COMPARATIVE STATICS BY ADAPTIVE DYNAMICS AND THE CORRESPONDENCE PRINCIPLE

By Federico Echenique[1]

1. INTRODUCTION

THE INTUITION BEHIND COMPARATIVE STATICS results is usually dynamic in nature. The economic explanation for why differences in endogenous variables result from variations in exogenous variables often takes the form of some sequential adjustment process. For example, consider the account by Milgrom, Qian, and Roberts (1991) of technological and organizational changes in modern manufacturing: "... the falling costs of high-speed data communication, data processing, and flexible, multitask equipment lead to increases in the directly affected activities, which through a web of complementarities then lead to increases in a set of related activities as well."

Despite the prevalence of dynamic economic explanations, the formal analysis is static. In games of strategic complementarities (GSC; Topkis (1979), Vives (1990)), parameterized by a variable that is complementary to the choice variables, the existing results on monotone comparative statics of equilibria are summarized by a theorem of Milgrom and Roberts (1990; MR hereafter): the largest and smallest equilibria in a parameterized GSC are increasing in the parameter.[2]

GSC can have a large number of equilibria and there are no a priori reasons to expect the largest or smallest equilibrium to be played. Thus the MR result only tells us how a rather coarse summary statistic of the equilibrium set behaves. Moreover, a local approach using the implicit function theorem can give a conclusion opposite to the MR result (see Section 2). I claim that if we refine away equilibria that are unstable for adaptive dynamics, we obtain unambiguous "monotone" comparative statics.

This paper presents results for economic models with complementarities; GSC are its most important application. The paper has two main results. First, if players behave adaptively after a parameter increase, their choices in each round of play will be larger than play before the increase. Second, if the GSC is parameterized by t and $e(t)$ is a selector of equilibria—for all t, $e(t)$ is an equilibrium—that is continuous but not increasing in t, then $e(t)$ selects equilibria that are unstable with respect to a broad class of adaptive dynamics. These two results rely only on complementarity assumptions, and require no topological structure on the models.

The second result above is a version of Samuelson's Correspondence Principle. Samuelson (1947) obtains unambiguous comparative statics by refining away unstable equilibria:

[2] See also Lippman, Mamer, and McCardle (1987), Sobel (1988), and Milgrom and Shannon (1994). Topkis (1998) contains an exposition of the theory.

he applies the Implicit Function Theorem to smooth "equilibrium conditions" and interior, stable, equilibria of simple one-dimensional economic models. It turns out that, in multi-dimensional models, the principle does not yield unambiguous comparative statics (see Arrow and Hahn (1971) and Echenique (2000)). In general, stability is not enough to determine the direction of comparative statics.

This paper shows that Samuelson's principle holds in models with complementarities, in particular in GSC and in general equilibrium models with gross substitutes. The Correspondence Principle presented here has the same advantage over Samuelson's as the new comparative statics methods—the theorems of Topkis and Milgrom and Shannon—have over the use of the Implicit Function Theorem (see Milgrom and Shannon (1994)). No convexity or smoothness of the maps or spaces involved is needed, no Inada conditions, no need to restrict to Euclidean spaces. Given the importance of increasing returns and other nonconvexities in many areas of economics, it is important to have methods that do not require convexity. A final advantage of this version of the Correspondence Principle is that the dynamics used encompass a wide array of behavioral assumptions. In this sense the results are robust to the specification of out-of-equilibrium dynamics. In Echenique (2000) I present Farrell and Saloner's (1985) network externalities model as an example that brings out these points: for comparative statics in network externalities models, none of the existing comparative statics methods are useful.

I also prove that comparative statics that is monotone selects equilibria that are stable—a converse to the Correspondence Principle. This result requires topological assumptions. Thus, with some qualifications, monotone comparative statics in models with complementarities is the same as stability.[3]

In GSC with continuous payoffs and compact choice sets, Vives's (1990) results on learning imply that Cournot dynamics converges to a larger equilibrium after an increase in a parameter. I isolate the effect of the parameter increase on the subsequent dynamics from the continuity and compactness necessary to obtain convergence to an equilibrium. The distinction is analogous to monotone comparative statics for decision problems—where topological conditions are needed to guarantee that optima exist but the comparative statics conclusion does not depend on them. I work with adaptive dynamics that is more general than Cournot. The resulting theorem allows me to prove a version of the Correspondence Principle that is free from topological assumptions and robust to the specification of (adaptive) learning.

The paper proceeds as follows. Section 2 motivates and gives some intuition for my results. Section 3 presents definitions and notation. Section 4 contains the main results. Section 5 discusses the limiting behavior of learning after a parameter change, and the converse to the Correspondence Principle.

2. MOTIVATION AND INTUITION

We find ourselves confronted with this paradox: in order for the comparative statics analysis to yield fruitful results, we must first develop a theory of dynamics.

Samuelson (1947, p. 262).

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[3] In a smooth model with local strategic complementarities, Dierker and Dierker (1999) show that local comparative statics is monotone if and only if a dominant diagonal condition is satisfied. This condition can be related to stability with respect to best-response dynamics; this is, to the best of my knowledge, the only precedent to my result.

Consider the following game. Two workers choose simultaneously the effort level $x \in$ $[0, 1]$ that they put into a common task. They use a common technology whose productivity is indexed by a real number t; a higher value of t implies a higher productivity. Let $\beta_i(x, t)$ be worker i's optimal choice of effort when the other worker chooses x, i.e. her best-response function. Then, $\beta(x, t) = \beta_1(\beta_2(x, t), t)$ is called the composed best-response function, and the Nash equilibria of the game coincide with the fixed points of $\beta(., t)$. Assume that the players' efforts are complementary so that $\beta_i(x, t)$ is increasing in x, and that higher productivity makes each agent want to work harder so that $\beta_i(x, t)$ is increasing in t. Figure 1 shows a typical best-response function for a GSC like the one described. The dotted graph represents the game after an increase in the parameter and $\mathscr{E}(t)$ is the equilibrium set.

