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A WEAK CORRESPONDENCE PRINCIPLE FOR MODELS WITH COMPLEMENTARITIES

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Abstract. I prove that, in models with complementarities, some non-monotone comparative statics must select unstable equilibria; and, under additional regularity conditions, that monotone comparative statics selects stable equilibria.

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1

1. Introduction

Comparative statics is the source of a model's testable implications, it says how the model's endogenous variables depend on its exogenous parameters. In this paper I derive general comparative statics results for a class of models where there are complementarities between the model's different endogenous variables, and between endogenous variables and exogenous variables. Many economic models fall in this class, mainly because many economic models are games of strategic complementarities (Topkis 1979, Vives 1990). See Topkis (1998) and Vives (1999) for economic examples.

To fix ideas, consider a collection of functions, indexed by $T \subset \mathbf{R}^n$, $f_t : [0,1] \to [0,1]$, $t \in T$. Here f_t is the reduced form of a model, for example the best-response function of a game, and the fixed points of f_t are the equilibria of the model. I shall impose that endogenous variables are complementary, which here means that f_t is monotone increasing for each t, and that there are complementarities between the endogenous and exogenous variables, which here means that $t \mapsto f_t(x)$ is monotone increasing for each x. The problem is how to obtain a *monotone comparative statics conclusion*: when $t < t'$ then the model's equilibria increase.

INSERT FIGURE 1

Figure 1

Consider the situation in Figure 1, where $t, t' \in T$ and $t < t'$. Assume that e_i is selected for t if and only if e'_i is selected for t', a kind of "local comparative statics" assumption. There are then three possible comparative statics conclusions; two are monotone ($e_1 \to e'_1$ and $e_3 \to e'_3$), and one is non-monotone ($e_2 \to e'_2$).

The non-monotone comparative statics $e_2 \to e'_2$ involves fixed points that are unstable for dynamics $x_n = f_t(x_{n-1})$. In this paper I prove that—in models with complementarities—non-monotone comparative statics must be selecting unstable fixed points.

Further, all monotone comparative statics involve stable fixed points, as e_1, e'_1, e_3 and e'_3 are stable. In this paper I prove that this converse result is also true in some generality. The converse requires complementarities, but also some additional regularity conditions.

In the rest of the Introduction, I shall place my results in the context of existing results on comparative statics.

Samuelson (1947) presents comparative-statics methods that rely on calculus, and smooth economic models. Samuelson's "correspondence principle" (CP) says that, if one rules out unstable equilibria, one obtains a local comparative statics conclusion—stable here means locally stable for some out-of-equilibrium dynamics. In Figure 1, there are two possible local comparative statics conclusions, but Samuelson's CP pins down exactly one.

A more recent approach is the comparative statics methods developed for models with complementarities (see e.g. Milgrom and Roberts (1990), Milgrom and Roberts (1994), Milgrom and Shannon (1994)). This literature relies on lattice programming, and Topkis's (1978) results on comparative statics for decision problems. In models with complementarities there is a largest and a smallest equilibrium; and if the exogenous parameters are complementary to endogenous variables, then the largest and smallest equilibria increase as the exogenous parameter increases. Figure 1 illustrates this result: the smallest and largest equilibria increase as we shift from t to t'.

The problem is that no equilibrium selection theory suggests that the smallest or largest equilibria are good predictions. A possible justification for focusing on these equilibria is that they have, in some models, certain optimality properties. For example, in a network-externalities model, or a coordination-failures model, the largest equilibrium is a Pareto-optimal outcome. But the very reason we are interested in these models is that Pareto inefficiencies may occur—e.g. in a network-externalities model, we think that a new, superior, technology may not be adopted by all players.

Echenique (2002), in turn, proves a strong version of the CP for models with complementarities. Echenique's result is: if $e(t)$ is an equilibrium for each t (so $t \mapsto e(t)$ is a selector of equilibria), $t \mapsto e(t)$ is continuous and *not* monotone increasing, then, for all interior t, $e(t)$ is unstable for a broad class of out-of-equilibrium dynamics. Echenique (2002) proves that, under some further restrictions, a converse result is true: if $t \mapsto e(t)$ is monotone increasing, then it must select stable equilibria.

The CP in Echenique (2002) does not explain the situation in Figure 1 because $f_{\hat{t}}$ for $t < \hat{t} < t'$ can be such that there is no way of connecting e_2 and e'_2 by a continuous selector. But both e_2 and e'_2 are unstable for the dynamics $x_n = f_t(x_{n-1})$. The "wrong" comparative statics of going from e_2 to e'_2 selects unstable equilibria, but because the comparison does not involve a continuous selector of equilibria, Echenique's version of the CP does not work.

