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FAIRNESS AND EFFICIENCY FOR ALLOCATIONS WITH PARTICIPATION CONSTRAINTS

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Abstract. We propose a notion of fairness for allocation problems in which different agents may have different reservation utilities, stemming from different outside options, or property rights. Fairness is usually understood as the absence of envy, but this can be incompatible with reservation utilities. It is possible that Alice's envy of Bob's assignment cannot be remedied without violating Bob's participation constraint. Instead, we seek to rule out *justified envy*, defined as envy for which a remedy would not violate any agent's participation constraint. We show that fairness, meaning the absence of justified envy, can be achieved together with efficiency and individual rationality. We introduce a competitive equilibrium approach with price-dependent incomes obtaining the desired properties.

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1. Introduction

We investigate the meaning of fairness in allocation environments with participation constraints and constrained consumption spaces. A special case is the random allocation problem in which agents have unit demand. Without participation constraints, we may regard all agents equally, and the absence of envy is a natural notion of fairness. In our model, different agents may have different reservation utilities, stemming from outside options or property rights. Participation constraints ensure that agents get at least their reservation utilities. Absence of envy may be incompatible with agents' participation constraints. In such environments, what does it mean to treat agents fairly?

It is well known that allocations satisfying both efficiency and envy-freeness exist (Varian, 1974; Hylland and Zeckhauser, 1979). In a model with participation constraints, the challenge is to make efficient and envy-free allocations compatible with agents' individual rationality. Our contribution is threefold. Our first contribution is to propose a notion of fairness that combines envy and individual rationality. We prove (Theorem 1) the existence of fair, efficient, and individually rational allocations. Our second contribution is to show that these fair and efficient outcomes can, under certain conditions, be viewed through *market outcomes* (Theorems 2 and 3), as in Varian and Hylland-Zeckhauser. Our third contribution (Theorem 4) is to accommodate quantitative constraints, such as those in course allocations (e.g. all students must take at least two math courses), or controlled school choice (e.g. a school seeks certain diversity objectives).

We understand fairness as the absence of justified envy, or as "ruling out envy that can be remedied within agents' individual rationality constraints." We do not want to say that an outcome is unfair if its unfairness can be traced to differences in agents' reservation utilities. Concretely, Alice envies Bob at an allocation x if she would rather have Bob's assignment in x than hers. To decide whether this envy is justified, we consider the possibility of swapping the assignments between Alice and Bob, since swapping is an obvious remedy for Alice's envy. We say that Alice's envy is justified if Bob could have received Alice's assignment without violating his participation constraint, and unjustified if Alice's assignment would put Bob below his reservation utility.[1]

[1]Our fairness notion is similar to the concepts introduced by Yılmaz (2010) and Athanassoglou and Sethuraman (2011) for object allocation problems with ordinal preferences. See Section 2 for a discussion.

Our notion of envy presumes that the obvious remedy for Alice's envy towards Bob is for them to switch assignments. Clearly, if Alice wants to bring the matter to court, the most natural and plausible remedy she could offer is for the two of them to switch assignments. One might devise more complicated remedies, with a fuller reallocation that would seek to eliminate Alice's envy, but these would necessarily be complicated and require Alice's complaint to rely on multiple agents. That said, our methods *do accommodate more general remedies* (Theorem 5).

Importantly, our notion of fairness is compatible with efficiency. We show that, under some conditions, our solution can be achieved as a market outcome. The idea seeks to generalize Varian's and Hylland and Zeckhauser's competitive equilibrium from equal incomes. The obvious solution would be to endogenize incomes. To this end, we construct price-dependent income functions. We have to be careful since, as shown by Hylland and Zeckhauser (1979), when incomes depend on prices, a Walrasian equilibrium might not exist. Our careful construction of income functions ensures individual rationality and fairness. This construction could be regarded as a minimal deviation from equal incomes that sustains individual rationality and no satiated agent overspending. If Alice envies Bob, then Bob's maximum achievable utility is his reservation utility (Lemma 4). Besides, if Alice has less money than Bob and she does not envy him, then she has just enough money to reach satiation. We provide an informal description of the income-function construction in Subsection 4.3.

We organize the paper as follows. We discuss related literature in Section 2, present our model and fairness notion in Section 3, and present main theorems in Section 4. We extend our theorems to allocation environments with general constraints and extended fairness notions in Section 5. We discuss on envy and manipulation in Section 6. In Section 7 we apply our result to economic environments of practical interest.

2. Related literature

Efficiency and fairness can be achieved in models without reservation utilities. Examples are the solutions of Varian (1974), Hylland and Zeckhauser (1979) and Bogomolnaia and Moulin (2001). Our problem is complicated, both conceptually and technically, by individual rationality constraints. Conceptually, the meaning of fairness among unequal agents is not obvious, while technically, implementation through market equilibrium may not be possible in economies with price-dependent incomes (see Hylland and Zeckhauser (1979)). Part of our contribution is to support fair, efficient and individually rational (IR) outcomes as competitive pseudo-market equilibria, as in Hylland-Zeckhauser.

Our notion of no justified envy is analogous to similar notions developed by Yılmaz (2010) and Athanassoglou and Sethuraman (2011). They assume that agents have ordinal preferences instead of cardinal utilities, and say that agent i justifiably envies agent j if i does not regard her allocation as first-order stochastically dominating j's, while any object with positive probability in her allocation is acceptable to j. Yılmaz considers the house allocation with existing tenants model in which some agents have deterministic endowments. He focuses on extending the probabilistic serial rule (Bogomolnaia and Moulin, 2001). Athanassoglou and Sethuraman consider the fractional endowment environment. Their purpose is to extend Yılmaz's mechanism and fairness notion. We work with cardinal utility, focus on market equilibrium instead of probabilistic serial, and use very different techniques. But we share some conceptual similarities with them that extend beyond the similarity in the definition of justified envy. These authors suggest a cake-eating algorithm that starts with all agents eating at the same speed, but when an agent is at risk of violating her IR constraint, only this agent has the right to eat, until she reaches her reservation utility or drops out of the algorithm. So only when IR binds for some agent is she allowed to eat at a higher speed than the others. In our competitive equilibrium method (see Theorem 2), our income functions seek to achieve similar ideas. Fairness pushes us towards equal incomes, but IR forces us to accept some inequality.

Schmeidler and Vind (1972) consider a model where IR constraints arise due to the presence of endowments. Starting from an initial endowment, they study *fair net trades*: trades leading to a Pareto optimal allocation in which no agent envies the trades made by others. Our model differs from theirs for two reasons. First, our model is primarily designed to address constrained consumption spaces, as in the random allocation problem. Fair net trades may not be feasible in such environments, leading to a weak notion of fairness. Specifically, under unit demand constraints there is no reason that one agent's net trade is feasible to any other agent.[2] Second, while reservation utilities in our model can arise due to the presence of endowments, they may also stem from other sources.

Balbuzanov and Kotowski (2019) explore the role of endowments for discrete allocation problems. Different from us, they interpret endowments as the rights to exclude others, and propose a new cooperative game solution concept. Although they allow for public ownership, or collective ownership by subgroups, endowments in their model are deterministic. As a result, their results are unrelated to ours.

Our results are applicable to school choice when we wish to use endowments instead of priorities to control children's rights towards schools. It is in particular applicable to controlled school choice. School choice was first introduced as an application of resource allocation models by Abdulkadiroğlu and Sönmez (2003). In the standard model of school choice, fairness and efficiency are generally incompatible. A lot of the school choice literature has been devoted to the resulting trade-off. In our solution, the trade-off is resolved. Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017) is the only paper we are aware of that emphasizes endowments in school choice. They assume that each child owns one seat of some school as endowment. Their goal is to design strategy-proof allocation mechanisms to meet the distributional constraint in the market and IR constraint of each child. Since they consider deterministic endowments and ordinal preferences, and their fairness notions are based on priorities, their results are unrelated to ours. The constraints we analyze have been discussed extensively in the literature on controlled school choice: see Ehlers (2010), Kojima (2012), Hafalir, Yenmez, and Yildirim (2013), Ehlers, Hafalir, Yenmez, and Yildirim (2014), and Echenique and Yenmez (2015); and the literature on distributional constraints (motivated by geographic distributional considerations): Kamada and Kojima (2015), Kamada and Kojima (2017), and Kamada and Kojima (2020), among others. Our approach of eliminating justified envy when it does not conflict with constraints is common to those papers. For example, Kamada and Kojima consider matchings where no blocking pair that would not violate distributional constraints are present.

Our results are also applicable to time banks where every agent demands others' services and also provides services to others. We are different from Andersson, Cseh,

[2]Suppose Bob is endowed with $1/3$ of good 1 and gets all of it; Alice is endowed with $2/3$ of good 1. Then Alice can never envy Bob's net trade, as adding Bob's net trade to her endowment, $2/3 + (1 - 1/3) > 1$, violates her unit demand constraint. Schmeidler and Vind's notion was never meant to be used in (what we term) random allocation problems.

Ehlers, and Erlanson (2021) in that they require integer quantities of services, while we allow for perfect divisible time.

In separate work (Echenique, Miralles, and Zhang, 2019), we give a direct Walrasian approach to allocation problems with constraints. Key is the idea of setting a price for each constraint. We properly embed constraint prices into each agent's budget constraint.

3. The model

3.1. **Notation and preliminary definitions.** In general, given a number $c \in \mathbf{R}_{++}$, we define the c-simplex $\Delta^n(c) \subseteq \mathbf{R}^n$ as $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j = c\}$, and define the c-subsimplex $\Delta^n_-(c)$ as $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j \leq c\}$. When n is understood, we simply use the notation $\Delta(c)$ and $\Delta_-(c)$.

We adopt the notational conventions of convex analysis: Denote by $\mathbf{R}_* = \mathbf{R} \cup \{-\infty\}$ the extended real numbers. A function $u : \mathbf{R}^n \to \mathbf{R}_*$ has *domain* $C = \{x \in \mathbf{R}^n : u(x) > -\infty\}$.[3] Throughout we work with functions that have a closed and convex domain. A function u with domain C is said to be continuous if it is continuous (in the relative topology on C) at every point $x \in C$, and monotone if for any $x, y \in C$, $x < y$ implies that $u(x) < u(y)$.

