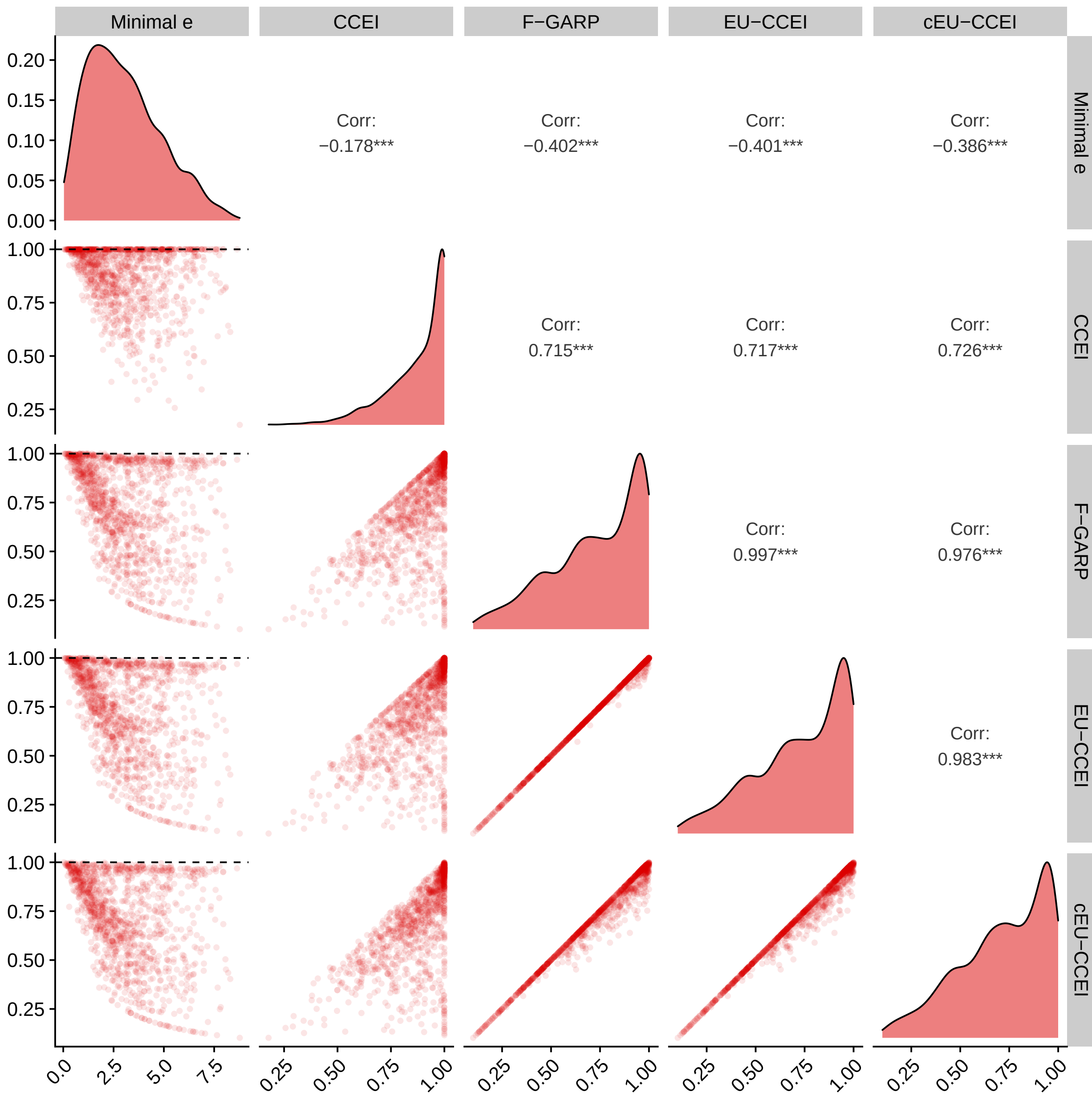


Figure F.15: Comparing measures of rationality in the CKMS data.