The requirement of a continuous selector is an important drawback in Echenique (2002) because it is difficult to guarantee that a continuous selector connects two fixed points. For example, consider Figure 1: how can one guarantee that a continuous selector connects e_2 and e_2'?

In the present paper I prove a weak version of the CP in Echenique (2002). The weak CP does not rely on continuous selections of equilibria, but it cannot guarantee that all non-monotone comparative statics selects unstable equilibria. Compared to Echenique (2002), I impose more structure on X and $x \mapsto f_t(x)$, and very little on T and $t \mapsto f_t(x)$ (Echenique requires $T \subset \mathbf{R}^n$ convex, here I only need T to be a partially ordered set). The results in Echenique (2002) apply to multidimensional endogenous variables, while this paper's strongest results are for one-dimensional endogenous variables. This is an important drawback, but **1.** there are models of interest with one-dimensional variables, like two-player games and two-good general equilibrium models; [1] **2.** my results shed light on the role of multi- vs. one-dimensional variables in the CP, they "explain" why one dimension helps in this context—famously, Samuelson's (1947) CP was discarded because it fails to hold with more than one dimension (Arrow and Hahn 1971); **3.** the CP presented here is more general than Samuelson's in models with complementarities, and applies to a broad class of out-of-equilibrium dynamics.

2. The Correspondence Principle

2.1. **Definitions.** Let L be a Banach lattice; let L^+ denote L's positive cone. For $x, y \in L$, say that $x \leq y$ if $y - x \in L^+$, that $x < y$ if $x \leq y$ and $x \neq y$, and that $x \ll y$ if $y - x$ is an interior point of L^+. Denote order intervals by $[x, y] = \{z \in L : x \leq z \leq y\}$. Letting X be a subset of L, a function $f : X \to X$ is ***monotone increasing*** if $x < y$ implies that $f(x) \leq f(y)$; it is ***strictly monotone increasing*** if $x \ll y$ implies that $f(x) \ll f(y)$.

Definition 1. Let T be a partially ordered set. A family of functions $(f_t : X \to X \mid t \in T)$ is an ***increasing family of continuous functions*** if each f_t is a continuous, monotone increasing function, and if, for each $x \in X$, $t \mapsto f_t(x)$ is monotone increasing.

In a family of functions, $(f_t : X \to X \mid t \in T)$, the set of fixed point of f_t is denoted by $\mathcal{E}(t) = \{x \in X \mid x = f_t(x)\}$. A fixed point $e(t)$ of f_t is ***isolated*** if there is a neighborhood V of $e(t)$ in X such that $V \cap \mathcal{E}(t) = \{e(t)\}$.

Definition 2. Let $f : X \to X$ be a function. A sequence $\{x_k\}$ in X is ***generalized adaptive dynamic from f starting at $x \in X$*** if

[1] General equilibrium with gross substitutes is a model with complementarities, in the sense of this paper.

$x_0 = x$ and there is some $\gamma \in \mathbf{N}$ such that

$$x_k \in [f(\inf\{x_{k-\gamma}, x_{k-\gamma+1}, \dots x_{k-1}\}), f(\sup\{x_{k-\gamma}, x_{k-\gamma+1}, \dots x_{k-1}\})]$$

for all $k \geq 1$; where by x_{-l} I mean x_0, for $l = 1, \dots \gamma - 1$.

To simplify notation, when a sequence $\{x_k\}$ is understood, I shall denote $\{x_{k-\gamma}, x_{k-\gamma+1}, \dots x_{k-1}\}$ by H_k^γ—the history of length γ at k. Let $\mathcal{D}(x_0, f)$ be the set of all sequences that are generalized adaptive dynamic from f starting at x_0.

As an illustration of generalized adaptive dynamic, consider a Bertrand pricing game with three firms, and let $\gamma = 2$. If firms 1 and 2 have set prices (1,5) and (5,1) in the last two periods then, in a generalized adaptive dynamic, firm 3 is "allowed" to set any price between its best response to (1,1) and its best response to (5,5). So, it can set any price between the best responses to the two extremal backward-looking conjectures about opponents' play. For a discussion of this class of learning processes, see Echenique (2000).