Figure 1 shows that, in accordance with the MR result, the smallest and largest equilibria increase after an increase in t. We might expect agents to be playing equilibrium e_2, in which case the results for extremal equilibria are silent. On the other hand, for small parameter changes the Implicit Function Theorem gives local comparative statics at each equilibrium. If we expect e_2 to be played we obtain a conclusion opposite to MR's result: e_2', the closest "new" equilibrium, is smaller than e_2.

Samuelson's Correspondence Principle (CP) says that, selecting equilibria that are stable for some reasonable out-of-equilibrium dynamics gives unambiguous comparative statics results. Note that e_2 and e_2' are unstable for the "Cournot best-response dynamics," $x_n = \beta(x_{n-1}, t)$. It seems then that, unless there is a reason for selecting extremal equilibria, the old methods of comparative statics coupled with the selection criterion of choosing stable equilibria have an advantage over the new literature.[4]
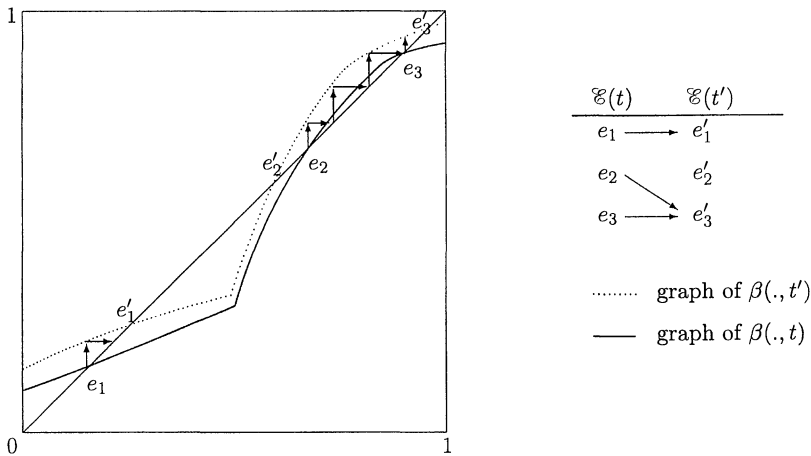

FIGURE 1.—An increase in t.

[4] Pareto optimality or coalition proofness are reasons to select extremal equilibria in, e.g., games of coordination failures. But also in many of these examples the interesting feature of the model is that the socially optimal equilibria may not be selected (as in network externalities or macroeconomic coordination failures).

In the example, since all increasing selections of equilibria pick stable points, the Implicit Function Theorem coupled with the CP yields unambiguous comparative statics results. MR's result does not give a conclusive answer to how the endogenous variables change after an increase in t. This paper shows that if the new methods are endowed with the CP, then they too yield unambiguous comparative statics. The two main ideas can be illustrated using the example:

1. If the workers are at an equilibrium and there is an increase in productivity, then each one will desire to increase her effort. If both agents realize this, then, because of complementarity between efforts, they will want to further increase their efforts. This suggests that any prediction of play after an increase in the parameter should involve larger efforts than the original equilibrium. For example, consider the "Cournot best-response dynamics," where, in each round, players select a best response to last round's play. In Figure 1 it is easy to see that this dynamic in the t'-game, starting at any of the three equilibria for the t-game, converges to a larger equilibrium (illustrated by the arrows in Figure 1).

2. What is wrong with equilibrium e_2? Since e_2 has arbitrarily close smaller equilibria corresponding to larger parameter values, then, by starting at any of these smaller equilibria, decreasing the parameter to t and reasoning as in item 1 we obtain a prediction of play that is yet smaller. For this reason, e_2 must be unstable under any dynamics obtained by reasoning as in item 1.

On the other hand, by looking at the change from e_1 to e_1' and from e_3 to e_3' it can be seen that these selections are increasing *and* select stable equilibria. Hence, in this picture, stability is the same as monotonicity. Since instability of equilibria usually leads game theorists (and probably players too) to doubt that players will select a particular equilibrium, my results imply that we should be at least suspicious about a selection of equilibria that is not monotone in the parameter.

3. DEFINITIONS AND NOTATION

3.1. *Standard Definitions*

A detailed discussion of the concepts defined in this subsection can be found in Topkis (1998). A set X with a transitive, reflexive, antisymmetric binary relation $\preceq$ is a *lattice* if whenever $x, y \in X$, both $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$ exist in X. It is *complete* if for every nonempty subset A of X, $\inf A, \sup A$ exist in X. A nonempty subset A of X is a *sublattice* if for all $x, y \in A$, $x \wedge_X y, x \vee_X y \in A$, where $x \wedge_X y$ and $x \vee_X y$ are obtained taking the infimum and supremum as elements of X (as opposed to using the relative order on A). A nonempty subset $A \subset X$ is *subcomplete* if $B \subset A, B \neq \varnothing$ implies $\inf_X B, \sup_X B \in A$, again taking inf and sup of B as a subset of X. I will use $\preceq$ to denote the order on lattices and $\leq$ to refer to the order on indexes and $\mathbf{R}$. The set $\{z \in X : x \preceq z\}$ will be denoted $[x, M]$.

Say that a correspondence $\phi : Z \twoheadrightarrow X$ is *weakly increasing* if, for any $x, x' \in X$ with $x \prec x'$, we have $\inf \phi(x) \preceq \inf \phi(x')$ and $\sup \phi(x) \preceq \sup \phi(x')$. Say that ϕ is *strongly increasing* if, for any $x, x' \in X$ with $x \prec x'$, we have $\sup \phi(x) \preceq \inf \phi(x')$. When ϕ is a function, i.e. single valued, both concepts coincide with the usual notion of "monotone increasing."