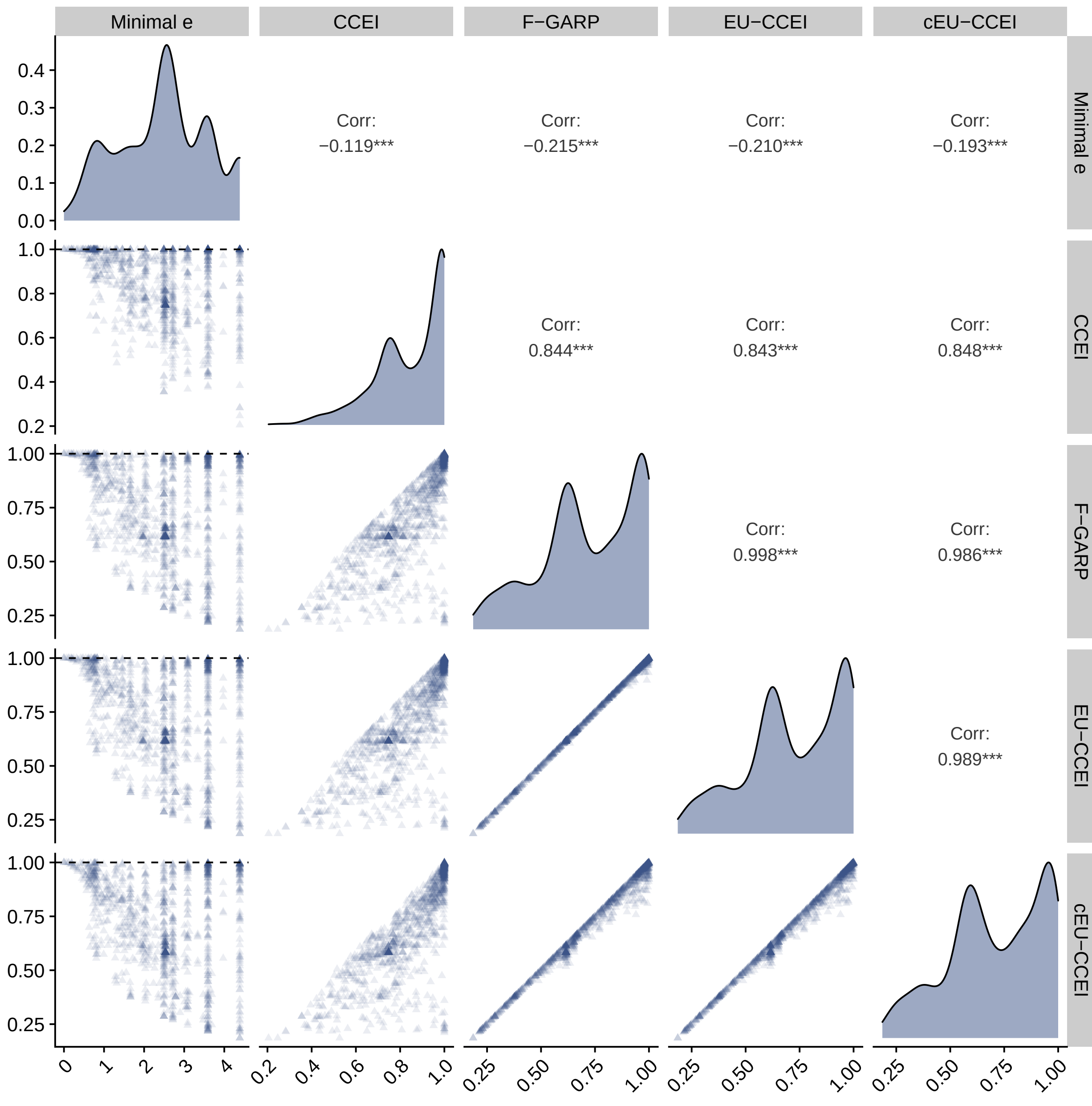


Figure F.16: Comparing measures of rationality in the CMW data.

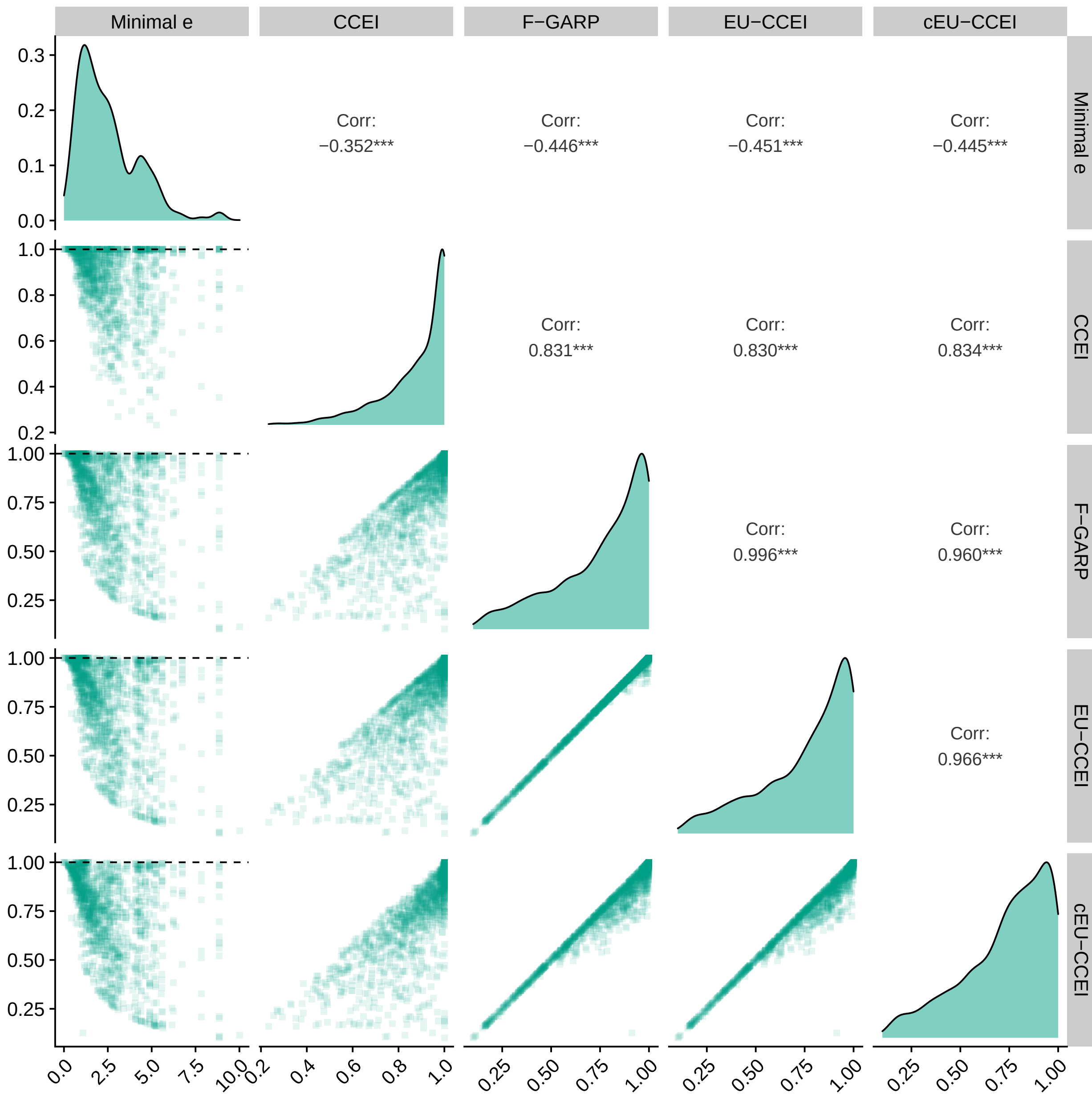


Figure F.17: Comparing measures of rationality in the CS data.

F.8 Choice Pattern: Additional Examples

Choice data from four subjects presented in Section 4.2, Figure 8, are not meant to be representative of the entire dataset consisting of more than 3,000 subjects. In this section, we present more examples to understand the similarity and differences between e_*, CCEI, and EU-CCEI.

We pick subjects from the CMW experiment, where all the subjects faced the same set of 25 budget lines. This feature of the design makes the variation of e_* smaller than in the other datasets (we observe several "jumps" in the empirical CDF of e_* in Figure 5), but the comparison across choice patterns becomes easier.