Definition 3. Let $f : X \to X$ be a function. A point $\hat{x} \in X$ is ***unstable for*** f if there is a neighborhood V of $\hat{x}$ such that, for all x in V, and all sequences $\{x_k\} \in \mathcal{D}(x, \phi)$, $\hat{x}$ is not an accumulation point of $\{x_k\}$. A point $\hat{x} \in X$ is ***stable for*** f if there is a neighborhood V of $\hat{x}$ in X such that, for all x in V and all sequences $\{x_k\} \in \mathcal{D}(x, \phi)$, $\hat{x} = \lim_k x_k$.

Note that there is an asymmetry in these definitions—a point is not unstable simply by failing to be stable. The asymmetry is because trajectories in $\mathcal{D}(x_0, f)$ are not uniquely defined by f.

2.2. Non-monotone comparative statics select unstable equilibria.

In the following, L is a Banach lattice such that the interior of L^+ is non-empty, $X \subset L$ is compact, and T is a partially ordered set.

Theorem 4. *Let $(f_t : X \to X \mid t \in T)$ be an increasing family of continuous functions, and fix $t, t' \in T$ with $t < t'$. Let $e \in \mathcal{E}(t)$, $\underline{e}, \overline{e} \in \mathcal{E}(t')$ with $\underline{e} < \overline{e}$ and $e \ll \overline{e}$. If $\nexists z \in \mathcal{E}(t')$ such that either $\underline{e} < z < \overline{e}$ or $e < z < \overline{e}$, then $\underline{e}$ is unstable for $f_{t'}$.*

INSERT FIGURE 2

Figure 2

Why is Theorem 4 a comparative statics result? Consider Figure 2, an example where $L = \mathbf{R}^2$. In part a), Theorem 4 implies that $\underline{e}$ is unstable. Thus, if one decides to refine away unstable equilibria, one can rule out the non-monotone comparative statics conclusion that selects e when the parameter is t, and $\underline{e}$ when it is t'. In part b), Theorem 4 implies that e_1, e_2, and e_3 are all unstable, but it is silent about $\underline{e}$. So, non-monotone comparative statics that goes from e to e_i $(i = 1, 2, 3)$ is flawed, but we cannot say that *all* non-monotone comparative statics are wrong. In Section 2.3, I give a reason for selecting $\overline{e}$: I give conditions under which $\overline{e}$ is stable.

Figure 2 suggests that the most interesting application of the theorem is when L is a chain, because when L is a chain we are always in a situation like the one on part a). I give the application to $L = \mathbf{R}$ in Corollary 5. Corollary 5 explains exactly the situation in Figure 1—where stability yields a comparative statics conclusion without the need to use calculus, or continuous selectors of equilibria.

Corollary 5. *Let $X = [a, b] \subset \mathbf{R}$, $(f_t : X \to X \mid t \in T)$ be an increasing family of continuous functions, and $t, t' \in T$ with $t < t'$. If $e \in \mathcal{E}(t)$, $e \notin \mathcal{E}(t')$ and $\underline{e}$ is the largest $z \in \mathcal{E}(t')$ with $z < e$, then $\underline{e}$ is unstable for $f_{t'}$.*

The proof of Theorem 4 exploits Dancer and Hess's (1991) use of topological index theorems in dynamical systems with the lattice structure that I have in this paper. In contrast, Echenique's (2002) CP can be interpreted as delivering a purely lattice-theoretical index theorem. [2]

Proof of Theorem 4. By Dancer and Hess's (1991) Proposition 1, either there exist z in $[\underline{e}, \overline{e}]$ arbitrarily close to $\overline{e}$ such that $f_{t'}(z) < z$ or there exist z in $[\underline{e}, \overline{e}]$ arbitrarily close to $\underline{e}$ such that $z < f_{t'}(z)$. I shall first rule out the first possibility, then I shall argue that Dancer and Hess's result implies that $\underline{e}$ is unstable.

Let $\{x_k\}$ be the sequence defined by $x_0 = e$, $x_k = f_{t'}(x_{k-1})$, $k \geq 1$. By Theorem 3 in Echenique (2002), $\lim_k x_k = \inf \{z \in \mathcal{E}(t') : e \leq z\} \in \mathcal{E}(t')$. Now, $\overline{e} \in \{z \in \mathcal{E}(t') : e \leq z\}$ so $\lim_k x_k \leq \overline{e}$, and there exist no $z \in \mathcal{E}(t')$ with $e \leq z < \overline{e}$ because $\mathcal{E}(t') \cap [\underline{e}, \overline{e}] = \{\underline{e}, \overline{e}\}$. So, $\lim x_k = \overline{e}$.

