Counting combinatorial choice rules

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Abstract

I count the number of combinatorial choice rules that satisfy certain properties: *Kelso–Crawford substitutability*, and *independence of irrelevant alternatives*. The results are important for two-sided matching theory, where agents are modeled by combinatorial choice rules with these properties. The rules are a small, and asymptotically vanishing, fraction of all choice rules. But they are still exponentially more than the preference relations over individual agents—which has positive implications for the Gale–Shapley algorithm of matching theory.
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1. Introduction

Consider hiring a team of workers from a set A of available workers. The decision of hiring worker x is not independent of the decision to hire worker y; the workers may be complements or substitutes. Let $C(A) \subseteq A$ be the workers hired. The function C is called a (combinatorial) choice rule. I shall give results on the number of functions C that satisfy various properties.

The main application I have in mind is the theory of matching markets (Roth and Sotomayor, 1990). In many-to-one, and many-to-many, matching theory, some agents, "firms," are matched to a set of individual "workers." Firms' behavior is modeled as a combinatorial choice rule: for each set of available workers they select a subset.

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The classical results on matching markets—among others, that the core is nonempty—require structure on the firms' choice rules: *substitutability* and *independence of irrelevant alternatives* (IIA). Substitutability was introduced, and applied to matching markets, by Kelso and Crawford (1982). It requires that, if worker x is chosen out of a set that contains worker y, then x is also chosen when y is not available. So x could not have been chosen because she was complementary to y, hence the name substitutability (see Section 2.1 for a formal definition and comparison with the assumption in Kelso and Crawford). IIA is a more basic rationality assumption. Substitutability and IIA are sufficient for the classical results on matching markets (this is clear from Roth, 1984 and Blair, 1988).

Substitutability and IIA have played a central role in the research on matching markets since Kelso and Crawford (1982), and on related problems, e.g. Gul and Stacchetti (1999, 2000) and Hatfield and Milgrom (2005) use substitutability in models that extend beyond matching models. But, while there are several analysis of substitutable combinatorial choice rules—Gul and Stacchetti (1999), Beviá et al. (1999), and Fujishige and Yang (2003) are recent examples—nobody has counted them.

I count the number of choice rules that satisfy substitutability and IIA. The results, and their main implications, are:

(1) The choice rules that satisfy substitutability are a small, and asymptotically vanishing, fraction of all choice rules. Arguably, then, substitutability is a strong assumption. In continuous models, one routinely disregards cases with Lebesgue-measure zero. The same logic suggests that substitutability is a strong assumption, in the sense that it is almost a "knife-edge" case. Concretely, for large groups of workers, the structure that ensures a stable matching is unlikely. I should make two qualifications:

First, an obvious caveat is: Even if they are scarce, the substitutable choice rules may nevertheless often occur. For example, because they are induced by certain behaviors—such as "responsive" preferences (Roth and Sotomayor, 1990, p. 173).

Second, one may also want to compare the number of substitutable rules to more restrictive sets of rules. With some knowledge about agents' behavior in the problem at hand, one can take some properties as given. It may then be that substitutability is not a strong additional assumption, if the substitutable rules are not scarce relative to the smaller set of rules.

(2) The choice rules that satisfy substitutability and IIA are exponentially more than the preference relations over individual workers. So the choice rules with the structure used in matching theory are few, *ma non troppo*.

This result has an important implication for the Gale–Shapley algorithm for finding a matching in the core of the many-to-one matching market. Segal (2003) proves that the Gale–Shapley algorithm requires approximately as much communication as communicating a preference relation over individual workers. My results and Segal's then imply that the algorithm requires exponentially less communication than full revelation of agents' choice rules. The implication helps explain why the Gale–Shapley algorithm is so widely used in practice. See Segal (2003) on what communication means, and why full revelation is the right benchmark.

The numbers involved are surprising. Suppose 8 objects can be chosen, much fewer than in actual matching markets. Already with 8 objects, the substitutable choice rules are a small fraction of the number of choice rules. There are 1.8×10^{308}, roughly a centillion, different

choice rules (more choice rules than protons in the universe). The number of substitutable choice rules is of the order 10^{99}.

It may be easier to think of bits: With 8 objects, it takes one kilobit (1028 bits) to communicate one choice rule, while communicating a substitutable rule takes 329 bits. Communicating a substitutable and IIA choice rule takes between 35 and 285 bits. And communicating a preference relation over individual objects takes 23 bits.[1]

My results exploit a connection between choice rules and a well-studied, but unsolved, problem in combinatorics: The problem of counting the monotone boolean functions (a Boolean function is a function from 2^X to $\{0, 1\}$), called "Dedekind's problem." I show that Dedekind's problem is equivalent to the problem of counting substitutable choice rules, and that Dedekind's problem gives bounds on the substitutable and IIA choice rules. Then, existing bounds and approximations to Dedekind's problem give bounds and approximations to the problem of counting choice rules.

I classify choice rules in two categories: combinatorial and non-combinatorial rules. Combinatorial rules model the choice of sets of objects. Non-combinatorial rules model the choice of mutually-exclusive individual objects. So, if C is a combinatorial rule, $\{x, y\} = C(\{x, y, z\})$ means that only the package $\{x, y\}$ is chosen out of $2^{\{x,y,z\}}$; if C is non-combinatorial, $\{x, y\} = C(\{x, y, z\})$ means that *either* x or y are chosen out of $\{x, y, z\}$. Non-combinatorial rules may thus choose a set of objects, but the interpretation is different: choosing x and y does not mean choosing them as a package, rather it means one is happy getting either of them. Mathematically, the two types of rules are identical, but the interpretation makes an important difference in studying their properties. There is an extensive theory of non-combinatorial choice rules in decision theory and social choice (see, for example, Moulin, 1991 or Austen-Smith and Banks, 1998).

My calculations for combinatorial choice rules have some obvious, but possibly interesting, implications for non-combinatorial choice rules.

Substitutability is formally identical to the condition called (Sen's) α (or Chernoff's condition) in the non-combinatorial model. Thus, the choice rules that satisfy α vanish as a proportion of all choice rules.