Figure F.18 is the scatterplot of e_* and EU-CCEI in the CMW data. Dashed lines represent the 25th, 50th, and 75th percentiles of e_* and EU-CCEI. Two shaded areas represent combinations of e_* and EU-CCEI that "disagree", in the sense that one measure says the subject is close to EU (relative to the median subject) but the other measure says the same subject is far from EU (again, relative to the median subject). Each subject's choice pattern is shown below.

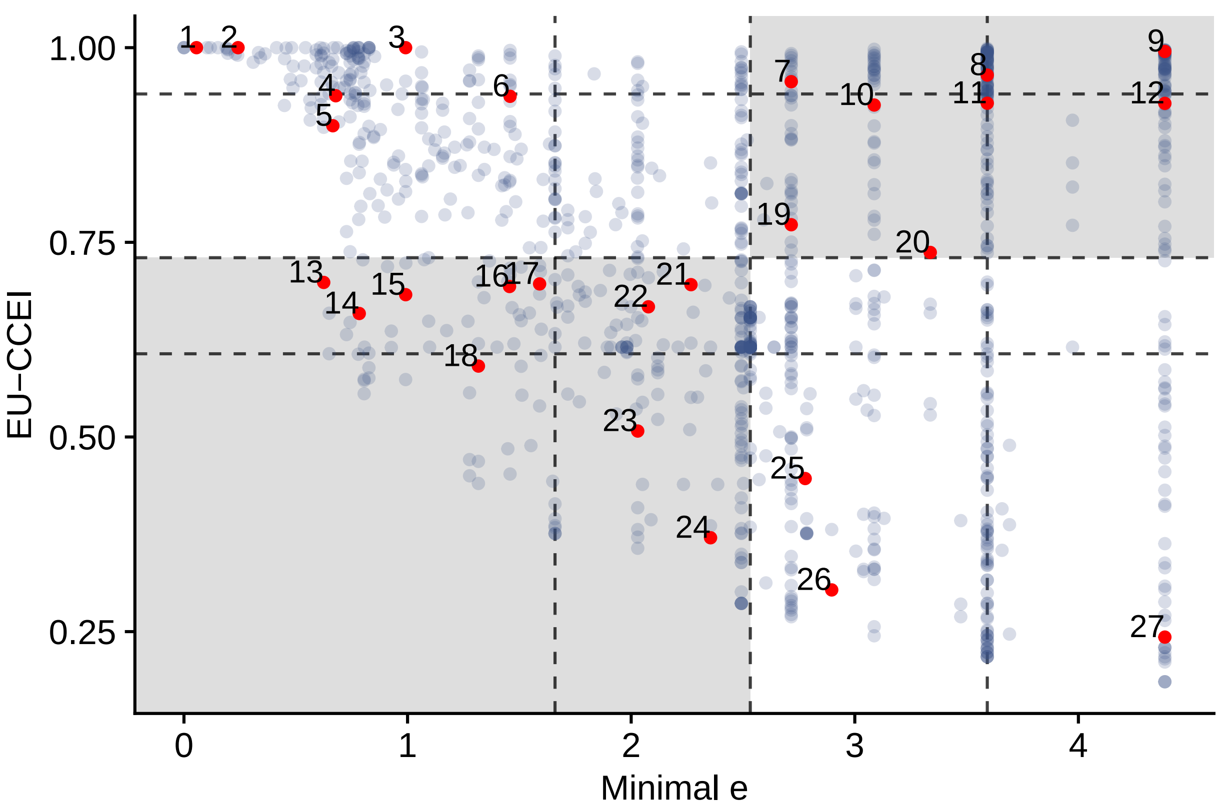
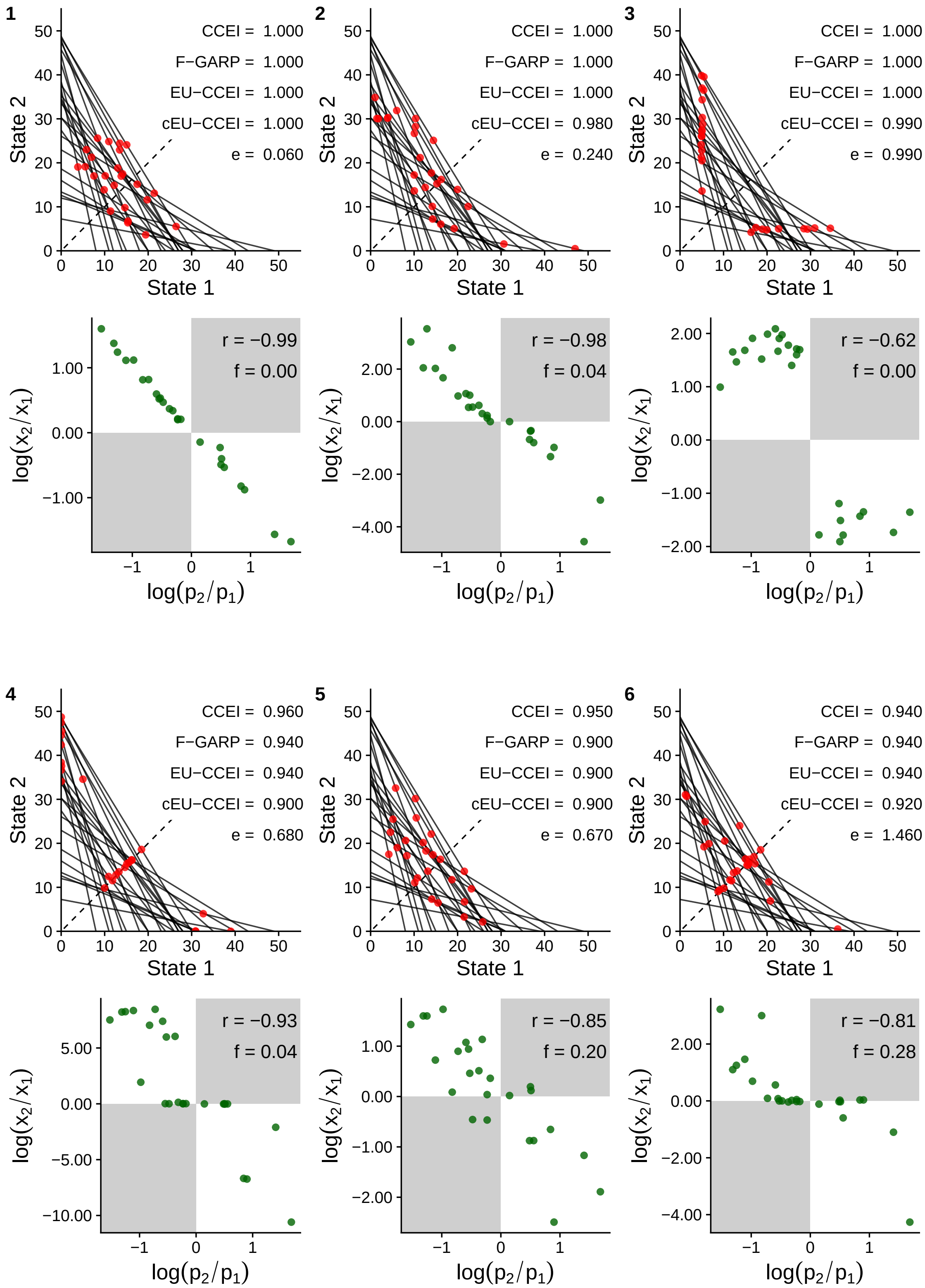
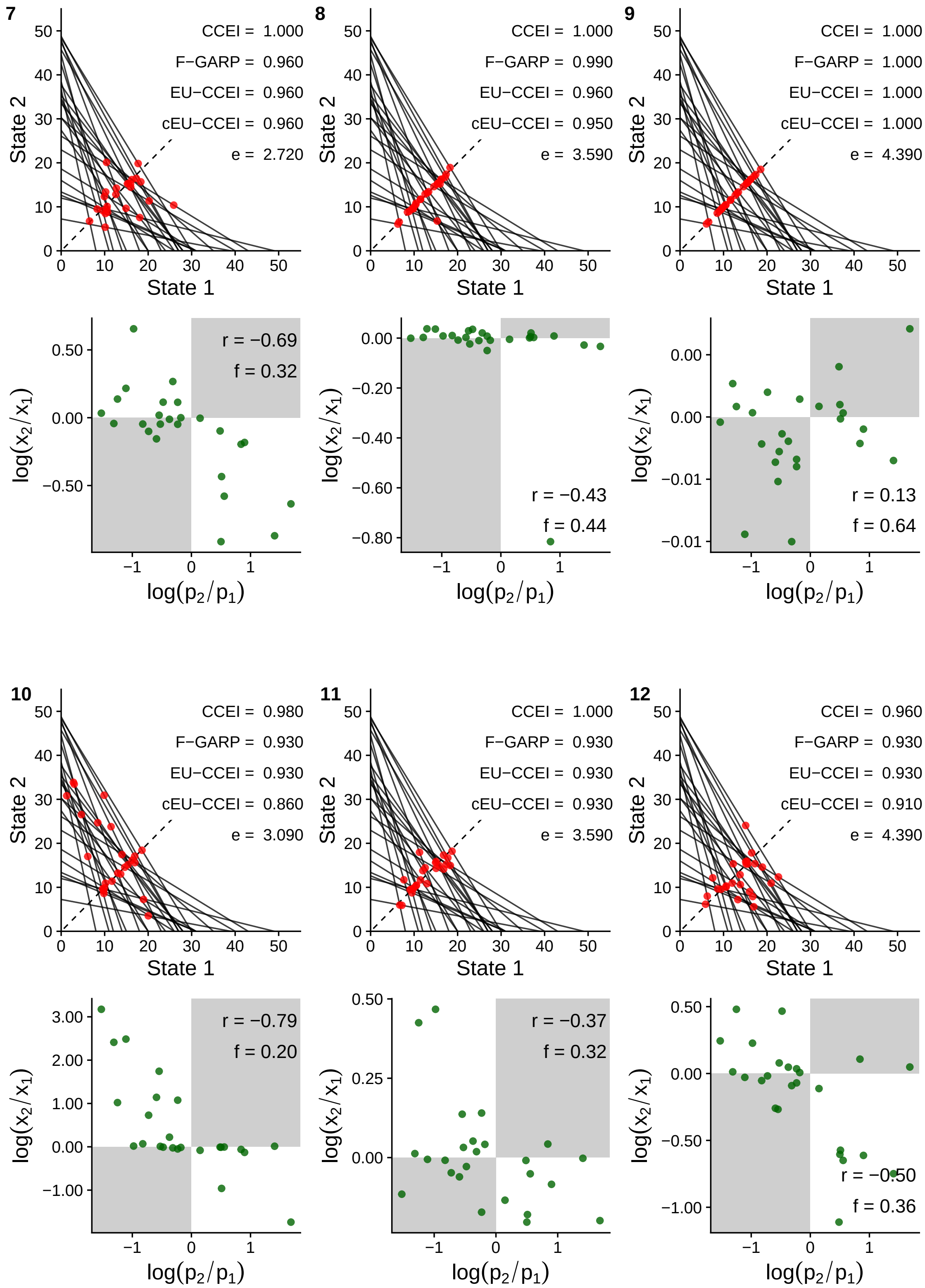
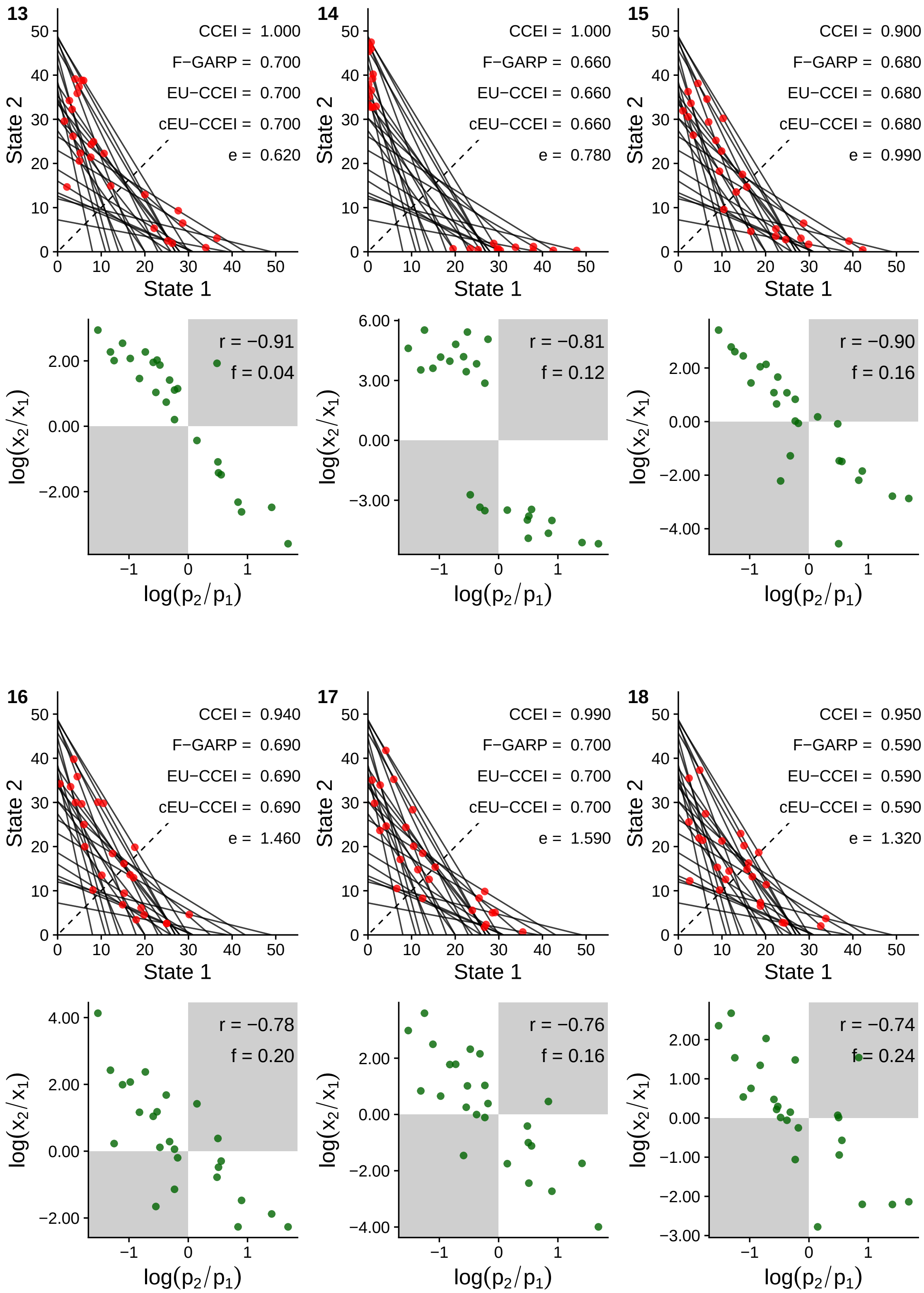