A function $u : \mathbf{R}^n \to \mathbf{R}$ is

- *quasi-concave* if, for all $x, y \in C$, and $\alpha \in (0,1)$, $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$.
- *strictly quasi-concave* if, for all $x, y \in C$ with $x \neq y$, and $\alpha \in (0,1)$, $u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$.
- *semi-strictly quasi-concave* if it is quasi-concave, and for all $x, y \in C$ with $u(x) \neq u(y)$ and $\alpha \in (0,1)$, $u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$.
- *concave* if, for all $x, y \in C$, and $\alpha \in (0,1)$, $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$.
- *linear* if we can identify $u(\cdot)$ with a vector $v \in \mathbf{R}^n$, so that $u(x) = v \cdot x$ for all $x \in C$. For random allocation problems, linear utility is interpreted as an *expected utility function*.
- *Lipschitz continuous* with constant $\theta > 0$ if for all $x, y \in C$, $|u(x) - u(y)| < \theta \|x - y\|$.
- satisfying the *Inada property* (at the axes) if, for all $x \in C$, $u(x) = u(0)$, unless $x \gg 0$.

[3] We adopt the conventions that $x + (-\infty) = -\infty$, and $\lambda \cdot (-\infty) = -\infty$ for any scalar λ.

- has l as its *favorite object* if, on its domain, decreasing consumption of any object $k \neq l$ by any amount $\epsilon > 0$ in favor of an increased consumption of ϵ of object l always leads to an increase in u. For example, if u is differentiable and $C = \mathbf{R}^n_{++}$, then l is a favorite object if $\frac{\partial u(x)}{\partial x_l} > \frac{\partial u(x)}{\partial x_k}$, $\forall k \neq l$. If u is linear, identified with $v \in \mathbf{R}^n$, then $v_l > v_k$, $\forall k \neq l$.

Sometimes we may require a utility function to have one of the aforementioned properties on a restricted domain, e.g. on a neighborhood of a point.

3.2. **Model.** A finite set of agents are to be assigned a finite set of objects. We assume that objects are perfectly divisible. In the random allocation problem, we would be allocating probabilities, and preferences would be defined on the set of probability distributions.

An *allocation problem* is a tuple $\Gamma = \{O, I, Q, (C^i, u^i, \tilde{u}^i)_{i \in I}\}$, where:

- $O = \{1, \dots, L\}$ represents a finite set of objects, or goods.
- $I = \{1, \dots, N\}$ represents a finite set of agents.
- $Q = (q_l)_{l \in O}$ is a capacity vector, and $q_l \in \mathbf{R}_{++}$ is the quantity of object l.
- For each agent i, $C^i \subset \mathbf{R}^L_+$ is i's consumption space, which denotes the set of assignments satisfying some exogenous constraints for i. We assume that C^i is nonempty, compact and convex.
- For each agent i, $u^i : \mathbf{R}^n \to \mathbf{R}_*$ is a continuous and monotone utility function with C^i as its domain.
- For each agent i, $\tilde{u}^i \in \mathbf{R}$ is her *reservation utility*.

Our model is general enough to accommodate various constraints imposed on every individual's consumption. An example is when agents are subject to a *unit demand constraint:* $C^i = \{x \in \mathbf{R}^L_+ : \sum_{j=1}^L x_j \leq 1\}$. More generally, we may have a bound on consumption $c^i \in \mathbf{R}_{++}$, so that $C^i = \{x \in \mathbf{R}^L_+ : \sum_{j=1}^L x_j \leq c^i\}$. For c^i large enough, the allocation problem is indistinguishable from a standard exchange economy, where the consumption space is $\mathbf{R}^L_+$. However, this family of constraints does not exhaust the kinds of constraints admitted by our model. For example, our model allows for floor and ceiling constraints in course allocation problems where students have to take at least k but at most $k' > k$ courses on some field of study.

An allocation problem is termed a *random allocation problem* if $C^i = \Delta_-$ for all i. This means that each agent faces a unit demand constraint, and is meant to consume a probability distribution over objects.

3.3. **Allocations.** Define $\mathcal{X} = \{x \in \mathbf{R}^{LN}_{+} : \sum_{i\in I} x^i_l = q_l \text{ for all } l \in O\}$. An *allocation* is an element $x \in \mathcal{X}$ such that $x^i \in C^i$ for all $i \in I$. For every i, x^i is i's assignment in x. Let $\mathcal{A}$ be the set of all allocations. We assume that $\{C^i\}_{i\in I}$ is such that $\mathcal{A}$ is nonempty, compact and convex.

In the random allocation problem, where $\sum_l q_l = N$, each agent's assignment is a probability distribution over O. When $x^i_l \in \{0,1\}$ for all i and all l, x is a deterministic allocation. The Birkhoff-von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953) ensures that every allocation is a convex combination of deterministic allocations.

3.4. **Individual rationality and Pareto optimality.** We regard agents as having the right to attain their reservation utilities. Reservation utilities may arise because agents have endowments of the objects in question, or more generally due to the presence of guaranteed outside options. Reservation utilities can arise from a policy that protects disadvantaged groups in school choice, for example.

An allocation x is *acceptable* to agent i if $u^i(x^i) \geq \tilde{u}^i$; x is *individually rational* (IR) if it is acceptable to all agents.

We assume that reservation utilities are such that an IR allocation exists. We say that Γ admits a *strictly positive IR allocation* if there is an IR allocation $\tilde{x} \in \mathbf{R}^{LN}_{++}$. All agents obtain strictly positive quantities of all goods in $\tilde{x}$.

An allocation x is *weakly Pareto optimal* (wPO) if there is no allocation y such that $u^i(y^i) > u^i(x^i)$ for all i. An allocation x is *Pareto optimal* (PO) if there is no allocation y such that $u^i(y^i) \geq u^i(x^i)$ for all i with at least one strict inequality. Given constrained consumption spaces in our model, wPO is compatible with wasteful situations where one can use existing resources to make some agents strictly better off, but cannot make all agents strictly better off because some agents are satiated.

Finally, we discuss the concept of *consistency (to economy reduction)*; see Thomson (forthcoming). Consistency provides an intermediate concept between Pareto efficiency and weak Pareto efficiency. Consistency of a property (to economy reduction) happens when the desired property holds in any reduced economy in which a subset of agents leave with their assignments. Formally, then, we say that an allocation x is *consistent weakly Pareto optimal* (cwPO) if there is no nonempty set of agents $I' \subset I$ and allocation y such that: 1) $y^i = x^i$ for all $i \notin I'$, and 2) $u^i(y^i) > u^i(x^i)$ for all $i \in I'$. In other words, no coalition of individuals may yield strict benefit for *all* of its members by letting agents outside the coalition leave with their assignments and reassigning the remaining resources among the members of the coalition.[4]

3.5. **Fairness.** Our notion of fairness rules out envy that cannot be justified by guaranteed reservation utilities. If an agent i envies another agent j at an allocation x (that is, i prefers x^j to x^i), our fairness notion regards the envy as not justified if switching their assignments would violate j's individual rationality constraint.

Formally, we say that an agent i has *justified envy* towards another agent j at an allocation x if

$$u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq \tilde{u}^j,$$

and the justified envy is said to be *strong* if $u^j(x^i) > \tilde{u}^j$. In words, if i envies j and j could have received i's assignment without violating j's individual rationality, then envy is justified. Note that because each agent's utility function takes her consumption space as its domain, $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) \geq \tilde{u}^j$ implicitly mean that $x^j \in C^i$ and $x^i \in C^j$.

We say that x has *no (strong) justified envy* (N(S)JE) if no agent has (strong) justified envy towards any other agent at x.

Fairness as defined by NJE provides a defense against possible objections raised by the agents. Imagine a social planner that proposes an IR allocation x. Suppose that an agent i objects because she would prefer to get what was assigned to agent j. An obvious remedy for i's complaint is a pairwise switch of their assignments. But if the envy is not justified, the planner's response to the objection is that the switch would violate j's right to attain her reservation utility.

Of course, one may imagine complaints that could be remedied through rearrangements more complicated than a pairwise switch. Such remedies may or may not be realistic, but in any case our methods easily accommodate much more general remedies. Specifically, one can devise cyclic rearrangements, where arbitrarily long sequences of agents collaborate in the satisfaction of an agent's envy, as long as the last agent's participation constraint is not violated. Theorem 5 in Section 5.2 extends our main result to cover this case.

In an IR and NJE allocation x, if $u^i = u^j$ and $\tilde{u}^i \geq \tilde{u}^j$, then it must be that $u^i(x^i) \geq u^j(x^j)$. That is, if i and j have equal preferences and i's reservation utility

[4]Miralles (2017) analyzes this concept for the random assignment problem with expected utilities. He concludes that cwPO is generically equivalent to PO in the space of linear preferences. See also Chambers and Hayashi (2020) for a study of consistency of core allocations in competitive economies.

is weakly higher than j's, then they both agree that i's assignment in x is also weakly better than j's. In particular, if $u^i = u^j$ and $\tilde{u}^i = \tilde{u}^j$, then it must be that $u^i(x^i) = u^j(x^j)$. So NJE and IR imply *equal treatment of equals* (called symmetry by Zhou (1990)).

3.6. **Approximate properties.** Our main results will prove that there exist allocations that satisfy individual rationality, Pareto optimality, and no justified envy. More precisely, some of our results are based on approximate versions of these properties: for any $\varepsilon > 0$, an allocation x satisfies

- *ε-individual rationality* (ε-IR) if $u^i(x^i) \geq \tilde{u}^i - \varepsilon$ for all $i \in I$.
- *ε-Pareto optimality* (ε-PO) if there is no allocation y such that $u^i(y^i) > u^i(x^i) + \varepsilon$ for all $i \in I$.
- *no ε-justified envy* (ε-NJE) if there do not exist two distinct agents i, j such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > \tilde{u}^j - \varepsilon$.

It is clear that ε-IR is weaker than IR, ε-PO is weaker than wPO, and ε-NJE is stronger than NJE.

4. Main results

4.1. **General individual constraints.** Let $\Gamma = \{O, I, Q, (C^i, u^i, \tilde{u}^i)_{i\in I}\}$ be an allocation problem under the assumptions that utility functions are continuous and monotone, and that an IR allocation exists.

Theorem 1. *Suppose that agents' utility functions in Γ are concave.*

(1) For any $\varepsilon > 0$, there exists an allocation that is ε-individually rational, ε-Pareto optimal and has no ε-justified envy;

(2) There exists an allocation that is individually rational, weak Pareto optimal and has no strong justified envy.