I also show that the choice rules that satisfy α are exponentially more than the choice rules that are rationalizable by a preference relation, and that there are more rules that satisfy (Sen's) β than there are rules that satisfy α. So there are also many more rules that satisfy β than there are rationalizable rules.

Section 2 presents the results on combinatorial choice rules. The proof of the main theorem is deferred to Section 4. Section 3 presents the results on non-combinatorial choice rules.

2. Combinatorial choice rules

2.1. The model

Let X be a finite set of cardinality n. Without loss of generality, let $X = \{1, \ldots, n\}$. A *choice rule* is a map $C : 2^X \to 2^X$ such $C(A) \subseteq A$ for all $A \subseteq X$. Note that $C(A) = \emptyset$ is a possibility. Let $\mathcal{C}(n)$ be the set of all choice rules, i.e.

$$\mathcal{C}(n) = \Pi_{A \subseteq X} 2^A.$$

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[1] I explain these calculations in Section 4.6.

For each choice rule C there is an associated choice rule R, defined by $R(A) = A \backslash C(A)$. If $C(A)$ are the chosen elements of A, $R(A)$ are the rejected elements.

A choice rule C satisfies *substitutability* if, for all $A, B \subseteq X$,

$$A \subseteq B \quad \text{implies} \quad C(B) \cap A \subseteq C(A). \tag{1}$$

The property actually used in Kelso and Crawford (1982) is different. Their model predicts salaries, as well as who is matched to whom. They state their property in terms of the set of workers demanded by a firm for a given vector of salaries. The property then requires that an increase in some salaries should not make the firm not demand a worker whose salary stays the same. The version of substitutability I use was introduced by Roth (1984); see Chapter 6 in Roth and Sotomayor (1990) for a discussion.

A choice rule C is *monotone* if, for all $A, B \subseteq X$,

$$A \subseteq B \quad \text{implies} \quad C(A) \subseteq C(B).$$

There is a simple relationship between substitutability and monotonicity: C satisfies substitutability if and only if its associated R is monotone (the proof is trivial). Hence the number of substitutable rules coincides with the number of monotone rules. As a consequence, my results extend directly to monotone rules.

A choice rule C satisfies *independence of irrelevant alternatives* (IIA) if, for every $A, B \subseteq X$, if $C(B) \subseteq A \subseteq B$ then $C(B) = C(A)$.

The theory of matching markets requires two hypotheses on choice: substitutability and IIA (see Kelso and Crawford, 1982; Blair, 1988 and Roth and Sotomayor, 1990).

2.2. Results

The calculation of $|\mathcal{C}(n)|$ is very simple, and probably known (I could not find a reference in print). I include it for use in the rest of the paper.

Theorem 1. $|\mathcal{C}(n)| = 2^{n2^{n-1}}$.

Proof. From the definition of $\mathcal{C}(n)$,

$$\left|\mathcal{C}(n)\right| = \Pi_{k=1}^n 2^{k\binom{n}{k}}.$$

So,

$$\begin{aligned}
\log\left|\mathcal{C}(n)\right| &= \sum_{k=1}^n k\binom{n}{k} \\
&= \sum_{k'=0}^{n-1} n \frac{(n-1)!}{(n-k'-1)!k'!} \\
&= n2^{n-1}
\end{aligned}$$

(log is logarithm base 2). Thus $|\mathcal{C}(n)| = 2^{n2^{n-1}}$. $\quad\square$

Let $\mathcal{C}_1(n)$ be the set of choice rules that satisfy substitutability. Let $\mathcal{C}_2(n)$ be the set of choice rules that satisfy substitutability and IIA; $\mathcal{C}_2(n)$ is the important class of rules for the theory of matching markets. If agents' choice rules are in $\mathcal{C}_2(n)$, the classical results on matching markets

follow. And all choice rules in $\mathcal{C}_2(n)$ are rationalizable by a preference relation over subsets of X.[2]

The theorem employs the following notation. Let $S(K, k)$ denote the number of partitions of $\{1, \ldots, K\}$ into k sets. $S(K, k)$ is called the Stirling number of the second kind (see e.g. Comtet, 1974). All logarithms in the paper are base 2, unless denoted by ln, in which case the base is e.

Theorem 2.

$$2^{n\binom{n-1}{\lfloor (n-1)/2 \rfloor}} \leqslant \left| \mathcal{C}_1(n) \right| \leqslant 2^{(n+\kappa \log(n))\binom{n}{\lfloor n/2 \rfloor}},$$

$$2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}} \leqslant \left| \mathcal{C}_2(n) \right|,$$

$$\left| \mathcal{C}_2(n) \right| \leqslant \sum_{0 \leqslant k \leqslant K \leqslant n} \binom{n}{K}\binom{K}{k} 2^{(1+\kappa \log(n)/n)\binom{n}{\lfloor n/2 \rfloor}} k! S\left(\binom{K}{\lfloor K/2 \rfloor}, k \right),$$

and $\log |\mathcal{C}_1(n)| \sim n\binom{n-1}{\lfloor (n-1)/2 \rfloor}$. κ *is a constant.*

The proof of Theorem 2 is in Section 4. Section 4.1, in particular, presents the main insight behind the proof.

I bring out the implications of Theorem 2 in two corollaries. Corollary 3 compares the substitutable rules to all rules, and Corollary 5 compares the IIA and substitutable rules to the preference relations over individual objects. These implications were discussed in the Introduction. The corollaries do not use the upper bound on $|\mathcal{C}_2(n)|$ from Theorem 2. I use the upper bound in my calculations (Section 4.6), but I do not know if it provides an asymptotic improvement over the upper bound given by $|\mathcal{C}_2(n)| \leqslant |\mathcal{C}_1(n)|$.

Corollary 3. $|\mathcal{C}_1(n)|/|\mathcal{C}(n)|$ *is* $o(2^{-2^n})$, *and* $\log |\mathcal{C}_1(n)|/\log |\mathcal{C}(n)|$ *is* $o(1)$.