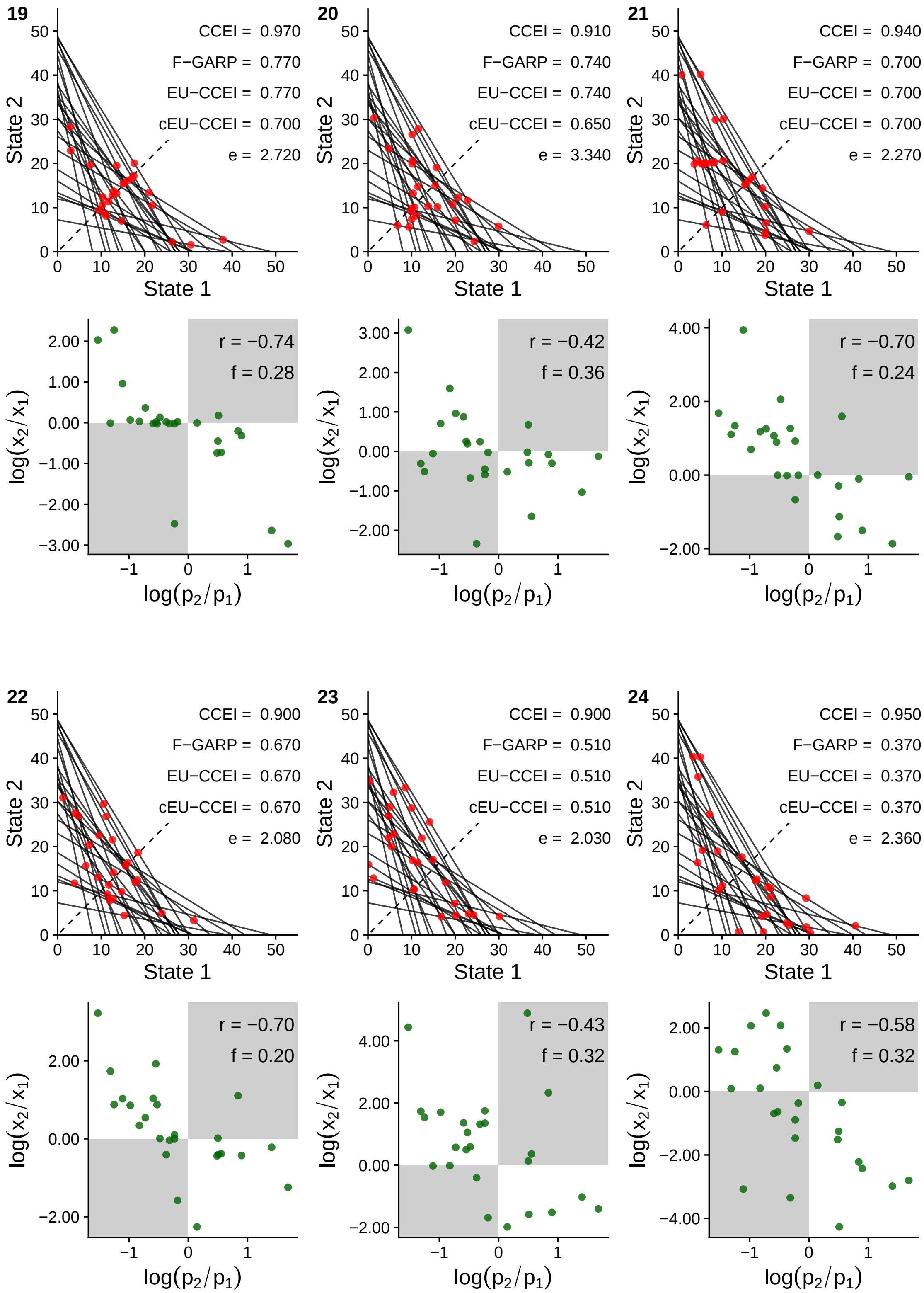
FIGURE F.18: e_* and EU-CCEI in CMW data. *Notes*: Vertical dashed lines represent the 25th, 50th, and 75th percentiles of e_*. Horizontal dashed lines represent the 25th, 50th, and 75th percentiles of EU-CCEI.