4.2. **Market foundations.** We then consider a special environment where every agent i is constrained by a maximum overall consumption $c^i \in \mathbf{R}_{++}$. Two special cases are worthy of emphasis. The first is the the unit demand environment of Hylland and Zeckhauser (1979), which is obtained when for all i, $C^i = \Delta_-$. The next is the standard model of an exchange economy, which obtains when c^i is large enough never to be bounded (formally, $\sum_{l\in O} q_l < \min_i c^i$).

Let $\Gamma = \{O, I, Q, (C^i, u^i, \tilde{u}^i)_{i\in I}\}$ be an allocation problem under the same assumptions as before, but where for every i, $C^i = \Delta_-(c^i)$ for some $c_i \in \mathbf{R}_{++}$. Let $v^i = \sup u^i(C^i)$ be the utility of agent i when she is satiated.

Theorem 2. *Suppose that agents' utility functions are quasi-concave, and that at least one of the following conditions hold:*

- $\sum_{l\in O} q_l < \min_i c^i$ *(c^i is sufficiently large for every $i \in I$).*
- u^i *satisfies the Inada property and $\tilde{u}^i > u^i(0)$, for every $i \in I$.*
- $u^i(x^i) = v^i$ *implies $x^i \gg 0$, for every $i \in I$, and a strictly positive IR allocation exists.*
- *There exists a common favorite object,*[5] *and a strictly positive IR allocation.*

Then there exists continuous income functions $m^i : \Delta \to \mathbf{R}_+$ and $(x,p) = ((x^i)_{i=1}^N, p)$, such that $p \in \Delta$ is a price vector, and

(1) $\sum_i x^i = Q$ (x is an allocation; or, "supply equals demand").
(2) x is individually rational, Pareto optimal and has no justified envy.
(3) $x^i \in argmax\{u^i(z^i) : z^i \in C^i$ and $p \cdot z^i \leq m^i(p)\}$.

We may start from an economy that does not satisfy the properties enumerated in Theorem 2, but by adding an artificial common favorite good to the economy and taking its supply to zero, we obtain an allocation with the desirable properties. This requires, however, strengthening our assumptions on utility functions:

Theorem 3. *Suppose every u^i is semi-strictly quasi-concave and Lipschitz continuous. Then there is an allocation x^* that is individually rational, consistent weak Pareto optimal and has no strong justified envy. Whenever x^* is also Pareto optimal, it has no justified envy. Moreover, if each u^i meets one of the following properties in some neighborhood of x^{i*}: 1) it is strictly quasi-concave, or 2) it is linear; then x^* is individually rational, Pareto-optimal and has no justified envy.*

The following is an immediate consequence of Theorem 3.

Corollary 1. *Suppose that each u^i is linear (e.g. random allocation problem), then there is an allocation that is individually rational, Pareto-optimal and has no justified envy.*

4.3. **Discussion.**

4.3.1. *Remarks on Theorem 1.* Theorem 1 gives the existence of allocations with the desired properties of efficiency, fairness, and individual rationality. The approximations in the theorem are due to limit arguments used in its proof. It is important

[5] An object is a *common favorite object* if every agent regards the object as favorite.

to emphasize that Theorem 1 allows for very general individual constraints, as long as the sets C^i are convex and compact.

The proof of Theorem 1 is based on weighted utilitarian maximization. We study the problem of maximizing

$$\sum_{i\in I} \lambda^i u^i(x^i) \tag{1}$$

over all IR allocations x, for each fixed vector $\lambda = (\lambda_1, \ldots, \lambda_N)$ of welfare weights. The trick is to find "fair" welfare weights. Ideally, one could proceed iteratively. For each λ, solve the weighted utilitarian maximization problem and check if there is any justified envy. If i justifiedly envies j, then adjust λ so as to decrease λ^j and increase λ^i. The iterative procedure does not quite work, but we use a related idea, based on the *Knaster-Kuratowski-Mazurkiewicz (KKM) lemma*.

The KKM lemma was used by Varian (1974) in proving the existence of Pareto efficient allocations with no envy whatsoever. Varian does not consider participation constraints, and works directly with allocations (more precisely, with the utility possibility frontier). Our approach using welfare weights (inspired by the Negishi approach to equilibrium existence), is quite different. Participation constraints introduce some technical difficulties, which necessitates an approximation argument. The presence of $\varepsilon > 0$ in the IR, efficiency and NJE properties are consequences of our approximation argument.

We briefly explain our use of the KKM lemma. Suppose there is a collection of closed sets, each one identified with a vertex of the simplex. Suppose that each face of the simplex is covered by the union of the sets identified with the vertices of such face. (Notice that the simplex is also a face of itself.) The KKM lemma says that such a collection of sets has non-empty intersection. In the proof of Theorem 1, we identify the simplex with the set of welfare weights λ. Thus each vertex of the simplex is the result of putting all weight on one agent in solving (1). Each set Λ^i corresponds to the set of weights yielding an ε-Pareto optimal allocation in which 1) agent i does not have ε-justified envy towards any other agent and 2) ε-individually rationality holds for agent i. We show that the collection $(\Lambda^i)_{i\in I}$ meets the conditions of the KKM lemma. Any point in the intersection of $(\Lambda^i)_{i\in I}$ meets the properties in the first statement of the theorem.

By taking the limit when $\varepsilon \to 0$, we obtain the second statement of Theorem 1. Observe that we only conclude that the obtained allocation is wPO. This is irrelevant in many allocation problems, in which wPO and PO are identical. There are environments, however, in which there is no such equivalence. Market design problems in which the consumption spaces are bounded (say, because of unit demand) constitute one example.

4.3.2. *Remarks on Theorem 2.* When agents' consumptions are simply bounded above ($C^i = \Delta_-(c^i)$), Theorem 2 provides conditions under which a fair, efficient, and IR allocation exists *and* can be supported as a form of market equilibrium (a "pseudo-market"). Our equilibria generalize Hylland and Zeckhauser's, or Varian's, notion of equilibrium with a fixed exogenous income. Here, income is not fixed. It is price dependent, and formulated through *income functions* m^i. These are carefully calibrated to ensure both IR and NJE. The use of competitive markets to achieve a fair and efficient allocation is inspired by Varian and Hylland and Zeckhauser, and more recently by Miralles and Pycia (2020), who establish Second Welfare Theorems for the kind of allocation problems studied in this paper.

Varian and Hylland-Zeckhauser assume fixed and equal incomes for all agents. The first complication in our paper is that equal incomes may not respect reservation utilities. Incomes must be price-dependent and constructed to satisfy IR and ensure NJE.[6] Our model suggests a "minimal departure" from equal incomes that satisfies IR: We allow an agent to have above-average income only in order to obtain exactly her reservation utility. A second complication is that a competitive equilibrium allocation with potentially satiated agents does not guarantee Pareto-optimality, unless expenses for satiated agents are minimal (Hylland and Zeckhauser, 1979). For this reason, we force an agent's income below average whenever the average lies above the minimal income providing her with satiation.

The main new idea in the proof of Theorem 2 lies in the construction of price-dependent incomes. The construction is done by, for each price vector, taking the median of three magnitudes: 1) a common income level, 2) the minimum expenditure guaranteeing satiation, and 3) the minimum expenditure guaranteeing reservation utility. Agents' incomes add up to the overall value of objects. An important property of our construction is that, when i's income is higher than j's, and j is not satiated, then i's income must be equal to the minimum expenditure ensuring her reservation utility. This naturally establishes no justified envy: If agent j envies i,

[6]When reservation utility arises from agents' endowments, one may be tempted to use Walrasian incomes. These are, of course, price dependent, and ensure IR. Unfortunately, there are simple examples of allocation problems with endowments where no Walrasian equilibria exist (Hylland and Zeckhauser, 1979). See Echenique, Miralles, and Zhang (2019) for a discussion.

then j must have less equilibrium income than i, so that i must find j's allocation unacceptable. Finally, it is clear that IR is ensured, since no agent's income lies below the minimum expenditure ensuring her reservation utility.

Once income functions are in place, existence follows standard ideas: first showing the existence of quasi-equilibrium (as in e.g Mas-Colell, Whinston, Green, et al. (1995), chapter 17, appendix B), and then exploiting the conditions stated in the theorem to bridge the gap between quasi-equilibrium and competitive equilibrium. To this end, the conditions stated in the theorem play a technical role. They serve to ensure that either all prices or incomes are strictly positive for all agents. The first condition consists of virtually unbounded (above) consumption spaces.[7] The second condition considers preferences for strictly positive bundles alongside with reservation utility above the minimal level. The third condition assumes that agents are satiated only at strictly positive bundles and there exists a strictly positive allocation that is IR. The fourth condition is based on the existence of a common favorite object, and a strictly positive IR allocation. The resulting allocation guarantees all the desired properties in their strongest sense.

One drawback of our approach, which it shares with other models based on market equilibrium, is that one cannot ensure the uniqueness of the allocation that satisfies our properties.

4.3.3. *Remarks on Theorem 3.* Once we adopt the framework of $C^i = \Delta_-(c^i)$ and adopt stronger assumptions on utilities, Theorem 3 allows us to improve on Theorem 1 and to generalize Theorem 2. When utilities are Lipschitz and semi-strictly quasi-concave, we provide a method for finding an allocation that is "generically" individually rational, Pareto optimal and with no justified envy. It suffices that, around each agent's final assignment, her indifference curves do not exhibit kinks, nor change from linearity to strict quasi-concavity.

In the proof of Theorem 3, we augment the economy Γ by adding an arbitrarily low amount of an artificial common favorite good. Lipschitz continuity is needed in order to facilitate such an addition. We consider a sequence of economies in which the supply of the artificial good tends to zero, obtaining the limit of a sequence of Pareto optimal, IR, NJE allocations. Under the generic conditions of the second statement in Theorem 3, the limit allocation is also Pareto optimal, NJE and IR.

[7]A corollary is that *if the consumption space is $\mathbf{R}^L_+$ for all agents, then our approach obtains an allocation that is Pareto optimal, IR and has no justified envy.*

5. Extensions

The advantage of Theorem 1 relative to Theorem 2 is its flexibility to deal with various constraints. In Section 5.1 we extend this approach beyond individual constraints, to environments where we take any closed and convex set $\mathcal{A}^{\mathcal{C}}$ of allocations as the primitive. We interpret $\mathcal{A}^{\mathcal{C}}$ as the set of allocations that comply with some given collection of constraints. In the random allocation problem, we assume that the allocations in $\mathcal{A}^{\mathcal{C}}$ can be implemented as randomized deterministic allocations. For example, if the constraints behind $\mathcal{A}^{\mathcal{C}}$ satisfy the condition of Budish, Che, Kojima, and Milgrom (2013), then $\mathcal{A}^{\mathcal{C}}$ is always implementable.