Proof. Using the upper bound on $|C_1|$ from Theorem 2, and Stirling's formula applied to $\binom{n}{\lfloor n/2 \rfloor}$,

$$\log \frac{2^{2^n} |\mathcal{C}_1(n)|}{|\mathcal{C}(n)|} \leqslant \left(n + \kappa \log(n) \right)\binom{n}{\lfloor n/2 \rfloor} + 2^n - n2^{n-1}$$

$$\sim \left\{ \left(\sqrt{n} + \left(\kappa/\sqrt{n} \right) \log(n) \right)\sqrt{2/\pi} + 1 - n/2 \right\} 2^n.$$

The term in brackets goes to $-\infty$, so $\frac{2^{2^n} |\mathcal{C}_1(n)|}{|\mathcal{C}(n)|} \to 0$, as $n \to \infty$. Thus $|\mathcal{C}_1(n)|/|\mathcal{C}(n)|$ is $o(2^{-2^n})$.

By a similar calculation,

$$\frac{\log |\mathcal{C}_1(n)|}{\log |\mathcal{C}(n)|} \leqslant \frac{(n + \kappa \log n)\binom{n}{\lfloor n/2 \rfloor}}{n2^{n-1}}$$

$$\sim \left(1/\sqrt{n} + \frac{\kappa \log n}{n^{3/2}} \right) \frac{2^{3/2}}{\sqrt{\pi}}.$$

So $\log |\mathcal{C}_1(n)|/\log |\mathcal{C}(n)|$ is $o(1)$. $\quad \square$

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[2] In fact, any $C \in \mathcal{C}_2(n)$ is rationalizable by any linear extension of the order $A > B$ if $A = C(A \cup B)$—the order used by Blair (1988). I thank Chris Chambers for pointing this out.

A *preference relation* over X is a linear, transitive, binary relation $\succ$ over X. Let $\mathcal{P}(n)$ be the set of all preference relations over X. The following theorem is due to Gross (1962) (Barthélémy, 1980 rediscovered the theorem, and presents a different proof).[3]

Theorem 4. $|\mathcal{P}(n)| \sim \frac{n!}{2(\ln 2)^{n+1}}$.

Proof. See Gross (1962). $\square$

Corollary 5. $\log |\mathcal{P}(n)| / \log |\mathcal{C}_2(n)|$ *is* $o(n^2 2^{-n})$.

Proof. Using the lower bound on $|\mathcal{C}_2(n)|$ from Theorem 2,

$$\frac{\log |\mathcal{P}(n)|}{\log |\mathcal{C}_2(n)|} \leqslant \frac{\log |\mathcal{P}(n)|}{\binom{n-1}{\lfloor (n-1)/2 \rfloor}} = h(n).$$

Using Theorem 4,

$$h(n) \sim \frac{\log n! - (n+1)\log(\ln 2) - 1}{2^n (n\pi/2)^{-1/2}}.$$

The result now follows from $\log n! \sim n \log n$, so $(\sqrt{n} \log n!)/n^2 \to 0$. $\square$

3. Non-combinatorial choice rules

3.1. The model

The model is the same as in Section 2.1: given a set X of alternatives, a choice rule is a function $C : 2^X \to 2^X$ with $C(A) \subseteq A$, for all $A \subseteq X$. But now the elements of X are mutually exclusive choices. Still, $C(A)$ may not be a singleton because the decision maker is happy choosing any of the elements of $C(A)$. In Section 2.1, $C(A)$ may not be a singleton because the decision maker chooses, as a package, $C(A)$ over any other subset of A.

Non-combinatorial choice rules have been studied extensively in the literature on individual and social choice (see e.g. Moulin, 1991, or Austen-Smith and Banks, 1998). My model differs in one aspect from the standard model of choice: I allow that $C(A) = \emptyset$. Some researchers allow $C(A) = \emptyset$ (Kim and Richter, 1986; Aizerman and Aleskerov, 1995), but the most commonly used model rules out $C(A) = \emptyset$.

There are four advantages to allowing $C(A) = \emptyset$. First, since $\emptyset$ is always a feasible choice, it allows one to model the presence of an outside option. Second, it gives a symmetric treatment of a choice rule C, and its associated R. The advantage of a symmetric treatment will be clear from Proposition 6. Third, while it implies some changes in the results on non-combinatorial choice (see Appendix A), it does not seem to violate the spirit of the results on non-combinatorial choice, as the nonempty-valued C remain a special case. Finally, of course, that the model coincides with combinatorial choice is an advantage.

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[3] See also Bailey (1998).

3.2. Results

When applied to non-combinatorial choice rules, Statement (1) on page 234 is not called substitutability. It is called *Sen's α* (or Chernoff's condition—for Chernoff, 1954, who introduced it). The difference is not only a matter of terminology: Statement (1) is *not* what α amounts to in the combinatorial interpretation of C. In fact, α amounts to IIA.

Now, knowing that Sen's α coincides with substitutability, one obtains the obvious restatements of the results in Section 2.2 in terms of the α property in non-combinatorial choice. I do not include the restatements in the paper.

A choice rule C satisfies *Sen's β* if, for all $A, B \subseteq X$, $A \subseteq B$ and $C(A) \cap C(B) \neq \emptyset$ imply that $C(A) \subseteq C(B)$. The following proposition is trivial; R is the rejection choice rule associated with C.

Proposition 6. *If R satisfies substitutability, then C satisfies β.*

Proposition 6 implies that there are more choice rules that satisfy β than choice rules that satisfy α (recall that for each C there is a unique R). So the results in Section 2.2 give also a lower bound on the choice rules that satisfy β.

Let C be a choice rule. Say that a preference relation $\succ$ over $X \cup \{\emptyset\}$ *rationalizes* C if, for all $A \subseteq X$, $x \in C(A)$ if and only if $x \succ \emptyset$ and $y \not\succ x$, for all $y \in A$. Let $\mathcal{C}_3(n)$ be the set of rules that are rationalized by some preference relation.