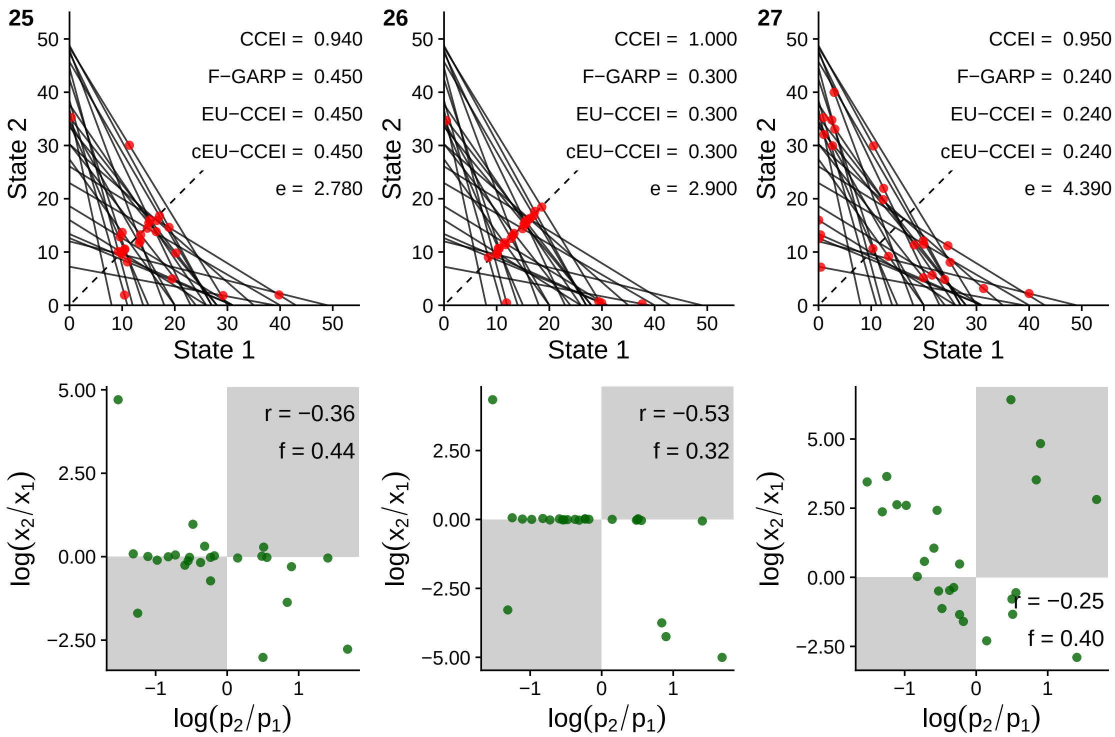

1
CCEI = 1.000
F−GARP = 1.000
EU−CCEI = 1.000
cEU−CCEI = 1.000
e = 0.060
State 2
State 1
r = −0.99
f = 0.00
log(x2/x1)
log(p2/p1)
2
CCEI = 1.000
F−GARP = 1.000
EU−CCEI = 1.000
cEU−CCEI = 0.980
e = 0.240
r = −0.98
f = 0.04
3
CCEI = 1.000
F−GARP = 1.000
EU−CCEI = 1.000
cEU−CCEI = 0.990
e = 0.990
r = −0.62
f = 0.00
4
CCEI = 0.960
F−GARP = 0.940
EU−CCEI = 0.940
cEU−CCEI = 0.900
e = 0.680
r = −0.93
f = 0.04
5
CCEI = 0.950
F−GARP = 0.900
EU−CCEI = 0.900
cEU−CCEI = 0.900
e = 0.670
r = −0.85
f = 0.20
6
CCEI = 0.940
F−GARP = 0.940
EU−CCEI = 0.940
cEU−CCEI = 0.920
e = 1.460
r = −0.81
f = 0.28



7
CCEI = 1.000
F−GARP = 0.960
EU−CCEI = 0.960
cEU−CCEI = 0.960
e = 2.720
State 2
State 1
r = −0.69
f = 0.32
log(x2/x1)
log(p2/p1)
8
CCEI = 1.000
F−GARP = 0.990
EU−CCEI = 0.960
cEU−CCEI = 0.950
e = 3.590
State 2
State 1
r = −0.43
f = 0.44
log(x2/x1)
log(p2/p1)
9
CCEI = 1.000
F−GARP = 1.000
EU−CCEI = 1.000
cEU−CCEI = 1.000
e = 4.390
State 2
State 1
r = 0.13
f = 0.64
log(x2/x1)
log(p2/p1)
10
CCEI = 0.980
F−GARP = 0.930
EU−CCEI = 0.930
cEU−CCEI = 0.860
e = 3.090
State 2
State 1
r = −0.79
f = 0.20
log(x2/x1)
log(p2/p1)
11
CCEI = 1.000
F−GARP = 0.930
EU−CCEI = 0.930
cEU−CCEI = 0.930
e = 3.590
State 2
State 1
r = −0.37
f = 0.32
log(x2/x1)
log(p2/p1)
12
CCEI = 0.960
F−GARP = 0.930
EU−CCEI = 0.930
cEU−CCEI = 0.910
e = 4.390
State 2
State 1
r = −0.50
f = 0.36
log(x2/x1)
log(p2/p1)