Our notion of NJE relies on the pairwise switch as being the remedy for envy. We think of such switches as natural, but want to emphasize that our results are, in fact, easily generalized to allow for more general remedies. In Section 5.2, we prove that Theorems 1, 2 and 3 hold without change if we carry out a chain of exchanges as the remedy for envy.

5.1. **General constraints.** Let $\mathcal{A}^{\mathcal{C}} \subset \mathcal{X}$ be a closed and convex set of feasible allocations. All assignments in $\mathcal{A}^{\mathcal{C}}$ are within agents' consumption spaces so that their utilities are well-defined. The definition of individual rationality is the same as before. We assume that there exists an IR allocation in $\mathcal{A}^{\mathcal{C}}$. The definition of efficiency extends naturally to $\mathcal{A}^{\mathcal{C}}$. An allocation $x \in \mathcal{A}^{\mathcal{C}}$ is *Pareto optimal* if there is no allocation $y \in \mathcal{A}^{\mathcal{C}}$ such that $u^i(y^i) \geq u^i(x^i)$ for all $i \in I$ with strict inequality for some agent; x is *weak Pareto optimal* (wPO) if there is no allocation $y \in \mathcal{A}^{\mathcal{C}}$ such that $u^i(y^i) > u^i(x^i)$ for all $i \in I$; and x is *ε-Pareto optimal* (ε-PO), for any $\varepsilon > 0$, if there is no allocation $y \in \mathcal{A}^{\mathcal{C}}$ such that $u^i(y^i) > u^i(x^i) + \varepsilon$ for all $i \in I$.

In our main model with only individual constraints, i's envy towards j is not justified if switching their assignments violates the participation constraint of j. Now, additional constraints provide another reason for negating i's envy: switching their assignments may not be feasible because it violates some constraints. To formalize this idea, let $x_{i\leftrightarrow j}$ denote the allocation obtained by switching the assignments of i and j in an allocation x; that is, $x^i_{i\leftrightarrow j} = x^j$, $x^j_{i\leftrightarrow j} = x^i$, and $x^k_{i\leftrightarrow j} = x^k$ for all $k \in I \setminus \{i, j\}$. Now an agent i has *justified envy* towards another agent j at an allocation $x \in \mathcal{A}^{\mathcal{C}}$ if

$$u^i(x^j) > u^i(x^i),\ u^j(x^i) \geq \tilde{u}^j, \text{ and } x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}.$$

Under this new definition, no justified envy between any two agents may no longer be compatible with efficiency and individual rationality. To overcome this difficulty, we classify agents into disjoint types. Informally, think of i and j as being of equal type if the constraints behind $\mathcal{A}^{\mathcal{C}}$ do not distinguish between them. We identify agents' types by checking whether switching their assignments in any feasible allocation is still feasible. Formally, we say that two agents i, j are of *equal type*, denoted by $i \sim j$, if for all $x \in \mathcal{A}^{\mathcal{C}}$, $x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$. The binary relation $\sim$ is reflexive and transitive.[8] Hence it partitions I into disjoint types. Then we say i has *equal-type justified envy* towards j at an allocation $x \in \mathcal{A}^{\mathcal{C}}$ if i has justified envy towards j, and i, j are of equal type. We say that x has *no equal-type justified envy* if no agent has equal-type justified envy towards any other agent. *No strong equal-type justified envy* and *no equal-type ϵ-justified envy* are defined in a similar way by stating that the relevant envy is absent in the allocation.

With the above definitions, we may generalize Theorem 1 as follows. Let Γ be an allocation problem with our running assumptions (utilities are continuous and monotone). Suppose that $\mathcal{A}^{\mathcal{C}}$ is a primitive set of allocations in Γ, and that there exists an IR allocation in $\mathcal{A}^{\mathcal{C}}$.

Theorem 4. *Suppose that agents' utility functions are concave.*

(1) For any $\varepsilon > 0$, there exists an allocation that is ε-individually rational, ε-Pareto optimal and has no equal-type ε-justified envy;

(2) There exists an allocation that is individually rational, weak Pareto optimal and has no strong equal-type justified envy.

There are two cases where Theorem 4 achieves fairness between *any* two agents. The first is when the constraints behind $\mathcal{A}^{\mathcal{C}}$ are *anonymous* in the sense that all agents are identified to be of equal type. Using the approach of Budish, Che, Kojima, and Milgrom (2013), suppose that every constraint takes the inequality form $\underline{q}_S \leq \sum_{(i,l)\in S} x^i_l \leq \bar{q}_S$ where $S \subseteq I \times O$ is a set of agent-object pairs, and $(\underline{q}_S, \bar{q}_S)$ is a pair of integers with $0 \leq \underline{q}_S \leq \bar{q}_S$. Then a collection of constraints are anonymous if every constraint set S is of the form $I \times O'$ where $O' \subset O$, because such constraints do not distinguish between the identities of agents. An example is the distributional constraints studied by Kamada and Kojima (2015, 2017) where

[8] Suppose $i \sim j \sim k$. For all $x \in \mathcal{A}^{\mathcal{C}}$, $x_{i\leftrightarrow k} = [(x_{i\leftrightarrow j})_{j\leftrightarrow k}]_{i\leftrightarrow j}$. $i \sim j$ implies that $x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$, $j \sim k$ implies that $(x_{i\leftrightarrow j})_{j\leftrightarrow k} \in \mathcal{A}^{\mathcal{C}}$, and $i \sim j$ implies that $[(x_{i\leftrightarrow j})_{j\leftrightarrow k}]_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$. So $x_{i\leftrightarrow k} \in \mathcal{A}^{\mathcal{C}}$.

the government wants to control the distribution of doctors in different geographic locations.

The second case where fairness is achieved between any two agents is captured by our original notion of individual constraints – the main model presented in Section 3. In that case, $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) \geq \tilde{u}^j$ implicitly mean that the two agents accept each other's assignments. So $x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$ is implied.

In general, however, $x \in \mathcal{A}^{\mathcal{C}}$ does not mean that $x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$. An example is controlled school choice in which the set of students I are partitioned into disjoint subsets $T_1, \ldots, T_K$ (which are interpreted as types), and for each school l, the desirable number of type k students, where $k \in \{1, \ldots, K\}$, is between $\underline{q}_{l,k}$ and $\bar{q}_{l,k}$. So for each school l and each type k, we have a constraint $\underline{q}_{l,k} \leq \sum_{i\in T_k} x^i_l \leq \bar{q}_{l,k}$. Then Theorem 4 says that there is an efficient and individually rational allocation that achieves fairness *within students of each type.*

5.2. **Justified envy by exchange.** Our notion of justified envy is based on pairwise switch, but the ideas behind our results can be applied more generally. Consider then envy that can be addressed by carrying out a chain of exchanges, each agent giving up her assignment in favor of an agent who envies her, and the last agent in the exchange being given the assignment of the first agent. If this reallocation does not violate the last agent's individual rationality constraint, then the envy is justified.

Formally, agent i has *justified envy by exchange* towards agent j at allocation x if there exists a sequence of distinct agents $(i_k)_{k=1}^K$ with

- $i_1 = i$ and $i_2 = j$;
- i_k envies i_{k+1}, $1 \leq k \leq K-1$;
- and $u^{i_K}(x^{i_1}) \geq \tilde{u}^{i_K}$.

The idea is that i could conjure a remedy for her envy towards j by proposing a coalition of agents and a reallocation of their assignments, such that all are made better off, with the possible exception of one agent whose participation constraint is not violated. We define *strong justified envy by exchange* and *ε-justified envy by exchange* similarly as before. We prove that our main theorems hold without change under this extended fairness notion.

Theorem 5. *Theorems 1, 2 and 3 hold if "justified envy" is replaced by "justified envy by exchange."*

6. Envy and manipulability

We now proceed with a discussion of envy and incentives, leading with an example that illustrates some of the issues. The example shows that fairness is more subtle than one might initially imagine. It also illustrates that the allocations we have advocated for in this paper may be manipulable. After presenting the example, we present some conditions that suffice to rule out profitable manipulations.

In our discussion, we are going to make some simplifying assumptions. We restrict attention to random allocation environments ($C^i = \Delta_-$) and linear utility functions. We can identify each agent i's utility function with a vector u^i. So her utility from consuming $x \in \Delta_-$ is $u^i \cdot x$. Reservation utilities are defined from endowments. There exists $\omega^i \in \Delta$ for each agent i with $\tilde{u}^i = u^i \cdot \omega^i$, and $\sum_i \omega^i = Q$.

Example 1. Suppose that there are three types of agents, and four agents of each type. Agents of the same type have equal preferences and endowments, and obtain equal assignments. Superscripts make reference to agent types. There are three objects, and the total (aggregate) endowment in this economy is $Q = (3, 2, 7)$.

The tables below exhibit agents' preferences, endowments and corresponding reservation utilities. Utilities are normalized in the sense of giving valuation 3 to the most preferred object and 1 to the least preferred one.

u^1	u^2	u^3
3	3	2
1	2	3
2	1	1

ω^1	ω^2	ω^3
3/8	3/8	0
1/4	1/4	0
3/8	3/8	1

$\tilde{u}^1$	$\tilde{u}^2$	$\tilde{u}^3$
17/8	2	1

The following is the *unique* competitive equilibrium delivered as in the proof of Theorem 2.[9] Prices (normalized to lie on the simplex) are $p^* = (1/2, 1/3, 1/6)$. Price-dependent incomes are $m^{1*} = m^{3*} = 1/4$ and $m^{2*} = 1/3$. Notice that m^{2*} has to be higher than the other agent types' incomes in order to satisfy individual rationality. The equilibrium allocation is: $x^{1*} = (1/4, 0, 3/4)$, $x^{2*} = (1/2, 0, 1/2)$, $x^{3*} = (0, 1/2, 1/2)$. Note that $u^2 \cdot x^{2*} = u^2 \cdot \omega^2 = \tilde{u}^2 = 2$, so type-2 agents' IR constraint binds. The allocation is (ex-ante) Pareto-efficient, IR and NJE. Importantly, type-1 agents envy type-2 agents, who in turn find type-1 assignments unacceptable.