Note that $\mathcal{C}_3(n)$ and $\mathcal{P}(n)$ are not isomorphic; since $\emptyset$ is an option, $\mathcal{C}_3(n)$ contains strictly more elements than $\mathcal{P}(n)$ (or $\mathcal{P}(n+1)$). In fact,

Theorem 7. $|\mathcal{C}_3(n)| = 2|\mathcal{P}(n)|$.[4]

Proof. Preference relations $\succ$ and $\succ'$ over $X \cup \{\emptyset\}$ give rise to the same C if and only if

(1) $\{x \in X: x \succ \emptyset\} = \{x \in X: x \succ' \emptyset\}$, and
(2) they coincide on all pairs $(x, y) \in X^2$ such that $x \succ \emptyset$ and $y \succ \emptyset$.

So there are as many C in $\mathcal{C}_3(n)$ as classes of preference relations that differ in either (1) or (2). These can be counted as follows: First choose $k + 1 = 1 \ldots n + 1$, the rank of $\emptyset$ in the rationalizing $\succ$. Second, choose the set of k elements of X that is ranked above $\emptyset$. Third, choose a preference relation over the elements that are ranked above $\emptyset$.

Let $w_k = |\mathcal{P}(k)|$ be the number of preference relations over $\{1, \ldots, k\}$. Then $|\mathcal{C}_3(n)| = \sum_{k=0}^{n} \binom{n}{k} w_k$.

Gross (1962) shows that $w_k = \frac{1}{2} \sum_{i=0}^{\infty} i^k 2^{-i}$. So,

$$|\mathcal{C}_3(n)| = \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} \sum_{k=0}^{n} \binom{n}{k} i^k$$

$$= \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} (1 + i)^n$$

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[4] Satoru Takahashi suggested this result. A previous draft contained a longer proof of a weaker statement.

$$= \frac{1}{2} \sum_{j=1}^{\infty} 2^{-(j-1)} j^n$$
$$= 2|\mathcal{P}(n)|. \qquad \square$$

In light of Theorem 7, the following corollary follows just like Corollary 5.

Corollary 8. $\log |\mathcal{C}_3(n)| / \log |\mathcal{C}_2(n)|$ *is* $o(n^2 2^{-n})$.

And, for α alone,

Corollary 9. $|\mathcal{C}_3(n)| / |\mathcal{C}_1(n)|$ *is* $o(2^{-2^n})$.

Proof. By the proof of Theorem 7,

$$|\mathcal{C}_3(n)| = \sum_{k=0}^{n} \binom{n}{k} |\mathcal{P}(k)| \leqslant n \binom{n}{\lfloor n/2 \rfloor} |\mathcal{P}(n)|.$$

So, using the lower bound on $\mathcal{C}_1(n)$ from Theorem 2,

$$\frac{|\mathcal{C}_1(n)|}{2^{2^n} |\mathcal{C}_3(n)|} \geqslant \frac{2^{n\binom{n-1}{\lfloor (n-1)/2 \rfloor}}}{2^{2^n} n \binom{n}{\lfloor n/2 \rfloor} |\mathcal{P}(n)|} \sim h(n) = \frac{2^{\frac{n}{\sqrt{n-1}} 2^{n-1} \sqrt{2/\pi}}}{2^{2^n} \sqrt{n} 2^n \sqrt{2/\pi} \frac{n!}{2(\ln 2)^{n+1}}}.$$

Where I have used Stirling's formula to estimate the $\binom{n}{\lfloor n/2 \rfloor}$ terms, and Theorem 4 for $|\mathcal{P}(n)|$.

In turn, and using log to simplify the expression,

$$\log h(n) = \frac{n 2^n}{\sqrt{(n-1)2\pi}} - 2^n - (1/2)\log n - n - \log n! + (n+1)\log(\ln 2) + \Omega,$$

where $\Omega = 1 - (1/2)\log(2/\pi)$.

Since $\log n!$ is $o(n^2)$, $\log h(n) \to \infty$. $\square$

4. Proof of Theorem 2

4.1. Idea of the proof, and auxiliary definitions

A collection $a \subseteq 2^X$ of subsets of X is an *antichain* if $A, B \in a$ implies $A \not\subseteq B$; antichains are also called *Sperner families*. A collection $a \subseteq 2^X$ of subsets of X is an *(order) filter* if $A \in a$ and $A \subseteq B$ implies $B \in a$. A filter is characterized by its minimal elements, and these form an antichain.

The number of antichains is equivalent to *Dedekind's problem*: finding the number of functions $f : 2^X \to \{0, 1\}$ that are monotone increasing. The equivalence between the number of antichains and Dedekind's problem is easy to show (see e.g. Engel, 1997); the argument is similar to that of the proof of Lemma 10 below. Dedekind's problem is an open problem. But there are known bounds, and asymptotic formulas.

The proof of Theorem 2 proceeds by relating the choice rules in $\mathcal{C}_1(n)$ and $\mathcal{C}_2(n)$ to collections of antichains, and thus to Dedekind's problem. The relation for $\mathcal{C}_1(n)$ is almost immediate, the complication is with $\mathcal{C}_2(n)$. The literature on Dedekind's problem then provides the bounds and asymptotic formulas reported in Theorem 2.

If C is a substitutable rule, the rejection (rule R) is monotone: if x is rejected from A, then x is rejected from B, for all $B \supseteq A$. Let T_x be the collection of sets from which x is rejected. Then T_x is a filter, so T_x is characterized by its minimal elements, and these form an antichain. For $C \in \mathcal{C}_2(n)$, IIA provides additional restrictions on T_x. Thus I characterize the substitutable C as assignments of an antichain to each element of X. The characterization allows me to use results on Dedekind's problem.

4.2. Results on filters

I now present a series of short lemmas on filters. The results give some necessary, and some sufficient, conditions for a system of filters to correspond to a choice rule that satisfies substitutability and IIA. But the results fall short of a characterization of the substitutable and IIA choice rules.

A *filter at x* is a filter T_x such that $x \in A$, for all $A \in T_x$.