Let W be a neighborhood of $\overline{e}$ with $e < z$ for all $z \in W$, such a neighborhood exists because $\overline{e} - e$ is an interior point of L^+. Let $z \in W \cap [\underline{e}, \overline{e}]$, and let $\{z_k\}$ be the sequence defined by $z_0 = z$ and $z_k = f_{t'}(z_{k-1})$, $k \geq 1$. By induction, monotonicity of $f_{t'}$ ensures that $x_k \leq z_k \leq \overline{e}$ for all k. Then

$$\overline{e} = \lim_k x_k \leq \liminf_k z_k \leq \limsup_k z_k \leq \lim_k x_k = \overline{e}.$$

[2] I thank Yakar Kannai for this interpretation.

Now, for any $z \in W$, $f_{t'} < z$ would imply that z is an upper bound on the range of the sequence $\{z_k\}$, constructed as before with $z_0 = z$. This is not possible, as $\underline{e} = \lim_k z_k$. I have then ruled out the first possibility in Dancer and Hess's result.

Let V be a neighborhood of $\underline{e}$. By Dancer and Hess's Proposition 1, there is $z \in V \cap [\underline{e}, \overline{e}]$ with $\underline{e} < z < f_{t'}(z)$. Let $\{z_k\} \in \mathcal{D}(z, f_{t'})$. I shall show by induction that z is a lower bound on $\{z_k\}$. Let γ be the bound on history implicit in $\{z_k\}$. First, $z \leq w$ for all $w \in H_1^\gamma$, and $z < f_{t'}(z) \leq z_1$ so $z \leq \inf H_2^\gamma$. Second, if $z \leq w$ for all $w \in H_{k-1}^\gamma$ then $z \leq \inf H_{k-1}^\gamma$ so $z < f_{t'}(z) \leq z_k$. Then, $z \leq \inf H_k^\gamma$. Hence, $\underline{e} < z \leq \liminf_k z_k$, so $\underline{e}$ is not an accumulation point of $\{z_k\}$, so $\underline{e}$ is unstable. □

2.3. **Monotone comparative statics select stable equilibria.** I show a partial converse to the CP: monotone comparative statics implies stable equilibria. Stability from the comparative statics properties of the selection of equilibria alone is a strong result; it comes at the cost of imposing stronger monotonicity assumptions, and a regularity condition on the selected equilibria.

Definition 6. Let $f : X \to X$ be a function. Say that a fixed point x of f is ***regular*** if there is a neighborhood U of x such that

(1) U contains no other fixed points of f, and
(2) the existence of some $y \in U$ with $y \ll f(y) \ll x$ implies that there is also $z \in U$ with $x \ll f(z) \ll z$.

INSERT FIGURE 3

Figure 3

I call the requirement "regularity" because, when $X \subset \mathbf{R}$ it coincides with the usual geometric notion of regularity—that f must cross the diagonal at the fixed point x, not only intersect it. So, if $X \subset \mathbf{R}$ and f is C^1, that the Jacobian of f has full rank at x is equivalent to x being regular. This is illustrated in Figure 3, part b). The first fixed point x_1 of f is not regular because there are smaller points y arbitrarily close—so that $y > f(y)$ while the reverse inequality is not true for points close to x_1 that are larger. On the other hand, it is easy to see that x_2 is regular both in the usual sense and in accordance with the definition above.

In general, though, regularity is stronger than the usual geometric notion of regularity. For instance, consider Figure 3 a). The map $g : x \mapsto x - f(x)$ in $\mathbf{R}^2$ is a counterclockwise smooth rotation of the unit disk that leaves the negative orthant (and 0) fixed but condenses the positive orthant in the $(0,-\infty)\times(0,\infty)$-cone and stretches the $(0,\infty)\times(0,-\infty)$-cone into the positive orthant. I have labeled the intersection of these cones with the disk A, B, C and D so the picture shows the action of the map. Clearly, this transformation can be made as smooth as desired, and 0 is a 0 of g, i.e. a fixed point of f. Then, g is a local diffeomorphism of 0, but 0 is not a regular point in the sense used here, because while all elements $y \in A$ satisfy $f(y) \ll y$, for all $z \in B$, $f(z) \not< z$.