The example serves to illustrate two points. First, $\omega^1 = \omega^2$ and $\tilde{u}^1 > \tilde{u}^2$. However, type-1 agents envy type-2 agents. In this economy, type-2 agents fare better than

[9]Observe that there is no competitive equilibrium from equal incomes guaranteeing individual rationality, which illustrates on the points we emphasized in our earlier discussion.

type-1s because they are, in a sense, more useful as trading partners. They do not have better outside options; one might think that agents' worth in an economy depend on the value of their endowments. Our example illustrates that there is another sense in which an agent may do well: because they contribute to other agents' welfare. For that reason, they are able to participate in more valuable exchanges.

The second point illustrated by the example is about manipulability. It turns out that if one type-1 agent impersonates type-2 preferences, then equilibrium prices do not change, yet equilibrium income for truthful type-1 and type-3 agents becomes $5/21$ – lower than when everyone is truthful. Equilibrium income for type-2 agents remains the same as in Example 1, that is, $1/3$, which is the income obtained by the manipulating agent. Equilibrium allocation probabilities become $(3/14, 0, 11/14)$ for truthful type-1 agents, $(33/70, 2/35, 33/70)$ for type-2 agents and for the manipulating agent, and $(0, 3/7, 4/7)$ for type-3 agents. It is easy to verify that such a manipulation is profitable for the manipulating agent, as compared to her allocated probabilities in the example.

Motivated by the opportunities for manipulability in the example, we proceed with a general discussion of incentives. We are going to propose a guide as to how to avoid the conclusion that our allocations are manipulable. In the end, however, the message is going to be fairly negative.

It is useful to think about incentives in the competitive equilibrium procedure highlighted by Theorem 2 because we can resort to the existing theory of incentives in markets. Now, there are two channels for incentive issues to arise. The first channel has to do with how agents' reported preferences affect equilibrium prices. The other is related to how reported preferences and reservation utilities affect income functions. We shall effectively shut down the first by resorting to the standard idea that atomistic agents do not affect prices. We also assume that either the endowment (see Proposition 1 below) or the reservation utilities (see Proposition 2 below) are known, or not subject to manipulation. As Example 1 illustrates, there is still scope for manipulation.

We say that prices are *sticky* to agent i's manipulation if they do not change at all upon individual manipulation of preferences by i. This is an extreme case of regularity *a la* Jackson (1992). Stickiness happens when the economy is a sufficiently large replica of an original economy were one type j of agents is pivotal (i.e. p^* is an affine transformation of u^j), and j-agents' demands can adapt to agent i's preference manipulation so as to keep the market cleared. Stickiness is taken as the best-case scenario against manipulation. In the classical Walrasian model with endowments, no benefit can be obtained at all by preference misreporting if prices are sticky. It is clear that we are taking a short-cut here instead of formally arguing about the limit of an economy as in Jackson (1992); a fully formal argument would lead us away from the main message of our paper.

We say that our competitive equilibrium procedure is manipulable by agent i at equilibrium prices p^* if she can obtain higher utility by reporting false preferences after recalculation of equilibrium prices.

Proposition 1. *Suppose that equilibrium prices p^* are sticky to agent i's manipulation, who has endowment ω^i and preferences u^i. This competitive equilibrium procedure is not manipulable by agent i at equilibrium prices p^* if and only if either i is satiated or $m^i(p^*) \geq p^* \cdot \omega^i$.*

Proof. We normalize prices to lie on the simplex, and identify the space of preferences with the interior of the simplex (which is possible by monotonicity of preferences). We argue that the best manipulation agent i can make is of the form $\hat{u}^i = \alpha u^i + (1-\alpha)p^*$, $\alpha \in (0,1)$, where either agent i obtains just enough income for satiation or, if not possible, α is arbitrarily close to 0.

Recall that incomes are given through income functions, not by the market value of endowments. We first find the highest income agent i can obtain by a preference manipulation, considering that it depends on the consequent minimum expenditure yielding individual rationality for such an agent. We solve the optimization problem

$$\sup_{u \in int(\Delta)} \min_{x \in \Delta} \{p^* \cdot x : u \cdot x \geq u \cdot \omega^i\}$$

Since $\omega^i \in \Delta$, $\min_{x\in\Delta}\{p^* \cdot x : u \cdot x \geq u \cdot \omega^i\} \leq p^* \cdot \omega^i$, for any $u \in int(\Delta)$. Since $p^* \in \Delta$, for any u arbitrarily close to p^* we obtain $\min_{x\in\Delta}\{p^* \cdot x : u \cdot x \geq u \cdot \omega^i\}$ arbitrarily close to $p^* \cdot \omega^i$. Thus we have the solution to the problem: By declaring preferences arbitrarily close to p^*, agent i obtains an income that is arbitrarily close to $p^* \cdot \omega^i$.

To conclude the proof, obtaining nearly maximal income has to be compatible with an allocation coinciding with an optimal choice under true preferences u^i. Suppose that i reports utility $\hat{u}^i = \alpha u^i + (1-\alpha)p^*$, $\alpha \in (0,1)$. Denoting with $x(u^i, p, m)$ the Marshallian demand of an agent with linear preferences u^i, under prices p, with income m, it is known that $x(\hat{u}^i, p, m) = x(u^i, p, m)$ for any $\alpha \in (0,1)$. So either agent i chooses α so that she obtains sufficient income for satiation, or, in case such an α does not exist, agent i would pick an arbitrarily small α. Such a nearly optimal manipulation is not possible if only if $m^i(p^*) \geq p^* \cdot \omega^i$. □

The preceding result is restricted to economies where the reservation utilities come from endowments. We also present a result with more general reservation utilities. The proof approximately follows that of Proposition 1, so it is omitted.

Proposition 2. *Consider a random allocation economy where our competitive equilibrium procedure delivers an equilibrium price vector p^*. Suppose that p^* is sticky to agent i's manipulation, who has reservation utility $\tilde{u}^i$ and linear preferences u^i. This competitive equilibrium procedure is* NOT *manipulable by agent i at equilibrium prices p^* if and only if either i is satiated or*

$$m^i(p^*) \geq \frac{\tilde{u}^i - \min_l u^i_l}{\max_l u^i_l - \min_l u^i_l} \max_l p^*_l + \frac{\max_l u^i_l - \tilde{u}^i}{\max_l u^i_l - \min_l u^i_l} \min_l p^*_l$$

The conclusion from these two propositions is rather negative. When reservation utilities are defined from endowments, even if prices are sticky to preference manipulation, our suggested procedure is manipulable unless no agent obtains no less budget than the market value of her endowment. This leaves little scope for income redistribution as suggested by our procedure, except for what is potentially unspent by satiated agents.

The difficulties implied by Proposition 2 seem more difficult to grasp. In order to get a clearer understanding, we normalize utilities and prices so that 1 is the maximum and 0 the minimum. Such normalization simplifies the condition in Proposition 2 to $m^i(p^*) \geq \tilde{u}^i$. Now, simple calculations imply that a wide range of preferences (those *not* meeting $u^i \leq p^*$) give $e^i(\tilde{u}^i, p^*) < \tilde{u}^i$ (here, as in the proof of Theorem 2, e^i is the expenditure function for agent i). For the sake of non-manipulability, not one of these agents may be envied, for our procedure would give them income $m^i(p^*) = e^i(\tilde{u}^i, p^*) < \tilde{u}^i$. As we know from the aforementioned Proposition, these agents could then profitably manipulate such equilibrium.

The fact that our procedure can be manipulated is not surprising if one compares it with the closely related literature. The mechanisms suggested by Athanassoglou and Sethuraman (2011) and Yılmaz (2010) for ordinal preferences are acknowledged as manipulable (in environments without endowments, the celebrated Probabilistic Serial is also manipulable; Bogomolnaia and Moulin (2001)). Such mechanisms consist of a cake-eating algorithm in which every agent eats at the same speed, yet agents in danger of not meeting individual rationality suddenly acquire infinite speed. It turns out that an agent could misreport preferences in order to acquire infinite eating speed earlier than with true preferences. Our suggested competitive equilibrium procedure, adapted to cardinal preferences, essentially departs from equal incomes only when individual rationality is at risk. Agents manipulate preferences in order to attain a higher income, de facto "eating at a higher speed."

In defense of our proposals, manipulation requires sophisticated agents.[10] Agents need to know the distribution of preferences and reservation utilities. She also needs to calculate equilibrium prices, crucial in determining the best way to manipulate preferences. In case of equilibrium multiplicity, she needs to know how one is selected. Nevertheless, it remains true that a sufficiently sophisticated agent with enough information is able to manipulate the allocation procedure.

7. Applications: Reservation Utilities in Allocation Problems

We provide a non-exhaustive list of applications studied in the economics literature, where it makes sense to consider minimum guaranteed utilities in allocation problems. The list includes school choice, time banks and housing markets.

7.1. **School choice.** School choice is the problem of allocating children to schools when we want to take into account children's (or their parents') preferences (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005; Abdulkadiroğlu, Pathak, and Roth, 2005; Pathak and Sönmez, 2013).

School districts want to give some children certain rights, like the right to attend a neighborhood school, or the right to go to the same school as an older sibling. In the current practice, such rights are achieved by giving children different priorities. However, priorities might not properly reflect such rights. In the following example, a student climbs up in the priority structure, but because the relative priorities of the other students change, she ends with a *worse* outcome in the student-optimal stable matching. The conveyed point is that a student's priority alone cannot ensure her outcome.

Example 2. Consider three schools $\{a, b, c\}$ and three children $\{1, 2, 3\}$. Suppose that school priorities, and children preferences are as follows.

[10]If reservation utilities are unknown and manipulable, manipulation can be easier and may not require high sophistication.

a	b	c		1	2	3
2	2	3		b	a	a
3	3	1		c	b	c
1	1	2		a	c	b

Then the student-optimal stable matching is $\mu(1) = b$, $\mu(2) = a$ and $\mu(3) = c$. If 1 and 2 switch roles in the priority ranking of school a, then the student-optimal stable matching becomes $\mu(1) = c$, $\mu(2) = b$ and $\mu(3) = a$. So 1 becomes worse off.

Given the absence of a direct connection between priorities and outcomes, we propose the use of endowments to control children's rights. We imagine that there is a lottery that gives an *initial* probabilistic allocation of children to schools. The lottery could reflect different objectives in controlled school choice, such as giving each child a higher chance of attending her neighborhood school, or giving each minority child a chance (literally, a positive probability) of attending the highest-ranked schools.[11] The initial allocation, or endowment, provides transparent and immediate reservation utility. A child who is endowed with a seat at her neighborhood school can simply choose to attend that school. Her right to attend that school does not depend on other children in any way.