A function $g : T \to X$ is *nowhere weakly increasing over an interval* $[\underline{t}, \overline{t}]$ if $t, t' \in [\underline{t}, \overline{t}]$ and $t \prec t'$ implies $g(x) \not\preceq g(x')$. This is not just the negation of weakly increasing; it rules out the existence of any subinterval of $[\underline{t}, \overline{t}]$ over which the function is increasing.

3.2. *The Model*

I present results for a parameterized family of models that possesses a complementarity property. The main examples of such a class of models are games of strategic complements (GSC). GSC were developed by Topkis (1979) and first introduced into economics by Vives (1990). GSC are very common in economic applications; see Topkis (1998) and Vives (1999) for an exposition of the theory and many economic examples.

Let T be a partially ordered set. An *increasing family of correspondences* $(\phi_t, t \in T)$ is a correspondence $\phi : X \times T \twoheadrightarrow X$ such that $x \mapsto \phi_t(x)$ is weakly increasing and $t \mapsto \phi_t(x)$ is strongly increasing. Let X be endowed with the order-interval topology (see Topkis (1998)). If, in addition, $x \mapsto \phi_t(x)$ is upper-hemicontinuous and subcomplete-sublattice-valued, then it will be called an *increasing family of uhc correspondences*. Let $\mathscr{E} = \{x \in X : x \in \phi(x)\}$ be the set of fixed points of ϕ. When $(\phi_t, t \in T)$ is a family of correspondences, the notation will be $\mathscr{E}(t) = \{x \in X : x \in \phi_t(x)\}$.

Each ϕ_t can be interpreted as the best-response correspondence of a parameterized game. The fixed points of ϕ_t are the Nash equilibria of the game when the parameter takes value t. In Echenique (2000) I show that if a game satisfies Milgrom and Shannon's (1994) complementarity assumptions (theirs is an ordinal generalization of the class of GSC), then its best-responses are an increasing family of correspondences. If, in addition, it satisfies Milgrom and Shannon's continuity assumptions, then its best-responses are an increasing family of uhc correspondences. This implies that the reduced forms of Topkis' (1979) and Vives' (1990) definitions of GSC fall within the framework of this paper. Besides generality and parsimony, an advantage of this model is that comparison with non-GSC results (such as Milgrom and Roberts (1994) and Villas-Boas (1997)) is easy.

3.3. *Dynamics*

Think of $\phi : X \twoheadrightarrow X$ as the best-response correspondence of a game. I shall use the information about the game contained in ϕ to specify sequences $\{x_k\}$ in X. For example, Cournot dynamics is given by $x_k \in \phi(x_{k-1})$; it is called an adaptive learning process because in each period k players adapt their choices to past play x_{k-1}. I shall work with families of such learning dynamics, specified adaptively, but in an otherwise fairly general manner.[5] Given a sequence $\{x_k\}$ in X, let $H_k = \{x_0, \ldots, x_{k-1}\}$ be the history at k.

DEFINITION 1: Let X be a lattice and $\phi : X \twoheadrightarrow X$ a correspondence. A sequence $\{x_k\}$ in X is called *generalized adaptive dynamics from* ϕ if $\inf \phi(\inf H_k) \preceq x_k \preceq \sup \phi(\sup H_k)$ for all $k \geq 1$. Let $\mathscr{D}(x_0, \phi)$ be the set of all sequences that are generalized adaptive dynamics from ϕ and start at x_0.

Consider a Bertrand oligopoly game and let ϕ be its best-response correspondence. A sequence $\{x_k\}$ of price choices by firms in Bertrand competition is in $\mathscr{D}(x_0, \phi)$ if: in each period k, firms choose prices that are larger than the smallest optimal prices if they conjecture that all other firms choose their lowest prices in the history of play; but smaller than the largest optimal prices if they conjecture that all the other firms will choose the largest prices they have chosen so far. Suppose there are three firms and let $k = 3$. If firms One and Two have set prices $(1, 3)$ and $(3, 1)$ in the last two periods, then firm Three is

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[5] When ϕ is the reduced form of a general equilibrium model, or an IS-LM model, instead of a game, the dynamics will have tatonnement-like interpretations.

"allowed" to set any price between its smallest best response to $(1, 1)$ and its largest best response to $(3, 3)$.

In GSC, examples of generalized adaptive dynamics include fictitious play (as best response to historic frequency of play) and, when players' choices are one-dimensional, also local "better-response" dynamics like gradient optimization algorithms. See Echenique (2000) for a comparison with the general adaptive dynamics in Milgrom and Roberts (1990) and Milgrom and Shannon (1994).[6, 7] An important example of adaptive dynamics is $\mathscr{A}(x, \phi) = \{\{x_k\}_{k=0}^{\infty} : x_0 = x, x_k \in \phi(x_{k-1}), k \geq 1\}$, the class of simple adaptive dynamics. The members of $\mathscr{A}(x, \phi)$ take the form of Cournot dynamics in games and tatonnement price adjustment in market models. Clearly, $\mathscr{A}(x, \phi) \subset \mathscr{D}(x, \phi)$.

A correspondence $\phi : X \twoheadrightarrow X$ here defines a class of dynamics, as opposed to the unique trajectories generated in the dynamical systems that are normally studied. This introduces ambiguity in the usual notions of stability. The following definitions capture this ambiguity.

DEFINITION 2: Let $\phi : X \twoheadrightarrow X$ and, for all $x \in X, \widetilde{\mathscr{D}}(x, \phi) \subset \mathscr{D}(x, \phi)$. A point $\hat{x} \in X$ is *best-case stable for* $\widetilde{\mathscr{D}}(., \phi)$ if there is a neighborhood V of $\hat{x}$ in X such that for all x in V, there is a sequence $\{x_k\} \in \widetilde{\mathscr{D}}(x, \phi)$ with $x_k \to \hat{x}$. A point $\hat{x} \in X$ is *worst-case stable for* $\widetilde{\mathscr{D}}(., \phi)$ if there is a neighborhood V of $\hat{x}$ in X such that for all x in V and all sequences $\{x_k\} \in \widetilde{\mathscr{D}}(x, \phi), x_k \to \hat{x}$.