Lemma 10. *We can identify each substitutable C with a collection of filters $(T_x\colon x \in X)$, such that T_x is a filter at x, for all $x \in X$. The isomorphism between C and (T_x) is: Given C, let (T_x) be*

$$T_x = \big\{A \subseteq X\colon x \in A \backslash C(A)\big\}.$$

And given (T_x) let

$$C(A) = \{x \in A\colon A \notin T_x\}.$$

Proof. Identify C with R, defined as $R(A) = A \backslash C(A)$. C is substitutable if and only if R is monotone. First, the collection (T_x) defined as above is a collection of filters. To see this, note that x is in all the elements of T_x by definition. Also, if $A \in T_x$ and $A \subseteq B$, then $x \in R(A) \subseteq R(B)$ by monotonicity of R, and $R(B) \subseteq B$, so $B \in T_x$.

Second, for each filter $(T_x\colon x \in X)$, define R by $R(A) = \{x \in X\colon A \in T_x\}$. Then $R(A) \subseteq A$ since $(A \in T_x \Rightarrow x \in A)$. And R is monotone since T_x is a filter. $\quad\square$

Now fix a collection of filters $(T_x\colon x \in X)$.

Lemma 11. (T_x) *satisfies IIA if and only if, for every $A, B \subseteq X$ with $A \subseteq B$, and every $x \in A$:*

$$\big((\forall z \in B \backslash A)(B \in T_z)\big) \wedge (B \in T_x) \quad \Rightarrow \quad (A \in T_x).$$

Proof. Let $A, B \subseteq X$ with $A \subseteq B$ and $C(B) \subseteq A$. By substitutability, $A \backslash C(A) \subseteq B \backslash C(B)$. Since $A \subseteq B$, $A \backslash C(A) \subseteq A \backslash C(B)$.

Note that, since $C(B) \subseteq B$, the statement $[(\forall z \in B \backslash A)(B \in T_z)]$ is equivalent to $C(B) \subseteq A$.

(Sufficiency) For every x, $C(B) \subseteq A$ and $B \in T_x$ imply $A \in T_x$. Thus $A \backslash C(A) \supseteq A \backslash C(B)$. So $A \backslash C(A) = A \backslash C(B)$. This shows that $C(A) = C(B)$.

(Necessity) The statement $[(\forall z \in B \backslash A)(B \in T_z)]$ is the same as $C(B) \subseteq A$. So IIA implies $A \backslash C(A) = A \backslash C(B)$. Hence, for all $x \in A$, if $B \in T_x$ then $A \in T_x$. $\quad\square$

For each $B \subseteq X$, let $Z^B = \{z \in B\colon B \in T_z\}$.

Lemma 12. (T_x) *satisfies IIA if and only if, for every* $B \subseteq X$, *for every* $z \in Z^B$,

$$(B \backslash Z^B) \cup \{z\} \in T_z.$$

Proof. (Necessity) Let $B \subseteq X$. Let $z \in Z^B$. Let $A = (B \backslash Z^B) \cup \{z\}$. Then for every $x \in B \backslash A$, $B \in T_x$. Further $B \in T_z$ so IIA, by Lemma 11, implies $A \in T_z$.

(Sufficiency) Let $A \subseteq B \subseteq X$ be such that, for all $x \in B \backslash A$, $B \in T_x$. Then $B \backslash A \subseteq Z^B$. Let $z \in A$ be such that $B \in T_z$. By hypothesis, $(B \backslash Z^B) \cup \{z\} \in T_z$. But $B \backslash A \subseteq Z^B$ implies that $(B \backslash Z^B) \cup \{z\} \subseteq A$. Then $A \in T_z$ since T_z is a filter. Hence, by Lemma 11, (T_x) satisfies IIA. □

Given $(T_x : x \in X)$, for each x, let M_x be the collection of minimal elements of T_x. So,

$$M_x = \big\{ B \in T_x : (A \subseteq B) \wedge (A \in T_x) \Rightarrow A = B \big\}.$$

Lemma 13. *If* (T_x) *satisfies IIA, then, for every* $x \in X$ *and every* $B \in M_x$, $\{x\} = Z^B$.

Proof. IIA implies (Lemma 12) that $(B \backslash Z^B) \cup \{x\} \in T_x$. Then $(B \backslash Z^B) \cup \{x\} \subseteq B$ implies that $(B \backslash Z^B) \cup \{x\} = B$. So $\{x\} = Z^B$. □

Lemma 14. *Let* $(T_x : x \in X)$ *satisfy IIA. For every* $x, y \in X$, *if* $x \neq y$ *then* $M_x \cap M_y = \emptyset$.

Proof. Immediate from Lemma 13. □

Lemma 15. *If* (T_x) *satisfies IIA, then,* $\cup \{M_x : x \in X\}$ *is an antichain.*

Proof. Let $A \in M_x$, $B \in M_y$ with $A \neq B$. If $x = y$ then $A \nsubseteq B$ since M_x is the collection of minimal sets in T_x. If $x \neq y$ then $A \subseteq B$ would imply that $B \in T_x$, as $A \in T_x$ and T_x is a filter. But then $x \in Z^B$. Impossible, as IIA implies that $Z^B = \{y\}$ by Lemma 13. □

Remark 16. By Lemmas 14 and 15, $\{M_x : x \in X\}$ is a partition of some antichain.

Lemma 17. *Let* $(T_x : x \in X)$ *satisfy IIA. For all* $x \in X$, *one of the following must be true*:

- $M_x = \emptyset$,
- $M_x = \{\{x\}\}$,
- $\exists A \in M_x$ *such that* $(\forall y \in A \backslash \{x\})(M_y = \emptyset)$.

Proof. Let $M_x \neq \emptyset$, and $M_x \neq \{\{x\}\}$, so $\{x\} \notin M_x$. We need to prove that $\exists A \in M_x$ such that $\forall y \in A \backslash \{x\}$, $M_y = \emptyset$.

Suppose, by way of contradiction, that for every $A \in M_x$, there is $y \in A$ with $y \neq x$ such that $T_y \neq \emptyset$.

Pick $A \in M_x$. We shall construct a sequence by induction. Let $y_1 \in A$ and $D_1 \in M_{y_1}$. Let $B_1 = A \cup D_1$, so $x, y_1 \in Z^{B_1}$. By IIA and Lemma 12 there is $A' \in M_x$ with $A' \subseteq (B_1 \backslash Z^{B_1}) \cup \{x\}$. Note $y_1 \notin A'$. By our assumption, there is $y_2 \in A'$ with $M_{y_2} \neq \emptyset$. Let $D_2 \in M_{y_2}$. Note $y_1 \neq y_2$.