7.2. **Time banks.** A time bank is a platform where agents exchange labor time without using money transfers. The key idea is that each unit of labor time devoted to the platform by an agent must be compensated exactly by the same amount of labor services received from other participants.

In our solution, the services an agent can supply are her endowments, and define her reservation utility. Our solution provides allocations that meet three desiderata: efficiency, individual rationality (agents consider participation beneficial) and no justified envy. Moreover, we show that it is possible to include additional constraints in the problem and still find allocations satisfying these desiderata.

Andersson, Cseh, Ehlers, and Erlanson (2021) solve a more complicated model in that time enters and is provided in integer quantities (say hours per week.) Our approach is simpler in that it allows for perfect divisible time. In compensation, our procedure is easily principled in terms of economic reasoning. It provides a market foundation for an exchange program in which real money is not accepted.

[11] "The Broken Promises of Choice in New York City Schools", *New York Times*, May 5th, 2017; and "Confronting Segregation in New York City Schools", *New York Times*, May 15th, 2017.

7.3. **Housing markets.** In the standard housing market model (Shapley and Scarf, 1974), each agent is endowed with a house, and they wish to exchange their endowments to enhance efficiency. Yılmaz (2010) considers an environment where only a subset of agents are endowed with houses, and proposes a solution to find ordinally efficient allocations under IR constraints of such agents. Athanassoglou and Sethuraman (2011) extend the solution to environments where agents are endowed with shares of houses. In all of these papers, agents have ordinal preferences.

Our paper provides the cardinal utility extension in parallel with Athanassoglou and Sethuraman (2011). Agents are endowed with shares of houses, and have preferences over all possible shares of houses, up to some limit in the total cumulative share an agent may enjoy. By the use of market foundation techniques, we provide solutions that are Pareto efficient in a cardinal sense, other than satisfying the fairness and participation constraints desiderata from the aforementioned papers. With the use of KKM techniques *a la* Varian (1974), we can include any set of additional constraints to our problem and still find a satisfactory solution.

8. Proof of Theorems 1 and 4

We prove Theorem 4, and then explain how an easy adaption proves Theorem 1.

For any given $\varepsilon > 0$, define

$$\mathcal{A}^* = \{x \in \mathcal{A}^{\mathcal{C}} : x \text{ is } \varepsilon\text{-individually rational}\}.$$

It is easy to see that $\mathcal{A}^*$ is nonempty and compact.

Let the N-dimensional simplex Δ be the domain of welfare weights.

For any $\lambda \in \Delta$, define

$$\phi(\lambda) = \text{argmax}\{\sum_{i\in I} \lambda^i u^i(x^i) - \delta \sum_{i\in I} \|x^i - \mathbf{1}\| : (x^i)_{i\in I} \in \mathcal{A}^*\},$$

where $\mathbf{1}$ is a vector of ones and $\delta > 0$ is small enough such that

$$\delta \max_{x\in\mathcal{A}^*} \sum_{i\in I} \|x^i - \mathbf{1}\| < \varepsilon.$$

Since all u^i are continuous and concave and $\sum_{i\in I} \|x^i - \mathbf{1}\|$ is continuous and strictly convex, the objective function $\sum_{i\in I} \lambda^i u^i(x^i) - \delta \sum_{i\in I} \|x^i - \mathbf{1}\|$ is continuous and strictly concave. Moreover, $\mathcal{A}^*$ is compact. Thus, $\phi : \Delta \to \mathcal{A}^*$ is a function (meaning it is singleton-valued), and, by the Maximum Theorem, continuous. Moreover, the choice of δ implies that ϕ is ε-Pareto optimal.

For any agent i, define

$$\Lambda^i = \{\lambda \in \Delta : \nexists j \in I \text{ s.t } i \text{ has equal-type } \varepsilon\text{-justified envy towards } j \text{ at } \phi(\lambda)\}.$$

The proof relies on an application of the so-called KKM Lemma (the lemma is due to Knaster, Kuratowski and Mazurkiewicz; see Theorem 5.1 in Border (1989)). In the following two lemmas we prove that $\{\Lambda^i\}_{i=1}^N$ is a *KKM covering* of the simplex Δ. This means that every Λ^i is closed and that for any $\lambda \in \Delta$ there is at least one Λ^i such that $\lambda^i > 0$ and $\lambda \in \Lambda^i$.

Lemma 1. *For every $i \in I$, Λ^i is closed.*

Proof. Let λ_n be a sequence in Λ^i such that $\lambda_n \to \lambda \in \Delta$. Let $x_n = \phi(\lambda_n)$. By continuity of ϕ, $x_n \to x = \phi(\lambda) \in \mathcal{A}^*$. Now we prove that $\lambda \in \Lambda^i$, that is, i does not have equal-type ε-justified envy towards any other agent. Suppose that there is an agent j of equal type with i such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > \tilde{u}^j - \varepsilon$. Since i and j are of equal type, $x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$, and $(x_n)_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$ for every n. By continuity of u^i and u^j, for n large enough we have $u^i(x_n^j) > u^i(x_n^i)$ and $u^j(x_n^i) > \tilde{u}^j - \varepsilon$. These mean that i has equal-type ε-justified envy towards j at x_n, which is a contradiction. Therefore, $\lambda \in \Lambda^i$ and Λ^i is closed. □

Lemma 2. *For every $\lambda \in \Delta$, $\lambda \in \cup_{i \in supp(\lambda)} \Lambda^i$.*

Proof. Suppose, towards a contradiction, that for some $\lambda \in \Delta$, $\lambda \notin \cup_{i\in \text{supp}(\lambda)} \Lambda^i$. Let $x = \phi(\lambda)$. Then for every $i \in \text{supp}(\lambda)$ there exists some j of equal type with i such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > \tilde{u}^j - \varepsilon$.

Suppose first that there exist some i and j in the aforementioned situation such that $j \notin \text{supp}(\lambda)$. Then consider the allocation $y = x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}$. y is ε-individually rational as x was ε-individually rational and $u^j(x^i) > u^j(\omega^j) - \varepsilon$. Note that $\lambda^j = 0$ and $u^i(x^j) > u^i(x^i)$ imply that $\sum_{h\in I} \lambda^h u^h(x^h) < \sum_{h\in I} \lambda^h u^h(y^h)$. We also have that $\sum_{h\in I} \|x^h - \mathbf{1}\| = \sum_{h\in I} \|y^h - \mathbf{1}\|$. Hence

$$\sum_{h\in I} \lambda^h u^h(x^h) - \delta \sum_{h\in I} \|x^h - \mathbf{1}\| < \sum_{h\in I} \lambda^h u^h(y^h) - \delta \sum_{h\in I} \|y^h - \mathbf{1}\|.$$

But it contradicts the definition of $x = \phi(\lambda)$.

The above argument means that every $i \in \text{supp}(\lambda)$ has equal-type ε-justified envy towards some $j \in \text{supp}(\lambda)$. Then, since the set of agents in $\text{supp}(\lambda)$ is finite, there must exist a subset of distinct agents $\{i_1, \dots i_K\} \subseteq \text{supp}(\lambda)$ such that i_1 has equal-type ε-justified envy towards i_2, i_2 has equal-type ε-justified envy towards i_3, and so on until i_K has equal-type ε-justified envy towards i_1. Then we can construct a new allocation y by letting agents in the cycle exchange their allocations. Since the agents in the cycle are of equal type, y must be feasible, that is, $y \in \mathcal{A}^{\mathcal{C}}$.[12] As before, we have that $\sum_{h\in I} \|x^h - \mathbf{1}\| = \sum_{h\in I} \|y^h - \mathbf{1}\|$ because y is obtained from x by permuting the assignments of agents in the cycle. Then we have

$$\sum_{h\in I} \lambda^h u^h(x^h) - \delta \sum_{h\in I} \|x^h - \mathbf{1}\| < \sum_{h\in I} \lambda^h u^h(y^h) - \delta \sum_{h\in I} \|y^h - \mathbf{1}\|.$$

As before, it is a contradiction. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. The proof is an application of the KKM lemma: see Theorem 5.1 in Border (1989).

By Lemmas 1 and 2, $\{\Lambda^i\}_{i=1}^n$ is a KKM covering of Δ. So there exists $\lambda^*_\varepsilon \in \cap_{i=1}^n \Lambda^i$. Let $x^*_\varepsilon = \phi(\lambda^*_\varepsilon)$. Then x^*_ε is ε-individually rational, ε-Pareto optimal and has no equal-type ε-justified envy.

Now let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n > 0$ for all n and $\varepsilon_n \to 0$. Let x^*_n be the allocation found above for each ε_n. Since the sequence $\{x^*_n\}$ is bounded, it has a subsequence $\{x^*_{n_k}\}$ that converges to some x^*. Since the set of feasible allocations is closed, x^* is a feasible allocation. We prove that x^* is individually rational, weak Pareto optimal and has no strong equal-type justified envy.

Since $u^i(x^{*i}_{n_k}) \geq \tilde{u}^i - \varepsilon_{n_k}$ for all n_k and all i, in the limit $u^i(x^{*i}) \geq \tilde{u}^i$ for all i. So x^* is individually rational. Suppose x^* is not weak Pareto optimal, then there exists a feasible allocation y such that $u^i(y^i) > u^i(x^{*i})$ for all i. For big enough n_k, $u^i(y^i) > u^i(x^{*i}_{n_k}) + \varepsilon_{n_k}$ for all i, which contradicts the ε_{n_k}-Pareto optimality of $x^*_{n_k}$. Suppose some agent i has strong equal-type justified-envy towards another agent j in x^*; that is, $u^i(x^{*j}) > u^i(x^{*i})$ and $u^j(x^{*i}) > \tilde{u}^j$. Then for big enough n_k, $u^i(x^{*j}_{n_k}) > u^i(x^{*i}_{n_k})$ and $u^j(x^{*i}_{n_k}) > \tilde{u}^j - \varepsilon_{n_k}$. But given that i and j are of equal type, this contradicts the property of no equal-type ε_{n_k}-justified envy of $x^*_{n_k}$. □

Remark 1 (Proof of Theorem 1). When there are only individual constraints, we remove the "equal type" qualification. All arguments will proceed as before.

[12] We can consider a sequence of allocations $\{x(k)\}_{k=0}^{K-1}$ with $x(0) = x$ and $x(k) = x_{i_k \leftrightarrow i_{k+1}}(k-1)$ for each $1 \leq k \leq K-1$. Since all agents in the cycle are of equal type, each $x(k) \in \mathcal{A}^{\mathcal{C}}$. We let $y = x(K-1)$.