Worst case is a (much) stronger notion of stability than best case. The results in this paper give the strongest possible conclusions: "wrong" comparative statics choose equilibria that are not even best-case stable, while "correct" comparative statics select worst-case stable equilibria.

4. DIRECTION OF DYNAMICS AND THE CORRESPONDENCE PRINCIPLE

The paper's main results are presented as Theorems 1 and 2. Theorem 1 captures, in a general framework, the intuition in Section 2 for GSC. It provides the comparative statics conclusion that the state of a system that is perturbed upwards is permanently larger. Think of e as the state of the system before the change, of $\phi(e)$ as the state immediately after the change, and suppose that the elements of $\mathscr{D}(e, \phi)$ describe the future evolution of the system.

THEOREM 1: *Let* $\phi : X \twoheadrightarrow X$ *be a correspondence on a lattice* X. *If* $e \preceq \phi(e)$ *and* ϕ *is weakly increasing on* $[e, M]$, *then* e *is a lower bound on any sequence* $\{x_k\}$ *in* $\mathscr{D}(e, \phi)$.

PROOF: I will show by induction that e is a lower bound on H_k for all k, which proves the theorem. First, since $\{e\} = H_1$ the statement is true for $k = 1$. Second, if e is a lower bound on H_{k-1}, then $\inf H_{k-1} \in [e, M]$. Since ϕ is weakly increasing in this interval,

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[6] The backward-looking behavior implicit in adaptive play may seem too naïve. The definition also allows some degree of forward-looking behavior. Any finite number of rounds of "I know that you know . . . that I play a best response to H_k" will satisfy the definition since they are just iterations of ϕ.

[7] Dynamics in $\mathscr{D}(x, \phi)$ need not be monotone, but, crucially, the "extremal" members of $\mathscr{D}(x, \phi)$—those that select the smallest and largest feasible play in each round—are monotone in the cases studied in the paper.

$\inf \phi(e) \preceq \inf \phi(\inf H_{k-1})$. Now, $e \preceq \inf \phi(e)$ and $\inf \phi(\inf H_{k-1}) \preceq x_k$ imply that $e \preceq x_k$. Since $H_k = H_{k-1} \cup \{x_k\}$, the inductive hypothesis and $e \preceq x_k$ imply that e is a lower bound on H_k.   Q.E.D.

For GSC with continuous payoffs and compact choice sets, that dynamics in $\mathcal{A}(e, \phi)$ converge to a larger equilibrium after an increase in a parameter can be obtained as a consequence of Vives's (1990) results on learning (his Theorem 5.1; see also Theorem 2.1 in Vives (1999)).[8] Theorem 1 isolates the effect of the increase on the subsequent dynamics from the continuity and compactness assumptions that are needed to ensure convergence to an equilibrium. It is a simple fact about the order structure of the problem; there is no implication for convergence to equilibria beyond the remark that *if* play converges it has to be to a point that is larger than e.

Theorem 2 proves that a continuous selector $e(t) \in \mathcal{E}(t)$ that is not monotone increasing must be picking unstable equilibria. The intuition is simple. If, in any neighborhood of $e(t)$ there is $e(t')$ with $t' \prec t$ and $e(t') \not\preceq e(t)$, then, starting the ϕ_t-dynamics at $e(t')$ Theorem 1 says that play is bounded below by $e(t')$ and hence cannot converge to $e(t)$. Thus $e(t)$ is unstable.

THEOREM 2: *Let $(\phi_t, t \in T)$ be an increasing family of correspondences on a lattice X and $T \subset \mathbf{R}^n$ be convex. Let $e : T \to X$ be a continuous selection from $(\mathcal{E}(t) : t \in T)$. If e is nowhere weakly increasing over some interval $[\underline{t}, \overline{t}]$ in T then, for all $t \in [\underline{t}, \overline{t}]$ with $\underline{t} \prec t \prec \overline{t}, e(t)$ is not best-case stable for $\mathcal{D}(., \phi_t)$.*

PROOF: Let $t \in [\underline{t}, \overline{t}]$ with $\underline{t} \prec t \prec \overline{t}$ and let V be a neighborhood of $e(t)$. Choose $\hat{t} \in e^{-1}(V) \cap [\underline{t}, \overline{t}]$ with $\hat{t} \prec t$, so $e(\hat{t}) \not\preceq e(t)$. Then $e(\hat{t}) \in V$. Let $\{x_k\} \in \mathcal{D}(e(\hat{t}), \phi_t)$. Now, $e(\hat{t}) \in \phi_{\hat{t}}(e(\hat{t}))$ and $\hat{t} \prec t$ so $e(\hat{t}) \preceq \inf \phi_t(e(\hat{t}))$. By Theorem 1, $e(\hat{t})$ is a lower bound on $\{x_k\}$, so any accumulation point α of $\{x_k\}$ satisfies $e(\hat{t}) \preceq \alpha$. Then $e(\hat{t}) \not\preceq e(t)$ implies $\alpha \neq e(t)$. In particular $x_k \not\to e(t)$.   Q.E.D.

REMARKS: (i) Theorem 2 does not impose any topological structure on the model. (ii) Theorem 2 applies when elements of $\{e(t) : t \in [\underline{t}, \overline{t}]\}$ are not ordered. (iii) Since $\mathcal{A}(e(t), \phi_t) \subset \mathcal{D}(e(t), \phi_t)$, nonincreasing selections are not best-case stable with respect to $\mathcal{A}(., \phi_t)$.