Given a sequence of different elements $y_1, y_2, \ldots, y_{k-1}$, $k \geqslant 2$, and corresponding sets $D_1, D_2, \ldots, D_{k-1}$ with $D_l \in M_{y_l}$, with $1 \leqslant l \leqslant k - 1$. Let

$$B_k = A \cup \left(\bigcup_{l=1}^{k-1} D_l \right).$$

So $x \in Z^{B_k}$ and $y_l \in Z^{B_l}$, $1 \leqslant l \leqslant k - 1$. By IIA and Lemma 12 there is $A' \in M_x$ with $A' \subseteq (B_k \backslash Z^{B_k}) \cup \{x\}$. By hypothesis, there is $y_k \in A'$ with $T_{y_k} \neq \emptyset$, so there is a corresponding $D_k \in M_{y_k}$. Since $y_k \in A'$, $y_k \neq y_l$, $1 \leqslant l \leqslant k - 1$.

But X is a finite set, so this has to stop. There must be a $A \in M_x$ with $T_y = \emptyset$ for all $y \in A$. $\square$

Lemma 18. *Let* $(T_x \colon x \in X)$ *satisfy IIA. Then, for all* $x \in X$,

$$\cap\{A \colon A \in M_x\} \subseteq \{x\} \cup \{y \in X \colon T_y = \emptyset\}.$$

Proof. Let $A \in M_x$ and $y \in A$ with $y \neq x$. Let $T_y \neq \emptyset$; I shall prove that $y \notin \cap\{A \colon A \in M_x\}$. Since $T_y \neq \emptyset$, there is $D \in M_y$. Note that $D \not\subseteq A$ because $D \subseteq A$ would imply that $y \in Z^A$, in contradiction with Lemma 13. Then $x, y \in Z^{A \cup D}$. By IIA and Lemma 12,

$$\big((A \cup D)\backslash Z^{A \cup D}\big) \cup \{x\} \in T_x.$$

So there is $A' \in M_x$ with

$$A' \subseteq \big((A \cup D)\backslash Z^{A \cup D}\big) \cup \{x\}.$$

Since $y \in Z^{A \cup D}$, there is $A' \in M_x$ with $y \notin A'$. Hence $y \notin \cap\{A \colon A \in M_x\}$. $\square$

Remark 19. Lemmas 10–18 imply that each $C \in \mathcal{C}_2(n)$ can be identified with an assignment of a collection of subsets $M_x \subseteq 2^X$ of X to each x, and a partition (Y, Z, W) of X such that:

(1) $(\forall x \in X)(\forall A \in M_x)(x \in A)$
(2) $(\forall y \in Y)(M_y = \emptyset)$;
(3) $(\forall z \in Z)(M_z = \{\{z\}\})$;
(4) $\cup\{M_x \colon x \in W\}$ is an antichain of subsets of $Y \cup W$;
(5) $(\forall x \in W)(\{x\} \notin M_x)$; so $|A| \geqslant 2$, for all $A \in M_x$.
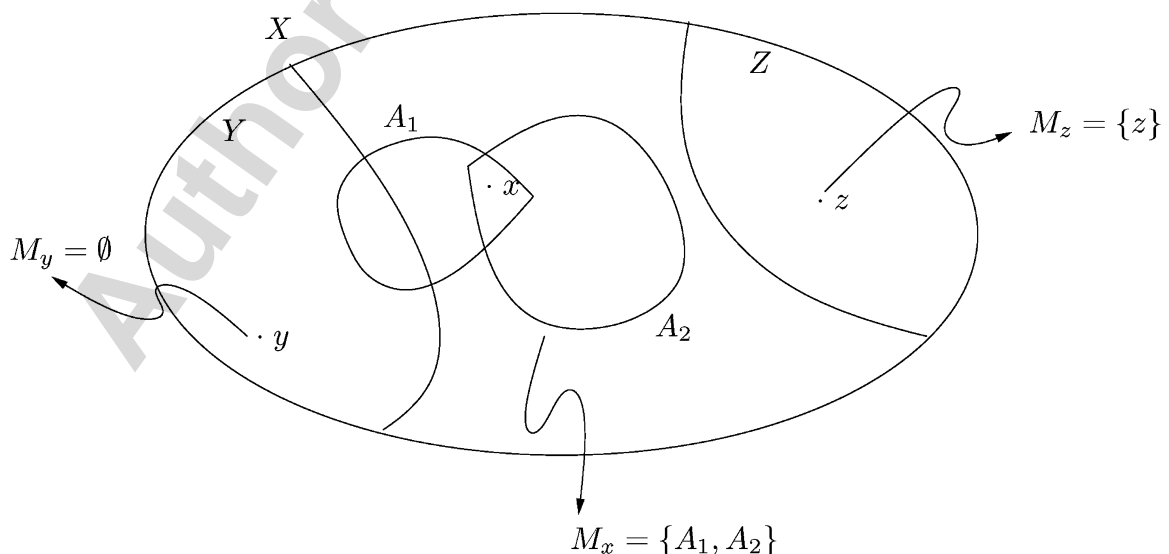
Figure 1 is a sketch of one such collection.

Fig. 1. A collection $(M_x \colon x \in X)$.

Now fix a partition (Y, W, Z) such that Y and W are nonempty. For each $x \in W$, let a_x be some antichain over Y. Construct assignment $(M_x\colon x \in X)$ by letting $M_x = \{A \cup \{x\}\colon A \in a_x\}$, for all x in W. Let $M_y = \emptyset$ for all $y \in Y$ and let $M_z = \{\{z\}\}$ for all $z \in Z$.