9. Proof of Theorem 2

We let the L-dimensional simplex Δ^L be the domain of prices.

9.1. **Incomes.** The key to the theorem is to carefully construct price-dependent income functions. For each consumer i, define i's *expenditure function* as

$$e^i(v,p) = \inf\{p \cdot x : x \in C^i, u^i(x) \geq v\},$$

for $p \in \Delta^L$ and $v \in \mathbf{R}$.

For any scalar $m \geq 0$ and $p \in \Delta^L$, let

$$\mu^i(m,p) = \text{median}(\{e^i(\tilde{u}^i,p), m, e^i(v^i,p)\}).$$

Consider the function

$$\varphi(m,p) = \sum_i \mu^i(m,p) - p \cdot Q.$$

Observe that

- $e^i(\tilde{u}^i,p) \leq e^i(v^i,p)$.
- μ^i is continuous and $m \mapsto \mu^i(m,p)$ weakly monotone increasing.
- φ is continuous and $m \mapsto \varphi(m,p)$ weakly monotone increasing.
- $\varphi(m,p) \leq 0$ for $m \geq 0$ small enough as $\sum_i e^i(\tilde{u}^i,p) \leq p \cdot Q$ (since an IR allocation x exists, $e^i(\tilde{u}^i,p) \leq p \cdot x^i$ for all i, and $\sum_i p \cdot x^i = p \cdot Q$.)

We shall define $m^i(p)$. First, in the case that $\sum_i e^i(v^i,p) < p \cdot Q$, we let $m^i(p) = e^i(v^i,p) + \frac{1}{N}[p \cdot Q - \sum_i e^i(v^i,p)]$. Second, in the case that $\sum_i e^i(v^i,p) \geq p \cdot Q$, we have that $\varphi(m,p) \leq 0$ for $m \geq 0$ small enough, and $\varphi(m,p) \geq 0$ for $m \geq 0$ large enough. Therefore there exists $m^* \geq 0$ with $\varphi(m^*,p) = 0$.

Now let $m^i(p) = \mu^i(m^*,p)$. To show that this is well defined, we need to prove that $m^i(p)$ is independent of the choice of m^*. To that end, suppose that there are $m_1, m_2 \in \mathbf{R}_+$ with $m_1 \neq m_2$ and $0 = \varphi(m_1,p) = \varphi(m_2,p)$. Suppose without loss of generality that $m_1 < m_2$. Now, since each μ^i is weakly monotone increasing as a function of m, we must have $\mu^i(m_1,p) = \mu^i(m_2,p)$ for all i. Then the definition of $m^i(p)$ is the same regardless of whether we choose m_1 or m_2.

Note that, in all cases, $p \cdot Q = \sum_i m^i(p)$.

Lemma 3. *m^i is continuous.*

Proof. Let $p^n \to p \in \Delta^L$. Note that if $\sum_i e^i(v^i,p) - p \cdot Q < 0$, then for n large enough we will have $\sum_i e^i(v^i,p^n) - p^n \cdot Q < 0$. Then $m^i(p^n) = e^i(v^i,p^n) + \frac{1}{N}[p^n \cdot$

$Q - \sum_i e^i(v^i, p^n)] \to e^i(v^i, p) + \frac{1}{N}[p \cdot Q - \sum_i e^i(v^i, p)] = m^i(p)$, by continuity of the expenditure function.

So suppose that $\sum_i e^i(v^i, p) - p \cdot Q \geq 0$, and let m be such that $\varphi(m, p) = 0$. We shall discuss two cases.

Case1: Consider the case that $\sum_i e^i(v^i, p^{n_k}) - p^{n_k} \cdot Q < 0$ for some subsequence p^{n_k}. Then $\sum_i e^i(v^i, p) - p \cdot Q = 0$. This means that if $\varphi(m, p) = 0$ then $m \geq e^i(v^i, p)$ for all i. Hence $m^i(p) = e^i(v^i, p)$ for all i. But since $m^i(p^{n_k}) = e^i(v^i, p^{n_k}) + \frac{1}{N}[p^{n_k} \cdot Q - \sum_i e^i(v^i, p^{n_k})]$, we get that $m^i(p^{n_k}) \to m^i(p)$.

Case 2: Now turn to a subsequence p^{n_k} with $\sum_i e^i(v^i, p^{n_k}) - p^{n_k} \cdot Q \geq 0$. Then there is m^{n_k} with $\varphi(m^{n_k}, p^{n_k}) = 0$. We can take this sequence to be bounded: consider any further convergent subsequence $m^{n'_k}$ and say that $m^{n'_k} \to m'$. Then $0 = \varphi(m^{n'_k}, p^{n'_k}) \to \varphi(m', p)$. Thus $m^i(p^{n'_k}) = \mu^i(m^{n'_k}, p^{n'_k}) \to \mu^i(m', p)$, as μ^i is continuous. Since the sequence $\{m^{n_k}\}$ is bounded, this implies that $m^i(p^{n_k}) \to m^i(p)$.

Cases 1 and 2 exhaust all possible subsequences of p^n. □

The role of the following lemma will be clear towards the end of the proof.

Lemma 4. *If $m^i(p) < \min\{m^j(p), e^i(v^i, p)\}$ then $m^j(p) = e^j(\tilde{u}^j, p)$.*

Proof. Since $m^i(p) < e^i(v^i, p)$, we must be in the case $\sum_i e^i(v^i, p) \geq p \cdot Q$ of the definition of income functions. So let $m^* \geq 0$ with $\varphi(m^*, p) = 0$.

Since $m^i(p) = \mu^i(m^*, p) < e^i(v^i, p)$, we must have $m^* \leq m^i(p)$. By hypothesis, $m^* < m^j(p)$. Then $m^j(p) = \mu^j(m^*, p)$ implies that $m^j(p) = e^j(\tilde{u}^j, p)$. □

9.2. **Existence of quasi-equilibrium.** We first establish the existence of a quasiequilibrium with $p^* \neq 0$. The argument is similar to Gale and Mas-Colell (1975). See also Mas-Colell, Whinston, Green, et al. (1995) (Chapter 17, Appendix B).

For any $p \in \Delta^L$, let $\underline{d}^i(p)$ be the set of vectors $x^{i\prime} \in C^i$ that satisfy the following properties:

$$\begin{aligned} p \cdot x^{i\prime} &\leq m^i(p) \\ u^i(x^{i\prime}) &\geq u^i(\hat{x}^i) \text{ for all } \hat{x}^i \in C^i \text{ with } p \cdot \hat{x}^i < m^i(p). \end{aligned}$$

We consider the correspondence $p \mapsto \underline{d}^i(p)$ with domain in Δ^L.

Observe that $\varnothing \neq \arg\max_{x^{i\prime} \in C^i} \{u^i(x^{i\prime}) : p \cdot x^{i\prime} \leq m^i(p)\} \subseteq \underline{d}^i(x, p))$. So $\underline{d}^i$ takes non-empty values.

Observe also that $\underline{d}^i$ is convex valued. To see this, let $z^i, y^i \in \underline{d}^i(p)$ and define $x^i(\alpha) = \alpha z^i + (1-\alpha)y^i$, for $\alpha \in [0,1]$. It is obvious that $x^i(\alpha) \in C^i$ and that $p \cdot x^i(\alpha) \leq m^i(p)$. For any $\hat{x}^i \in C^i$ with $p \cdot \hat{x}^i < m^i(p)$, $\min\{u^i(z^i), u^i(y^i)\} \geq u^i(\hat{x}^i)$ and quasi-concavity of u^i imply that $u^i(x^i(\alpha)) \geq u^i(\hat{x}^i)$. Thus $x^i(\alpha) \in \underline{d}^i(p)$.

A third observation is that $\underline{d}^i(p)$ is upper-hemicontinuous. To this end, consider a sequence p_n in Δ^L with $p_n \to p \in \Delta^L$. Consider $z^i_n \in \underline{d}^i(p_n)$ such that $z^i_n \to z^i$. Clearly, $z^i \in C^i$ and $p \cdot z^i \leq m^i(p)$ as m^i is continuous (Lemma 3). Moreover, for any $\hat{x}^i \in C^i$ with $p \cdot \hat{x}_i < m^i(p)$, we have that $p_n \cdot \hat{x}_i < m^i(p_n)$ for n large enough (again by Lemma 3). Thus $u^i(z^i_n) \geq u^i(\hat{x}^i)$ for n large enough, which by continuity of u^i implies that $u(z^i) \geq u^i(\hat{x}^i)$. Hence $z^i \in \underline{d}^i(p)$.

For any $x \in \times_i C^i$ and $p \in \Delta^L$, let

$$\bar{\pi}(x,p) = \text{argmax}\{p \cdot \left(\sum_i x^i - Q\right) : p \in \Delta^L\}.$$

Consider the correspondence

$$\xi : \times_i C^i \times \Delta^L \twoheadrightarrow \times_i C^i \times \Delta^L$$

defined by $\xi(x^1, \ldots, x^N, p) = (\times_i \underline{d}^i(p)) \times \bar{\pi}(x,p)$.

By the previous observations, and the maximum theorem, ξ satisfies the hypotheses of Kakutani's fixed point theorem. Let (x^*, p^*) be a fixed point of ξ.

We argue that (x^*, p^*) is a Walrasian quasiequilibrium. We have that $p^* \cdot x^{i*} \leq m^i(p^*)$ for every i, by construction of ξ. By definition of m^i, we have $\sum_i m^i(p^*) = p^* \cdot Q$. Hence, $p^* \cdot (\sum_i x^{i*} - Q) \leq 0$. This implies $\sum_i x^{i*} - Q \leq 0$ since otherwise, by definition of $\bar{\pi}$, we would have

$$p^* \cdot \left(\sum_i x^{i*} - Q\right) = \max_{p' \in \Delta^L} \{p' \cdot \left(\sum_i x^{i*} - Q\right)\} > 0.$$

We show that $\sum_i x^{i*} - Q = 0$. We first consider the case $\sum_i e^i(v^i, p^*) < p^* \cdot Q$. By definition of m^i, all agents i have $m^i(p^*) > e^i(v^i, p^*)$ so they must be satiated following the definition of $\underline{d}^i$. By monotonicity of preferences, we observe $\sum_l x^{i*}_l = c^i$. Hence

$$\sum_i \sum_l x^{i*}_l = \sum_i c^i \geq \sum_l q_l$$

where the inequality comes from the no overall excess supply assumption. Given that we knew $\sum_i x^{i*} - Q \leq 0$, we conclude $\sum_i x^{i*} - Q = 0$.