13
CCEI = 1.000
F−GARP = 0.700
EU−CCEI = 0.700
cEU−CCEI = 0.700
e = 0.620
State 2
State 1
r = −0.91
f = 0.04
log(x_2/x_1)
log(p_2/p_1)
14
CCEI = 1.000
F−GARP = 0.660
EU−CCEI = 0.660
cEU−CCEI = 0.660
e = 0.780
State 2
State 1
r = −0.81
f = 0.12
log(x_2/x_1)
log(p_2/p_1)
15
CCEI = 0.900
F−GARP = 0.680
EU−CCEI = 0.680
cEU−CCEI = 0.680
e = 0.990
State 2
State 1
r = −0.90
f = 0.16
log(x_2/x_1)
log(p_2/p_1)
16
CCEI = 0.940
F−GARP = 0.690
EU−CCEI = 0.690
cEU−CCEI = 0.690
e = 1.460
State 2
State 1
r = −0.78
f = 0.20
log(x_2/x_1)
log(p_2/p_1)
17
CCEI = 0.990
F−GARP = 0.700
EU−CCEI = 0.700
cEU−CCEI = 0.700
e = 1.590
State 2
State 1
r = −0.76
f = 0.16
log(x_2/x_1)
log(p_2/p_1)
18
CCEI = 0.950
F−GARP = 0.590
EU−CCEI = 0.590
cEU−CCEI = 0.590
e = 1.320
State 2
State 1
r = −0.74
f = 0.24
log(x_2/x_1)
log(p_2/p_1)


19
CCEI = 0.970
F−GARP = 0.770
EU−CCEI = 0.770
cEU−CCEI = 0.700
e = 2.720
r = −0.74
f = 0.28
20
CCEI = 0.910
F−GARP = 0.740
EU−CCEI = 0.740
cEU−CCEI = 0.650
e = 3.340
r = −0.42
f = 0.36
21
CCEI = 0.940
F−GARP = 0.700
EU−CCEI = 0.700
cEU−CCEI = 0.700
e = 2.270
r = −0.70
f = 0.24
22
CCEI = 0.900
F−GARP = 0.670
EU−CCEI = 0.670
cEU−CCEI = 0.670
e = 2.080
r = −0.70
f = 0.20
23
CCEI = 0.900
F−GARP = 0.510
EU−CCEI = 0.510
cEU−CCEI = 0.510
e = 2.030
r = −0.43
f = 0.32
24
CCEI = 0.950
F−GARP = 0.370
EU−CCEI = 0.370
cEU−CCEI = 0.370
e = 2.360
r = −0.58
f = 0.32
State 1
State 2
log(x2/x1)
log(p2/p1)


25
CCEI = 0.940
F−GARP = 0.450
EU−CCEI = 0.450
cEU−CCEI = 0.450
e = 2.780
26
CCEI = 1.000
F−GARP = 0.300
EU−CCEI = 0.300
cEU−CCEI = 0.300
e = 2.900
27
CCEI = 0.950
F−GARP = 0.240
EU−CCEI = 0.240
cEU−CCEI = 0.240
e = 4.390
State 1
State 2
r = −0.36
f = 0.44
r = −0.53
f = 0.32
r = −0.25
f = 0.40
$\log(p_2/p_1)$
$\log(x_2/x_1)$

F.9 Estimating Risk Aversion

We used the dataset of choices from linear budget lines to measure the degree of deviations from OEU. We can use the same dataset for the estimation of individual-level risk aversion parameters assuming some functional form (Choi et al., 2007; Friedman et al., 2022).

Let us assume that the Bernoulli function takes the power form

$$u(x) = \begin{cases} \dfrac{x^{1-\alpha}}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log x & \text{if } \alpha = 1 \end{cases}$$

where $\alpha \geq 0$ is the Arrow-Pratt coefficient of relative risk aversion. The interior solution to the maximization problem satisfies the first-order condition

$$\frac{x_1}{x_2} = \frac{(p_1/\mu_1)^{-1/\alpha}}{(p_2/\mu_2)^{-1/\alpha}}.$$

We can take logarithms to obtain

$$\log\left(\frac{x_1}{x_2}\right) = -\frac{1}{\alpha}\log\left(\frac{p_1/\mu_1}{p_2/\mu_2}\right).$$

Assume that each subject i makes her choices (x_1^{ik}, x_2^{ik}) given prices (p_1^{ik}, p_2^{ik}), $k = 1, \ldots, K$, according to the above log-linearized relationship with some additive mean-zero error. That is,

$$\log\left(\frac{x_1^{ik}}{x_2^{ik}}\right) = \beta_i \log\left(\frac{p_1^{ik}/\mu_1}{p_2^{ik}/\mu_2}\right) + \nu^{ik},$$

where ν^{ik} is a mean-zero error term. We can estimate the model with ordinary least squares. The parameter of interest, α, is recovered via a nonlinear transformation $\widehat{\alpha}_i = -1/\widehat{\beta}_i$ and its standard error is calculated using the Delta method.

A limitation of this approach is that the dependent variable is not defined at corner solutions. Following Choi et al. (2007) and Friedman et al. (2022), we incorporate observations at the corners by replacing the zero component with a small constant 10^{-3}.

The estimation result is summarized in Table F.4. As a benchmark, Choi et al. (2007) report three quartiles of estimated α of 0.597, 0.826, and 1.426, respectively.

In the following figures F.19 and F.20, we show how estimated risk aversion parameters and their standard errors, as well as measures of goodness-of-fit, are related to e_* and CCEI.

Table F.4: Summary statistics of estimated coefficient of relative risk aversion.

	N	Q1	Median	Q3	$\widehat{\alpha}_i < 0$
CKMS	1182	0.534	1.130	2.035	144
CMW	1119	0.522	1.074	2.141	91
CS	1423	0.328	0.710	1.479	54

Note: Q1 and Q3 indicate the first and the third quartile, respectively. The last column shows the number of subjects whose estimated $\widehat{\alpha}_i$ is smaller than zero.