The meaning of Theorem 2 is that, if $t \mapsto e(t)$ is not monotone, a perturbation in t will move the system away from $e(t)$. The continuity assumption in $t \mapsto e(t)$ casts this fact as instability. Alternatively, we could not impose continuity on $t \mapsto e(t)$ and say that nonmonotone selectors will make predictions that are not robust to perturbations in the parameter.[9]

5. LIMITS OF ADAPTIVE DYNAMICS AND A CONVERSE TO THE CORRESPONDENCE PRINCIPLE

I present results on the limit behavior of adaptive dynamics after a system is subject to a parameter change. I use these results to prove a converse to the Correspondence Principle.

[8] The first to use complementarities to obtain a result of this kind seem to be Deneckere and Davidson (1985), in an analysis of mergers in a differentiated Bertrand setting (I am grateful to a referee for pointing this out).
[9] For results on stability without imposing continuity of $t \mapsto e(t)$, see Echenique (2000).

To obtain results on the limit behavior of learning dynamics, the effect of events that occurred a long time ago must eventually disappear. Given a sequence $\{x_k\}$ in X, let $H_k^\gamma = \{x_{k-\gamma}, \ldots, x_{k-1}\}$ denote the history of length γ at time k (set $x_{-l} = x_0$ for $1 \leq l \leq \gamma$ to simplify notation). Let $\mathscr{D}^f(x_0, \phi)$ be the set of sequences $\{x_k\}$ for which there is $\gamma \in \mathbf{N}$ with the property that $\inf \phi(\inf H_k^\gamma) \preceq x_k \preceq \sup \phi(\sup H_k^\gamma)$ for all $k \geq 1$. Thus, $\mathscr{D}^f(x_0, \phi)$ are the sequences of choices that are justified in terms of, possibly long, but bounded histories of play. Note that $\mathscr{A}(x, \phi) \subset \mathscr{D}^f(x, \phi) \subset \mathscr{D}(x, \phi)$. I will denote the set of limits of adaptive dynamics starting at $x \in X$ by $F(x, \phi) = \{z \in X : \exists \{x_k\} \in \mathscr{D}^f(x, \phi) \text{ s.t. } z = \lim_k x_k\}$.

Theorem 3 requires an uhc correspondence. The main application is to continuous functions and best-response correspondences arising from GSC with continuous payoffs. For GSC with continuous payoffs, item 3 in the Theorem follows from Vives (1990, Theorem 5.1). But Theorem 3 provides additional information about the limits of adaptive behavior after an increase in a parameter: the limits have a largest and a smallest element that are equilibria larger than the state before the parameter increase—and all accumulation points of adaptive behavior after a parameter increase are bounded by these extremal equilibria. The proof of Theorem 3 is in the Appendix.

THEOREM 3: *Let X be a complete lattice and $x \in X$. Let $\phi : X \twoheadrightarrow X$ be an uhc correspondence that is weakly increasing on $[x, M]$. If $x \preceq \inf \phi(x)$, then:*

1. *$F(x, \phi)$ has a smallest and a largest element, $\inf F(x, \phi)$ and $\sup F(x, \phi)$, with $\inf F(x, \phi)$, $\sup F(x, \phi) \in \mathscr{E}$ and $\inf F(x, \phi) = \inf\{z \in \mathscr{E} : x \preceq z\}$;[10]*

2. *for all $\{x_k\} \in \mathscr{D}^f(x, \phi)$, $\inf F(x, \phi) \preceq \liminf_k x_k \preceq \limsup_k x_k \preceq \sup F(x, \phi)$;*

3. *if, in addition, ϕ is strongly increasing over $[x, M]$, then for all $\{x_k\} \in \mathscr{A}(x, \phi)$, $\lim x_k$ exists, $x \preceq \lim x_k$, and $\lim x_k \in F(x, \phi) \cap \mathscr{E}$.*

Theorem 3 implies that $\{z \in \mathscr{E} : x \preceq z\}$ is nonempty. This fact yields a simple proof of MR's comparative statics result for extremal equilibria (and of Milgrom and Shannon's (1994) generalization to ordinal GSC).

COROLLARY 1 (Milgrom and Roberts (1990), Milgrom and Shannon (1994)): *Let $(\phi_t, t \in T)$ be an increasing family of uhc correspondences. Let $t, t' \in T$ such that $t \prec t'$. Then $\inf \mathscr{E}(t) \preceq \inf \mathscr{E}(t')$ and $\sup \mathscr{E}(t) \preceq \sup \mathscr{E}(t')$.*

PROOF: Suprema and infima are well defined since by Zhou (1994) the set of fixed points is a complete lattice. Let $e = \sup \mathscr{E}(t)$. Then, by Theorem 3, $e \preceq \inf F(e, \phi_{t'}) \preceq \sup \mathscr{E}(t')$ since $\inf F(e, \phi_{t'}) \in \mathscr{E}(t')$. The result for infima follows analogously.   Q.E.D.

I show that monotone comparative statics imply stable equilibria. It is rather strong that stability follows from the comparative statics property of the selection of equilibria alone; this comes at the cost of imposing more structure on the problem: Euclidean spaces, stronger monotonicity assumptions and locally isolated equilibria.

A fixed point $e(t) \in \mathscr{E}(t)$ of the correspondence ϕ_t is *isolated* if there is a neighborhood V of $e(t)$ in X such that $V \cap \mathscr{E}(t) = \{e(t)\}$. The interior of the interval $[\underline{t}, \overline{t}]$ in $\mathbf{R}^n$ is denoted by $[\underline{t}, \overline{t}]^o$. The role of local isolation and strict monotonicity is to produce "asymptotic stability"—that the dynamics converge back to the equilibrium after a perturbation. A weaker stability conclusion, that the dynamics remain "close by," can be obtained without these two hypotheses.

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[10] $X \subset \mathbf{R}^n$, then $\inf F(x, \phi)$ is also the closest larger equilibrium.