Clearly, $(M_x\colon x \in X)$ defines a system of filters $(T_x\colon x \in X)$, by $B \in T_x$ if and only if there is $A \in M_x$ such that $A \subseteq B$. It turns out that

Lemma 20. $(T_x\colon x \in X)$ *satisfies IIA.*

Proof. Let $B \subseteq X$. If $x \in Z^B$ then there is some $A \in a_x$ with $A \cup \{x\} \subseteq B$. Since $A \subseteq Y$, $T_y = \emptyset$ for all $y \in A$. Hence $A \subseteq B \backslash Z^B$. Then $(B \backslash Z^B) \cup \{x\} \in T_x$.

So (T_x) satisfies IIA by Lemma 12. $\square$

4.3. Upper and lower bounds

Let $\mathcal{A}(k)$ be the collection of antichains of the set $\{1, 2, \ldots, k\}$. Let $\mathcal{B}(k, m)$ be the collection of antichains $a \in \mathcal{A}(k + m)$ such that $(\forall A \in a)(|A| \geqslant 2)$ and $\cup a \supseteq \{1, \ldots, k\}$.

Let $\eta(k, m)$ be the multinomial coefficient,

$$\eta(k, m) = \left[\frac{n!}{m! k! (n - m - k)!} \right].$$

Lemma 21. $|\mathcal{A}(\lfloor n/2 \rfloor)|^{n - \lfloor n/2 \rfloor} \leqslant |\mathcal{C}_2(n)|$, *and*

$$\left| \mathcal{A}(n - 1) \right| \leqslant \left| \mathcal{C}_2(n) \right| \leqslant \sum_{m=1}^{n} \sum_{k=1}^{n-m} \eta(k, m) \theta(k, m),$$

where

$$\theta(k, m) = \sum_{a \in \mathcal{B}(m, k)} \left\{ \sum_{\left\{ \substack{i_1, \ldots, i_k: \\ i_1 + \cdots + i_k = |a|} \right\}} \frac{|a|!}{i_1! \cdots i_k!} \right\}.$$

Proof. First, I prove the lower bounds on $|\mathcal{C}_2(n)|$ in Lemma 21. The bounds follow from constructions like those in Lemma 20.

Fix the partition $(Y, W, Z) = (\{1, \ldots, \lfloor n/2 \rfloor\}, \{\lfloor n/2 \rfloor + 1, \ldots, n\}, \emptyset)$. For each $x \in W$, let $a_x \in A(\lfloor n/2 \rfloor)$ be some antichain. The assignment $M_x = \{A \cup \{x\}\colon A \in a_x\}$ defines

$$C(A) = \left\{ x \in X\colon (\nexists A' \in M_x)(A' \subseteq A) \right\}.$$

Lemma 20 implies that C is substitutable and IIA.

So there are at least as many IIA and substitutable C as maps $A(\lfloor n/2 \rfloor)^{\{\lfloor n/2 \rfloor + 1, \ldots, n\}}$. This gives the first lower bound in Lemma 21. Letting $(Y, W, Z) = (\{1, \ldots, n - 1\}, \{n\}, \emptyset)$, and reasoning as before, one proves the second lower.

Second, I shall prove the upper bound on $\mathcal{C}_2(n)$ in Lemma 21. By Remark 19, there are at most as many C in $\mathcal{C}_2(n)$ as partitions (Y, W, Z) of X, and assignments of antichains to the elements of W that satisfy the five statements in Remark 19.

Fix integers m and k, smaller than n. There are $\binom{n}{m}$ possible choices for Y of cardinality m. For each such choice, there are $\binom{n-m}{k}$ possible choices for W of cardinality k. The product $\binom{n}{m}\binom{n-m}{k}$ is $\eta(k, m)$. Given Y and W, there are at most $\theta(k, m)$ ways of assigning to the elements

of W an antichain over $Y \cup W$ that satisfies (4) and (5) in Remark 19. This gives the upper bound on $\mathcal{C}_2(n)$. $\quad \square$

Lemma 22. $|\mathcal{C}_1(n)| = |\mathcal{A}(n-1)|^n$

Proof. By Lemma 10, $|\mathcal{C}_1(n)|$ equals the number of systems of filters $(T_x: x \in X)$. For each x, there are $|\mathcal{A}(n-1)|$ ways of assigning an antichain M_x to x; each antichain M_x is identified with a filter. $\quad \square$

Let $S(K, k)$ denote the number of partitions of $\{1, \ldots, K\}$ into k sets. $S(K, k)$ is called the Stirling number of the second kind (see e.g. Comtet, 1974).

Lemma 23.

$$\sum_{K=0}^{n} \binom{n}{K} \sum_{k=0}^{k} \binom{K}{k} |\mathcal{A}(K)| k! S\left(\binom{K}{\lfloor K/2 \rfloor}, k\right)$$

is an upper bound on $|\mathcal{C}_2(n)|$.

Proof. Choose a number $K \in \{0, \ldots, n\}$, and choose a subset $(W \cup Y)$ of X of cardinality K. Choose a number $k \in \{0, \ldots, K\}$. Choose a subset W of $(W \cup Y)$ of cardinality k.

Choose an antichain $a \in \mathcal{A}(K)$ over $W \cup Y$. An assignment of a to the elements of W gives a partition of a. The number of such partitions is $S(|a|, |W|)$, the Stirling Number of the Second Kind. By Sperner's Theorem (Engel, 1997), $|a| \leqslant \binom{K}{\lfloor K/2 \rfloor}$. Hence, $S(|a|, |W|) \leqslant S(\binom{K}{\lfloor K/2 \rfloor}, k)$.

Now, each partition can be assigned in $k!$ many ways to the elements of W. And there are $|\mathcal{A}(K)|$ antichains over $W \cup Y$. So there are at most $|\mathcal{A}(K)| k! S(\binom{K}{\lfloor K/2 \rfloor}, k)$ many assignments of antichain over $W \cup Y$ to the elements of W. The stated upper bound follows. $\quad \square$

4.4. Proof of Theorem 2

First, that $2^{\binom{n}{\lfloor n/2 \rfloor}} \leqslant |\mathcal{A}(n)|$ is immediate, since any subset of the collection of subsets of $\{1, \ldots, n\}$ with $\lfloor n/2 \rfloor$ elements is an antichain (Engel, 1997). Then, $2^{\binom{n}{\lfloor n/2 \rfloor}} \leqslant |\mathcal{A}(n)|$ together with the second lower bound on $|\mathcal{C}_2(n)|$ in Lemma 21, provide the lower bound on $|\mathcal{C}_2(n)|$ in Theorem 2.