Theorem 7. *Let $(f_t : X \to X \mid t \in T)$ be an increasing family of continuous functions, and fix $t, t' \in T$ with $t < t'$. Let $e \in \mathcal{E}(t)$, $\overline{e} \in \mathcal{E}(t')$, $e \ll \overline{e}$, and let $f_{t'}$ be strictly increasing on $[e, \overline{e}]$. Suppose $\nexists z \in \mathcal{E}(t')$ such that $e < z < \overline{e}$. If $e \ll f_{t'}(e)$, and $\overline{e}$ is regular, then $\overline{e}$ is stable for $f_{t'}$.*

Proof. Let $\{x_k\} \in \mathcal{D}(e, f_{t'})$ be defined by $x_0 = e$, $x_k = f_{t'}(x_{k-1})$, $k \geq 1$. It is easy to see by induction that $\{x_k\}$ is a strictly increasing sequence, and that $x_k \ll \overline{e}$ for all k. First, $x_0 = e \ll \underline{e}$ implies that $x_0 = e \ll f_{t'}(e) = x_1 \ll f_{t'}(\overline{e}) = \overline{e}$, as $f_{t'}$ is strictly increasing on $[e, \overline{e}]$ and $e \ll f_{t'}(e)$. Second, if $x_{k-2} \ll x_{k-1} \ll \overline{e}$, then $f_{t'}(x_{k-2}) = x_{k-1} \ll f_{t'}(x_{k-1}) = x_k \ll f_{t'}(\overline{e}) = \overline{e}$. Let U be the neighborhood of $\overline{e}$ in the definition of regularity. By Theorem 3 in Echenique (2002), $x_k \to \overline{e}$, so there is $K \in \mathbf{N}$ such that $x_K \in U$. Since $x_K \ll \overline{e}$, and $\overline{e}$ is regular there is $y \in U$ with $\overline{e} \ll f_{t'}(y) \ll y$. Let $\{y_k\} \in \mathcal{D}(y, f_{t'})$, be defined by $y_0 = y$, $y_k = f_{t'}(y_{k-1})$, $k \geq 1$. By Theorem 3 in Echenique (2002), $y_k \to \overline{e}$.

I shall denote the interior of L^+ by L^{+o}. Since $x_K \ll \overline{e} \ll y$ there are open sets O_1 and O_2 with $y - \overline{e} \in O_1 \subset L^{+o}$ and $\overline{e} - x_K \in O_2 \subset L^{+o}$. Let $O = (y - O_1) \cap (x_K + O_2)$, so that O is an open neighborhood of $\overline{e}$. Let $z \in O$, then $y - z \in L^{+o}$ and $z - x_K \in L^{+o}$, so we have $x_K \ll z \ll y$.

Finally, let $\{z_k\} \in \mathcal{D}(z, f_{t'})$. It is easy to see by induction that $x_{K+k} \leq z_k \leq y_k$ for all $k \in \mathbf{N}$. (First, $x_K \leq z_0 \leq y$. Second if $x_{K+k-1} \leq z_{k-1} \leq y_{k-1}$, then $f_{t'}(x_K) \leq f_{t'}(z) \leq f_{t'}(y)$). This implies that

$$\overline{e} = \lim_k x_k \leq \liminf_k z_k \leq \limsup_k z_k \leq \lim_k y_k = \overline{e}.$$

Proving that for any $z \in O$, and any $\{z_k\} \in \mathcal{D}(z, f_{t'})$, $z_k \to \overline{e}$. □

Theorems 4 and 7 explain why the CP works in one-dimensional models, even without using the Implicit Function Theorem (like in Samuelson), or continuous selectors of equilibria. Let $X = [a, b] \subset \mathbf{R}$,

$T = [\underline{t}, \overline{t}] \subset \mathbf{R}^n$. Suppose that $(x, t) \mapsto f_t(x)$ is C^1, $\partial_x f_t(x) \geq 0$, and $\partial_t f_t(x) > 0$.

Corollary 8. *There is $\hat{T} \subset T$, $T \backslash \hat{T}$ has Lebesgue measure zero, such that, for all $t, t' \in \hat{T}$ with $t < t'$, and any $e \in \mathcal{E}(t)$:*

- *$\underline{e}$, the largest $z \in \mathcal{E}(t')$ with $z \leq e$, is unstable for $f_{t'}$;*
- *$\overline{e}$, the smallest $z \in \mathcal{E}(t')$ with $e \leq z$, is stable for $f_{t'}$;*

Corollary 8 is a simple consequence of Theorems 4, 7, and standard results in differential geometry.