THEOREM 4: *Let $X \subset \mathbf{R}^m$ and $(\phi_t, t \in T)$ be an increasing family of uhc correspondences with $T \subset \mathbf{R}^n$ convex.[11] Let $e : T \to X$ be a continuous selection from $(\mathscr{E}(t) : t \in T)$. If e is strictly increasing[12] over some interval $[\underline{t}, \overline{t}]$ and $e(t)$ is isolated, with $t \in [\underline{t}, \overline{t}]^o$, then $e(t)$ is worst-case stable for $\mathscr{D}^f(., \phi_t)$.*

PROOF: Let $t \in [\underline{t}, \overline{t}]^o$ be such that $e(t)$ is isolated. Let N be a neighborhood of $e(t)$ with $N \cap \mathscr{E}(t) = \{e(t)\}$, and let $P = \{x \in \mathbf{R}^m : 0 \preceq x\}$ be the positive cone in $\mathbf{R}^m$. Let B_r and B_{2r} be open balls contained in N with center $e(t)$ and radii r and $2r$, respectively. Take $t_1, t_2 \in e^{-1}(B_r) \cap [\underline{t}, \overline{t}]^o$ with $t_1 \ll t \ll t_2$. Note then that $[e(t_0), e(t_1)] \subset B_{2r}$. To see this, set $e(t) = 0$ without loss of generality. If $x \in [e(t_0), e(t_1)]$, then $x \vee 0 \preceq e(t_1)$ and $(-x) \wedge 0 \preceq -e(t_0)$. Then

$$|x| = x \vee 0 + (-x) \wedge 0 \preceq e(t_1) - e(t_0)$$
$$= 0 \vee (e(t_1) + e(t_0)) - 0 \wedge (e(t_1) + e(t_0)) = |e(t_0) + e(t_1)|$$

so, since $\| \cdot \|$ is a lattice norm, $\|x\| \le \|e(t_0) + e(t_1)\|$. But $e(t_0), e(t_1) \in B_r$ so

$$\|x\| \le 2 \max\{\|e(t_0)\|, \|e(t_1)\|\} \le 2r.$$

Hence $x \in B_{2r}$.

Now, $e(t_0) \ll e(t) \ll e(t_1)$, i.e. $e(t) \in e(t_0) + P^o$ and $e(t) \in e(t_1) - P^o$. Let $V = (e(t_0) + P^o) \cap (e(t_1) - P^o) \cap X$, an X-relatively open neighborhood of $e(t)$. The claim is that V satisfies the definition of worst-case stability.

Let the sequences $\{y_n\}$ and $\{z_n\}$ be such that $y_0 = e(t_0), z_0 = e(t_1)$, and $y_n = \inf \phi_t(y_{n-1}), z_n = \sup \phi_t(z_{n-1})$ for all $n \ge 1$. Then $\{y_n\}$ and $\{z_n\}$ are simple adaptive play from ϕ_t and, by Theorem 3, $y_n \to e'$ and $z_n \to e''$ with $e', e'' \in \mathscr{E}(t)$. Now, for any $x \in V$, let $\{x_k\}$ be some arbitrary generalized adaptive play from ϕ_t starting at $x_0 = x$. By Lemma 1 in the Appendix applied twice, $y_n \preceq \liminf_k x_k \preceq \limsup_k x_k \preceq z_n$ for all n. This implies that $e' \preceq \liminf_k x_k \preceq \limsup_k x_k \preceq e''$. But $e', e'' \in [e(t_0), e(t_1)] \subset B_{2r} \subset N$ and therefore $e' = e'' = e(t)$ by local isolation. Then, $e(t) \preceq \liminf_k x_k \preceq \limsup_k x_k \preceq e(t)$, so that $x_k \to e(t)$. Thus V satisfies the definition of worst-case stability. Q.E.D.

Facultad de Ciencias Sociales, Universidad de la República, and Departamento de Economía, Universidad Torcuato di Tella, Miñones 2177, C1428ATG Buenos Aires, Argentina

APPENDIX: PROOF OF THEOREM 3

The following lemma is a version of MR's Theorem 8 for the present context. It is used here as an auxiliary result in the proof of Theorem 3.

Define iterations of the lower selections from ϕ by $\underline{\phi}^0(x) = \inf \phi(x)$, $\underline{\phi}^n(x) = \inf \phi(\underline{\phi}^{n-1}(x))$ for all x and $n \ge 1$. Define iterations of the upper selections $\overline{\phi}^n(x)$ similarly.

[11] The theorem is true when X is a subset of a Banach lattice whose order cone has a nonempty interior; the proof uses only this structure.

[12] A function f on a Euclidean space is strictly increasing if $x \ll y$ implies $f(x) \ll f(y)$.

LEMMA 1: *Let X be a complete lattice and $\phi : X \twoheadrightarrow X$. Fix $x \in X$ and $\{x_k\} \in \mathscr{D}^f(x, \phi)$. If $x \preceq \inf \phi(x)$ and ϕ is a uhc correspondence that is weakly increasing on $[x, M]$, then, for all $n \in \mathbf{N}$, there exists $K_n \in \mathbf{N}$ such that $k \geq K_n$ implies $x_k \in [\underline{\phi}^n(x), \bar{\phi}^n(M)]$.*