Second, Kleitman and Markowsky (1975) proved that

$$|\mathcal{A}(n)| \leqslant 2^{(1+\kappa \log(n)/n)\binom{n}{\lfloor n/2 \rfloor}},$$

for some constant κ. Lemma 22 then implies the upper bound on $|\mathcal{C}_1(n)|$, and Lemma 23 the upper bound on $|\mathcal{C}_2(n)|$, in Theorem 2.

Third, Kleitman (1969) proved that

$$\log|\mathcal{A}(n)| \sim \binom{n}{\lfloor n/2 \rfloor},$$

which, by Lemma 22, implies $\log |\mathcal{C}_1(n)| \sim n \binom{n-1}{\lfloor (n-1)/2 \rfloor}$.

4.5. Note

One can use other bounds from the literature on Dedekind's problem, and obtain variations on the results that I present—see Engel (1997) for references to advances on Dedekind's problem.

The best asymptotic estimate for $|\mathcal{A}(n)|$ is due to Korshunov (1981). Clearly, Korshunov's estimate applies to estimating $|\mathcal{C}_1(n)|$ (by Lemma 22).

4.6. Explanation of the calculations in the introduction

The calculation of $|\mathcal{C}_1(n)|$ is from Lemma 22 and the result that $|\mathcal{A}(7)| = 2414682040998$ from Comtet (1974). The calculation for $|\mathcal{C}_2(n)|$ follows from the bounds in Theorem 2, but using the known values of $|\mathcal{A}(k)|$. The values $|\mathcal{A}(k)|$, for $k = 1, \ldots, 7$, are from Comtet, while $|\mathcal{A}(8)|$ is from Wiedemann (1991). Gross (1962) calculates that there are 7087261 preference relations over a set with 9 elements (8 elements and $\emptyset$).

Acknowledgments

Appendix A. Rationalizability

This appendix treats non-combinatorial rules. Substitutability and β do not imply that C is rationalizable. In fact, substitutability and β do not imply that C satisfies IIA. For example: Let $X = \{1, 2\}$. Let $C(X) = \emptyset$ and $C(\{i\}) = \{i\}$, for $i = 1, 2$. Then C violates IIA, since $C(X) \subseteq \{1\}$ but $C(\{1\}) \neq C(X)$. It is easy to verify that C satisfies substitutability and β.

Theorem 24 clears this issue. And it shows another use for the lemmas developed in Section 4.2. Kim and Richter (1986) present a different characterization of rationalizable non-combinatorial choice rules.

Theorem 24. *C satisfies substitutability, β and IIA if and only if C is rationalizable.*

Proof. It is immediate that a rationalizable C satisfies substitutability, β, and IIA.

Let C satisfy substitutability, β, and IIA. Let $(M_x : x \in X)$ be the system of antichains associated to C. Let $\succ$ be the binary relation defined on $X \cup \{\emptyset\}$ by:

- $y \succ x$ if $x \neq y$ and $\{x, y\} \in M_x$.
- $\emptyset \succ x$ if $\{x\} \in M_x$; $x \succ \emptyset$ otherwise.

I shall prove that $\succ$ is transitive. Then I show that $\succ$ rationalizes C.
First, I need to prove the following statement:

$$(\forall A \subseteq X)(\forall B \subseteq X)\big[(A \subseteq B) \Rightarrow \big(C(A) \subseteq C(B)\big) \vee \big(A \subseteq R(B)\big)\big].$$

Let $A \subseteq B$, and $A \nsubseteq R(B)$. By substitutability, $R(A) \subseteq R(B)$; so $A \nsubseteq R(B)$ implies $C(A) \cap C(B) \neq \emptyset$. Now β implies $C(A) \subseteq C(B)$.

Second, I shall prove that, $\forall x \in X$, $\forall A \in M_x$, $|A| \leqslant 2$. Suppose not. Let $x \in X$ and $A \in M_x$ with $|A| > 2$. Then there is a set A' with $\{x\} \subsetneq A' \subsetneq A$. Since A is a minimal set from which x is rejected, $x \in C(A')$. Now, IIA and $x \notin C(A)$, imply $C(A) \nsubseteq C(A')$. Then, using the statement we proved first, $A' \subseteq R(A)$. Let $y \in A' \backslash \{x\}$; so $A \in T_y$. But then $y \in Z^A$ (using the notation introduced before Lemma 12). On the other hand, by Lemma 13, $\{x\} = Z^A$, for all $A \in M_x$. A contradiction.

Third, I prove that $y \succ x$ and $z \succ y$ implies $z \succ x$. Let $A = \{x, y, z\}$. $y \succ x$ is $\{x, y\} \in M_x$, so $A \in T_x$; similarly, $A \in T_y$. So $\{x, y\} \subseteq Z^A$.

Now, $A \backslash Z^A = \emptyset$ would imply that $\{x\} \in T_x$, as Lemma 12 says that $\{x\} \cup A \backslash Z^A \in T_x$. But that would contradict that $\{x, y\} \in M_x$.

We must have $A \backslash Z^A \neq \emptyset$; so $A \backslash Z^A = \{z\}$. By Lemma 12, $\{x, z\} = \{x\} \cup A \backslash Z^A \in T_x$. We know $\{x\} \notin M_x$, as $\{x, y\} \in M_x$. So $\{x, z\}$ is minimal in T_x. Hence $\{x, z\} \in M_x$. Hence $z \succ x$, and $\succ$ is transitive.

Finally I prove that $\succ$ rationalizes C. Fix $A \subseteq X$. Let $x \in C(A)$. Then $\{x\} \notin M_x$, as $A \notin T_x$. So $x \succ \emptyset$. Let $y \in A$. Then again $A \notin T_x$ implies $\{y, x\} \notin M_x$. So $y \nsucc x$. $\square$

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