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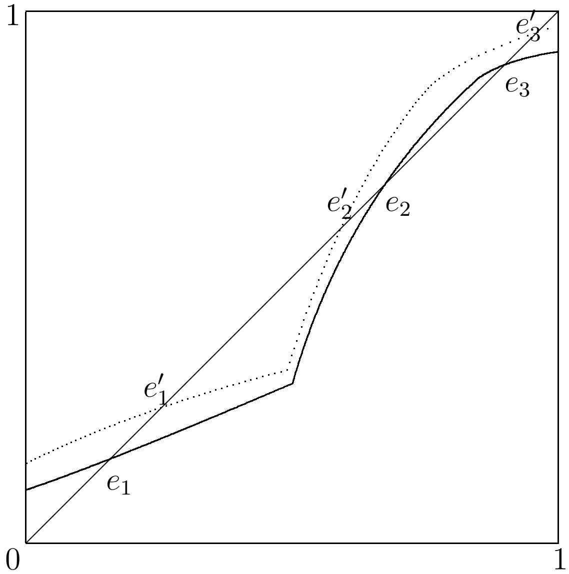
A Weak Correspondence Principle For Models With Complementarities
Figures



$\mathcal{E}(t)$	$\mathcal{E}(t')$
e_1	e_1'
e_2	e_2'
e_3	e_3'

(dotted line) graph of $f_{t'}$

(solid line) graph of f_t

FIGURE 1. An increase in t

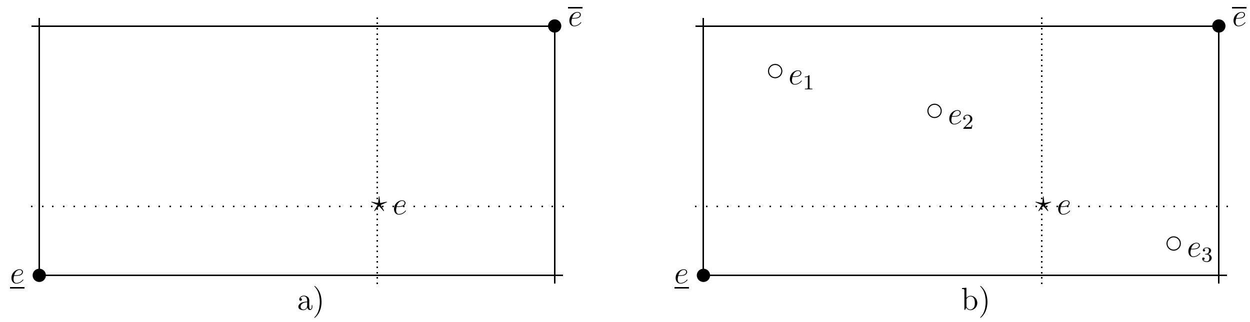


FIGURE 2. $e \in \mathcal{E}(t)$. In a), $\{\underline{e}, \overline{e}\} = \mathcal{E}(t')$, $\underline{e}$ is unstable. In b) $\{\underline{e}, e_1, e_2, e_3, \overline{e}\} = \mathcal{E}(t')$, e_1, e_2, and e_3 are unstable.

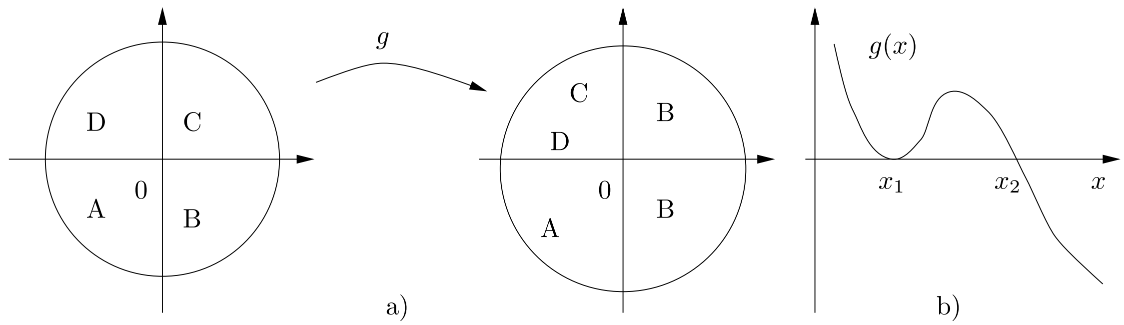


FIGURE 3. $g(x) = x - f(x)$