PROOF: The proof proceeds by induction on n. To get the result for $n = 0$, do induction on k using $K_0 = 1$: First, note that $H_1^\gamma = \{x_0\}$ and $x_0 = x$ so $x_1 \in [\underline{\phi}(\inf H_1^\gamma), \bar{\phi}(\sup H_1^\gamma)]$ implies that $\underline{\phi}(x) \preceq x_1 \preceq \bar{\phi}(x)$. Also, by weak monotonicity on $[x, M]$, $\bar{\phi}(x) \preceq \bar{\phi}(M)$ so $x_1 \in [\underline{\phi}(x), \bar{\phi}(M)]$. Suppose now that $\underline{\phi}(x) \preceq x_l$ for all $1 \leq l \leq k - 1$, so $\underline{\phi}(x)$ is a lower bound on H_k^γ. Then $x \preceq \inf \phi(x) = \underline{\phi}(x) \preceq \inf H_k^\gamma$. So, $\inf H_k^\gamma \in [x, M]$ and weak monotonicity of ϕ gives $\inf \phi(x) \preceq \inf \phi(\inf H_k^\gamma)$. But then $\inf \phi(\inf H_k^\gamma) \preceq x_k$. Also, M is an upper bound on H_k^γ and $x \preceq \inf H_k^\gamma \preceq \sup H_k^\gamma \preceq M$. By monotonicity on $[x, M]$, $x_k \preceq \bar{\phi}(\sup H_k^\gamma) = \sup \phi(\sup H_k^\gamma) \preceq \sup \phi(M) = \bar{\phi}(M)$. Hence, $x_k \in [\underline{\phi}(x), \bar{\phi}(M)]$. This establishes the result for $n = 0$ with $K_0 = 1$.

Now, let K_{n-1} work for $n - 1$ in the statement of the lemma. Set $K_n = K_{n-1} + \gamma$. Pick any $k \geq K_n$. By the inductive hypothesis, for any $x_l \in H_k^\gamma$, $\underline{\phi}^{n-1}(x) \preceq x_l$. Thus $\underline{\phi}^{n-1}(x)$ is a lower bound on H_k^γ so we get $\underline{\phi}^{n-1}(x) \preceq \inf H_k^\gamma$. This implies that $\inf \phi(\underline{\phi}^{n-1}(x)) \preceq \inf \phi(\inf H_k^\gamma)$ because ϕ is weakly increasing on $[x, M]$ (and $x \preceq \underline{\phi}^{n-1}(x)$ by $x \preceq \inf \phi(x) \preceq \underline{\phi}^{n-1}(x)$). Thus, $\underline{\phi}^n(x) \preceq \underline{\phi}(\inf H_k^\gamma) \preceq x_k$. Similarly, by the inductive hypothesis, $\bar{\phi}^{n-1}(M)$ is an upper bound on H_k^γ and therefore $\sup \phi(\sup H_k^\gamma) \preceq \sup \phi(\bar{\phi}^{n-1}(M))$. This gives $x_k \preceq \bar{\phi}^n(M)$.   Q.E.D.

LEMMA 2: *If $\{x_k\}$ is a monotone sequence in a complete lattice X, then $\{x_k\}$ is convergent and $\lim_k x_k = \bigvee_k x_k$.*[13]

PROOF: Let A be the range of $\{x_k\}$ and $x^* = \sup A = \bigvee_k x_k$. Note that x^* is also the supremum of the range of any subsequence of $\{x_k\}$ since by monotonicity the range of any subsequence has the same set of upper bounds. Let V be any neighborhood of x^*. The claim is that eventually $x_k \in V$ for all k. Since V^c is closed and the closed order intervals are a sub-basis for the closed sets in the order interval topology, there is a collection $\{\bigcup_{m=1}^{n_i}[a_m^i, b_m^i] : i \in I\}$ with $V^c = \bigcap_{i \in I} \bigcup_{m=1}^{n_i}[a_m^i, b_m^i]$ (this is without loss of generality since any a_m^i or b_m^i may be $\inf X$ or $\sup X$ because X is complete). But $x^* \notin V^c$ so there is $j \in I$ with $x^* \notin \bigcup_{m=1}^{n_j}[a_m^j, b_m^j]$. Now, for any $m = 1 \dots n_j$ there can only be a finite number of elements of $\{x_k\}$ in $[a_m^j, b_m^j]$. To see this note that if there is a subsequence $\{x_{k_l}\}$ with $a_m^j \preceq x_{k_l} \preceq b_m^j$ for all $l \in \mathbf{N}$, then $a_m^j \preceq x_{k_l} \preceq x^*$ and b_m^j is an upper bound on the subsequence so $x^* \preceq b_m^j$. Hence $x^* \in [a_m^j, b_m^j]$, a contradiction. Since n_j is finite, there can only be a finite number of elements of $\{x_k\}$ in $\bigcup_{m=1}^{n_j}[a_m^j, b_m^j]$. Hence, eventually, $x_k \notin \bigcap_{i \in I} \bigcup_{m=1}^{n_i}[a_m^i, b_m^i] = V^c$. Since V was an arbitrary neighborhood, $x_k \to x^*$.   Q.E.D.

PROOF OF THEOREM 3: Define the sequences $\{x_k\}, \{y_k\}$ by $x_0 = y_0 = x$ and $x_k = \underline{\phi}^k(x)$ and $y_k = \bar{\phi}^k(x)$ for $k \geq 1$ (all infima and suprema are well defined by the completeness of X). First I will show by induction that $\{x_k\}$ and $\{y_k\}$ are sequences in $[x, M]$. Since $x \preceq \inf \phi(x) \preceq \sup \phi(x)$, $x_0 = x \preceq x_1 \preceq y_1$. If $x \preceq x_{k-1}$, then $x, x_{k-1} \in [x, M]$. By weak monotonicity of ϕ on $[x, M]$, then, $x \preceq \inf \phi(x) \preceq \inf \phi(x_{k-1})$. This implies that $x_k \in [x, M]$. The argument for $\{y_k\}$ is identical.

Now, ϕ is weakly increasing on $[x, M]$ and $x_0 \preceq x_1 \preceq y_1$, so $x_{k-1} \preceq x_k$ and $y_{k-1} \preceq y_k$ for all k. Thus, $\{x_k\}$ and $\{y_k\}$ are monotone sequences in X. By Lemma 2, $x_k \to x^* = \bigvee_k x_k$ and $y_k \to y^* = \bigvee_k y_k$.