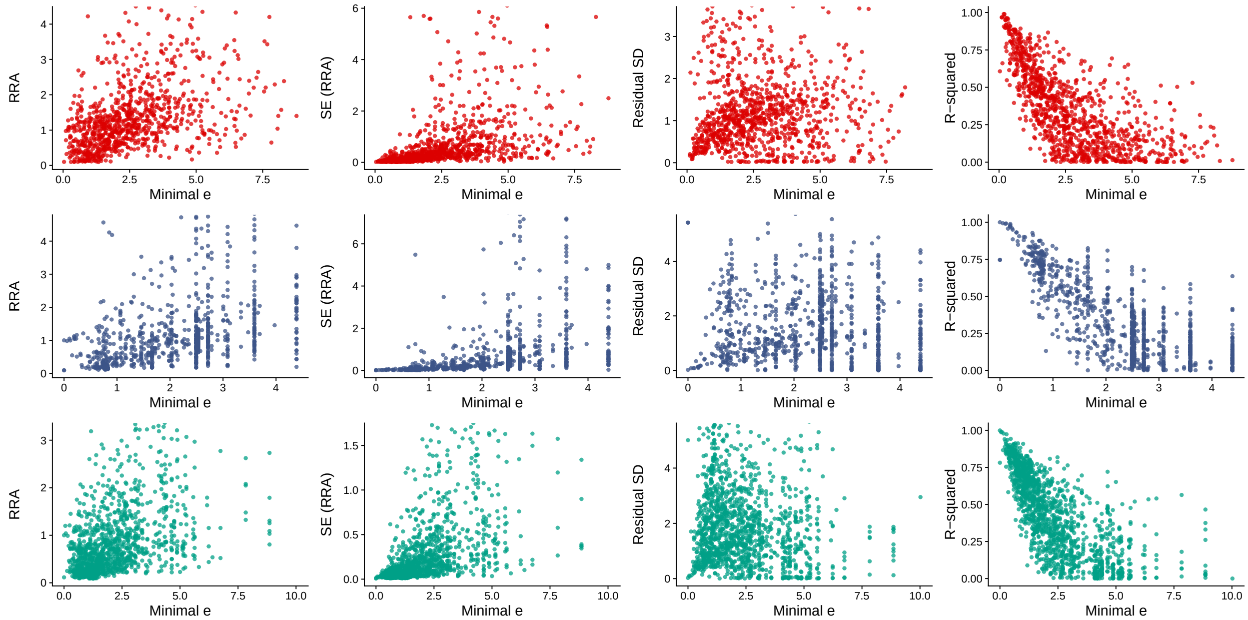


FIGURE F.19: Parametric estimation of relative risk aversion and e_*.

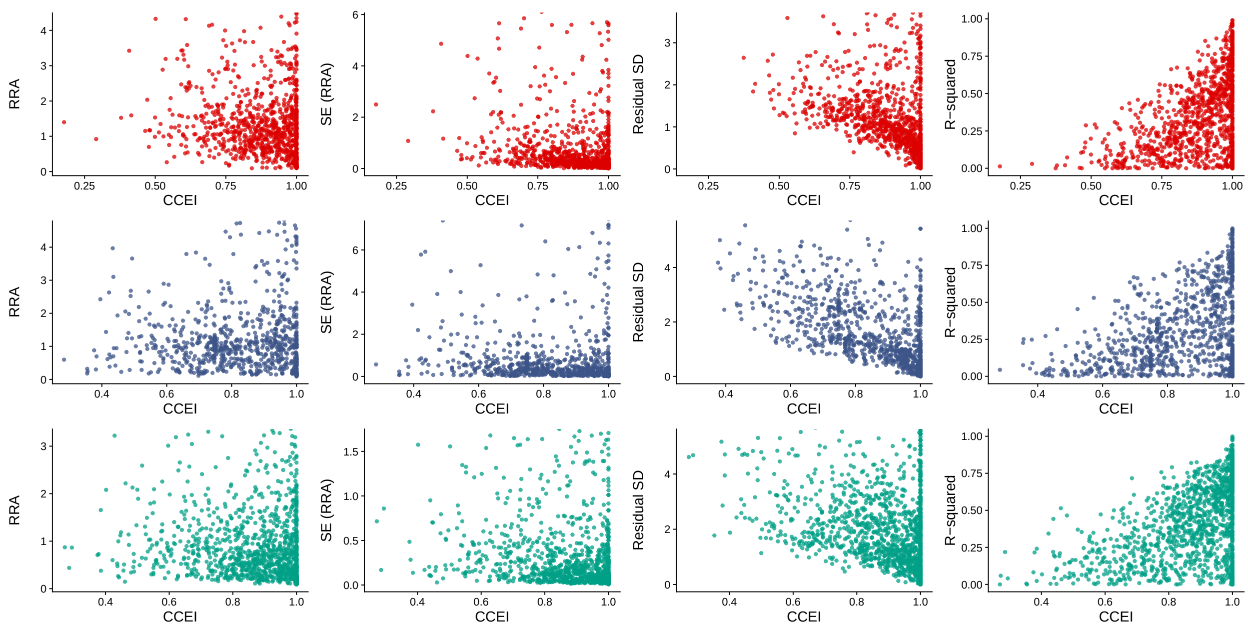


FIGURE F.20: Parametric estimation of relative risk aversion and CCEI.

References

Bronars, S. G. (1987): "The Power of Nonparametric Tests of Preference Maximization," *Econometrica*, 55, 693–698.

Chambers, C. P. and F. Echenique (2014): "On the Consistency of Data with Bargaining Theories," *Theoretical Economics*, 9, 137–162.

Choi, S., R. Fisman, D. Gale, and S. Kariv (2007): "Consistency and Heterogeneity of Individual Behavior under Uncertainty," *American Economic Review*, 97, 1921–1938.

Choi, S., S. Kariv, W. Müller, and D. Silverman (2014): "Who Is (More) Rational?" *American Economic Review*, 104, 1518–1550.

Echenique, F., T. Imai, and K. Saito (2016): "Testable Implications of Models of Intertemporal Choice: Exponential Discounting and Its Generalizations," Caltech HSS Working Paper 1388.

——— (2019): "Decision Making under Uncertainty: An Experimental Study in Market Settings," arXiv:1911.00946.

Echenique, F., S. Lee, and M. Shum (2011): "The Money Pump as a Measure of Revealed Preference Violations," *Journal of Political Economy*, 119, 1201–1223.

Echenique, F. and K. Saito (2015): "Savage in the Market," *Econometrica*, 83, 1467–1495.

Friedman, D., S. Habib, D. James, and S. Crockett (2022): "Varieties of Risk Preference Elicitation," *Games and Economic Behavior*, 133, 58–76.

Nishimura, H., E. A. Ok, and J. K.-H. Quah (2017): "A Comprehensive Approach to Revealed Preference Theory," *American Economic Review*, 107, 1239–1263.

Polisson, M., J. K.-H. Quah, and L. Renou (2020): "Revealed Preferences over Risk and Uncertainty," *American Economic Review*, 110, 1782–1820.

Stoer, J. and C. Witzgall (1970): *Convexity and Optimization in Finite Dimensions*, Berlin: Springer-Verlag.

Varian, H. R. (1985): "Non-Parametric Analysis of Optimizing Behavior with Measurement Error," *Journal of Econometrics*, 30, 445–458.