Also, ϕ is subcomplete and sublattice valued on $[x, M]$, so $x_k \in \phi(x_{k-1})$ for all $k \geq 1$ and thus $\{x_k\} \in \mathscr{A}(x, \phi) \subset \mathscr{D}^f(x, \phi)$. This implies that $x^* \in F(x, \phi)$, and thus $F(x, \phi)$ is nonempty. Now, set $z_k = x_{k+1} \in \phi(x_k)$ for all $k \geq 1$. Since $x_k \to x^*$ and ϕ is upper hemicontinuous on $[x, M]$ and closed valued (see Theorem 14.17 in Aliprantis and Border (1994)), there is $z \in \phi(x^*)$ and a subsequence $\{z_{k_l}\}$ of $\{z_k\}$ such that $z_{k_l} \to z$. But $\{z_{k_l}\}$ is also a subsequence of $\{x_k\}$, and the order interval topology on X is Hausdorff because X is a complete lattice, so $z = x^*$. Then $x^* \in \phi(x^*)$ so $x^* \in \mathscr{E}$. Clearly, $x = x_0 \preceq x^*$ since $x^* = \sup A$. The reasoning for $\{y_k\}$ is analogous and gives $y_k \to \bigvee_k y_k \in F(x, \phi) \cap \mathscr{E}$.

[13] The notation $\bigvee_k x_k$ refers to the supremum of the range of the sequence $\{x_k\}$.

Let $\{z_k\} \in \mathscr{D}^f(x, \phi)$ and let $\{x_k\}$ and $\{y_k\}$ be defined as above. I will show by induction that $\{y_k\}$ is pointwise larger than $\{z_k\}$. First, $x_0 \preceq z_0 \preceq y_0$ trivially. If $x_l \preceq z_l \preceq y_l$ for all $1 \le l \le k - 1$, then $z_{k-1} \in [x, M]$, and $y_{k-1} = \sup\{y_l : 1 \le l \le k - 1\}$ is an upper bound on H_k^γ. Then $\sup H_k^\gamma \preceq y_{k-1}$. By weak monotonicity, then $z_k \preceq \sup \phi(\sup H_k^\gamma) \preceq \sup \phi(y_{k-1}) = y_k$. Hence, $\{y_k\}$ is pointwise larger than $\{z_k\}$, which implies that the set of upper bounds of the range of $\{y_k\}$ is contained in the set of upper bounds of $\{z_k\}$. But $y_k \to y^* = \bigvee_k y_k$ so $\limsup_k z_k \preceq y^*$. By Lemma 1, for all n, $x_n = \underline{\phi}^n(x) \preceq \liminf z_k$. Thus $\liminf z_k$ is an upper bound on the range of $\{x_n\}$. But $x_n \to \bigvee_n x_n = x^*$, so we get $x^* \preceq \liminf z_k$. Hence, $x^* \preceq \liminf z_k \preceq \limsup z_k \preceq y^*$. In particular, if $\{z_k\} \in \mathscr{D}^f(x, \phi)$ is convergent the corresponding limit will also be in $[x^*, y^*]$. Thus, x^* and y^* are, respectively, lower and upper bounds on $F(x, \phi)$ and since $x^*, y^* \in F(x, \phi)$ this implies $x^* = \inf F(x, \phi)$ and $y^* = \sup F(x, \phi)$. Hence $F(x, \phi)$ is nonempty and has a smallest element, $\inf F(x, \phi) \in \mathscr{C}$, and a largest element, $\sup F(x, \phi) \in \mathscr{C}$. This proves the first half of item 1 and item 2.

To finish the proof of item 1, first note that $x \preceq \inf F(x, \phi)$, since $\inf F(x, \phi) = \bigvee_k x_k$. Together with $\inf F(x, \phi) \in \mathscr{C}$ this implies that $\{z \in \mathscr{C} : x \preceq z\} \ne \varnothing$. Let $e \in \{z \in \mathscr{C} : x \preceq z\}$. By induction I show that e is an upper bound on the range of $\{x_k\}$. First note that $x_0 = x \preceq e$ implies $x_0, e \in [x, M]$. Then, $x_{k-1} \preceq e$ and weak monotonicity of ϕ on $[x, M]$ imply that $x_k = \inf \phi(x_{k-1}) \preceq \inf \phi(e)$ and $x_k \in [x, M]$. But $e \in \phi(e)$ so $\inf \phi(e) \preceq e$. Thus, $x_k \preceq e$ for all k. Since $\inf F(x, \phi) = \bigvee_k x_k$ this implies that $\inf F(x, \phi)$ is a lower bound on $\{z \in \mathscr{C} : x \preceq z\}$. But we proved that $\inf F(x, \phi) \in \{z \in \mathscr{C} : x \preceq z\}$, thus proving item 1 of Theorem 3.

Finally, assume that ϕ is strongly increasing over $[x, M]$. Let $\{z_k\} \in \mathscr{A}(x, \phi)$. By the argument above, $\{z_k\}$ is a sequence in $[x, M]$ and $z_0 = x \preceq z_1$. Since ϕ is strongly increasing, $z_{k-2} \preceq z_{k-1}, z_{k-1} \in \phi(z_{k-2})$ and $z_k \in \phi(z_{k-1})$, we conclude that $z_{k-1} \preceq z_k$. Inductively, then, $\{z_k\}$ is monotone. By repeating the argument made above for the infimum selection $\{x_k\}$, we obtain that $\lim_k z_k$ exists and $\lim_k z_k \in \mathscr{C}$. Since $\{z_k\} \in \mathscr{A}(x, \phi) \subset \mathscr{D}^f(x, \phi), \lim_k z_k \in F(x, \phi)$. This proves item 3 in Theorem 3.

$Q.E.D.$

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