We then consider the case $\sum_i e^i(v^i, p^*) \geq p^* \cdot Q$. We claim that $p^* \cdot x^{i*} = m^i(p^*)$ for every i, since by definition of m^i we have $m^i(p^*) \leq e^i(v^i, p^*)$. Indeed, suppose that

$p^* \cdot x^{i*} < m^i(p^*) \leq e^i(v^i, p^*)$. Since x^{i*} does not satiate the agent, for an arbitrarily small ball B around x^{i*} there is $x^{i\prime} \in B$ with $u^i(x^{i\prime}) > u^i(x^{i*})$ and $p^* \cdot x^{i\prime} < m^i(p^*)$, contradicting $x^{i*} \in \underline{d}^i(x^*, p^*)$. Observe that, as a consequence of the above,

$$p^* \cdot Q = \sum_i m^i(p^*) = \sum_i p^* \cdot x^{i*}. \tag{2}$$

Consequently, $p^* \cdot (\sum_i x^{i*} - Q) = 0$. Since $\sum_i x^{i*} - Q \leq 0$, we obtain $p_l^* = 0$ for any l with $\sum_i x_l^{*i} - q_l < 0$ (underdemanded objects). Then, since preferences are monotonic, it is without loss of generality to assume that $\sum_i x^{i*} - Q = 0$ by consuming the remaining units of underdemanded objects for free.

This proves that (x^*, p^*) is a Walrasian quasiequilibrium.

9.3. **Existence of equilibrium.** We prove now that (x^*, p^*) is a Walrasian equilibrium in the cases considered in the Theorem. In all cases we prove that, for each agent i, either $m^i(p^*) > 0$ or else the null bundle 0 is the only affordable one. A standard argument follows converting such quasiequilibrium into an equilibrium. Suppose $m^i(p^*) > 0$ and there exists $y^i \in C_i$ such that $u^i(y^i) > u^i(x^{i*})$ and $p^* \cdot y^i \leq m^i(p^*)$. Then, for $\lambda < 1$ sufficiently close to 1, $\lambda y^i \in C_i$, $p^* \cdot \lambda y^i < m^i(p^*)$ and, by continuity of preferences, $u^i(\lambda y^i) > u^i(x^{i*})$, contradicting $x^{i*} \in \underline{d}^i(p^*)$. For the remaining case $m^i(p^*) = 0$, if 0 is the sole affordable bundle, then $0 = x^{*i}$ trivially is i's optimal choice subject to her budget constraint.

In all cases, we skip the possibility of $\sum_i e^i(v^i, p^*) < p^* \cdot Q$. By definition of m^i, all agents i would have $m^i(p^*) > e^i(v^i, p^*)$, so they would certainly be satiated following the definition of $\underline{d}^i$. Therefore we would trivially have an equilibrium. Note that, by skipping such a possibility, we have $p^* \cdot x^{i*} = m^i(p^*)$ for all i.

We first consider the case $\min_i c^i > \sum_l q_l$. We show that $p^* >> 0$. Suppose, by way of contradiction, that $p_l^* = 0$ for some good l. Since $p^* \cdot Q > 0$ and $\sum_i x^{i*} - Q = 0$, there must be an individual j with $p^* \cdot x^{j*} = m^j(p^*) > 0$. Take a vector δ containing zeros in all coordinates but l, where it contains $\epsilon > 0$. Notice that $c^j > \sum_l q_l$ implies $x^{j*} + \delta \in C^i$ for ϵ small enough. By monotonicity and continuity of preferences, and since $x^{j*} + \delta$ is also affordable, a standard argument shows that x^{j*} is not a quasiequilibrium allocation for j under prices p^*. We conclude that $p^* >> 0$. Consequently, for each agent i with $m^i(p^*) = 0$, the 0 bundle is her only affordable bundle.

We now consider the case when both Inada condition $u^i(x^i) = u^i(0)$ unless $x^i >> 0$ and $\tilde{u}^i > u^i(0)$ hold for all i. $p^* \in \Delta^L$ contains at least one strictly positive price, thus $m^i(p^*) \geq e^i(\tilde{u}^i, p^*) > 0$ for all i.

We consider the case in which satiation implies a strictly positive bundle for each satiated agent, and moreover there is a strictly positive IR allocation, namely $\tilde{x}$. Since satiation only happens for strictly positive bundles and $p^* \in \Delta$, we have $e^i(v^i, p^*) > 0$ for all i.

Our next step is to establish that $\underline{m} = \min\{m^i(p^*) : 1 \leq i \leq I\} > 0$. First, if $\underline{m} = \min\{e^i(v^i, p^*) : 1 \leq i \leq I\}$ then we are done because $e^i(v^i, p^*) > 0$ for all i. Ruling out this case, there must exist i with $m^i(p^*) < e^i(v^i, p^*)$, which implies that $\sum_i m^i(p^*) = p^* \cdot Q = \sum_i p^* \cdot \tilde{x}^i$. Now, if

$$\underline{m} = \min\{e^i(\tilde{u}^i, p^*) : 1 \leq i \leq I\}, \tag{3}$$

then there is h with $\underline{m} = m^h(p^*) \leq e^i(\tilde{u}^i, p^*)$ for all i; which implies by the definition of the income functions that $m^i(p^*) = e^i(\tilde{u}^i, p^*)$ for all i. But $e^i(\tilde{u}^i, p^*) \leq p^* \cdot \tilde{x}^i$ for all i and

$$\sum_i e^i(\tilde{u}^i, p^*) = \sum_i m^i(p^*) = p^* \cdot Q = \sum_i p^* \cdot \tilde{x}^i$$

imply that $e^i(\tilde{u}^i, p^*) = p^* \cdot \tilde{x}^i$ for all i. So $m^i(p^*) = p^* \cdot \tilde{x}^i$ for all i. Because $\tilde{x}^i >> 0$, $m^i(p^*) > 0$ for all i.

Finally, if Equation (3) does not hold, then $0 \leq \min\{e^i(\tilde{u}^i, p^*) : 1 \leq i \leq I\} < \underline{m}$. So $m^i(p^*) > 0$ for all i.

Lastly, we consider the case that a common favorite object l and a strictly positive IR allocation $\tilde{x}$ both exist. We argue that $p^*_l > 0$. Suppose that $p^*_l = 0$. Since $p^* \in \Delta^L$, there must be an object $k \neq l$ with $p^*_k > 0$. For any agent i who is consuming object k, substituting his consumption of object k for an equal consumption of object l saves expenses and increases utility. Hence $x^{i*} \notin \underline{d}^i(p^*)$. This contradiction shows that $p^*_l > 0$. Notice that, in consequence, $e^i(v^i, p^*) > 0$ for all i. The rest of the proof for this case just mimics that of the previous case (i.e. satiation implies strictly positive bundle.)

9.4. **Properties of a competitive equilibrium allocation x^*.**

9.4.1. *Pareto optimality.* We disregard the case $\sum_i e^i(v^i, p^*) < p^* \cdot Q$ in which clearly every agent i is satiated since $m^i(p^*) > e^i(v^i, p^*)$. In the cases that follow below, any satiated agent i must have $m^i(p^*) = e^i(v^i, p^*)$. Suppose that $y^i \in C^i$ and that $u^i(y^i) \geq u^i(x^{i*})$. Then we must have $p^* \cdot y^i \geq m^i(p^*)$ because otherwise $p^* \cdot y^i <$

$m^i(p^*) \leq e^i(v^i, p^*)$ and if i is satiated, then $p^* \cdot y^i < e^i(v^i, p^*)$ and $u^i(y^i) \geq v^i$ contradicts the definition of e^i; if i is not satiated, there would exists z^i with $p^* \cdot z^i < m^i(p^*)$ and $u^i(z^i) > u^i(y^i) \geq u^i(x^{i*})$.

Obviously if $y^i \in C^i$ and $u^i(y^i) > u^i(x^{i*})$ then $p^* \cdot y^i > m^i(p^*)$. So if $y = (y^i)$ Pareto dominates x^* then $\sum_i p^* \cdot y^i > \sum_i m^i(p^*)$. But by Equation (2) this is impossible if y is an allocation.

9.4.2. *Individual rationality.* To show that x^* is individually rational it suffices to notice that $m^i(p^*) \geq e^i(\tilde{u}^i, p^*)$ for all i.

9.4.3. *No justified envy.* Suppose that i envies j at x^*. This implies that i is not satiated, hence $m^i(p^*) < e^i(v^i, p^*)$. It also implies that $m^i(p^*) < m^j(p^*)$ as $m^i(p^*) < p^* \cdot x^{j*} \leq m^j(p^*)$. By Lemma 4, then, $m^j(p^*) = e^j(\tilde{u}^j, p^*)$.

We obtain that

$$p^* \cdot x^{i*} = m^i(p^*) < m^j(p^*) = e^j(\tilde{u}^j, p^*),$$

and hence $u^j(x^i) < \tilde{u}^j$ by definition of expenditure function. So i's envy is not justified.

10. Proof of Theorem 3

We denote a constant of Lipschitz continuity common to all utility functions with θ. Let y be an IR allocation, which exists by assumption. Consider an additional object $e \notin O$, and an α-extended economy, for any $\alpha \in (0,1)$, with the capacity vector Q^α:

$$q_l^\alpha = (1-\alpha) q_l, \ \forall l \in O, \text{and } q_e^\alpha = \alpha \sum_i c_i.$$

Preferences for each agent $i \in I$ in this extended economy are defined to be:

$$U^i((x_l)_{l \in O}, x_e) = u^i((x_l)_{l \in O}) + \theta x_e.$$

along the extended consumption space $\Delta_-^{L+1}(c^i)$. Notice that under this construction, e is a common favorite good in this extended economy. By Lipschitz continuity, the allocation y^α with $y_l^{i\alpha} = (1-\alpha) y_l^i$ for $l \in O$ and $y_e^{i\alpha} = \alpha c_i$ meets $U^i(y_i^\alpha) > \tilde{u}^i$ for all $i \in I$. Consider the allocation z^α with $z^{i\alpha} = c^i Q^\alpha / \sum_j c^j >> 0$. By continuity of preferences, for $\beta > 0$ low enough, the allocation $\beta z^\alpha + (1-\beta) y^\alpha$ is a strictly positive IR allocation in the extended economy.

Therefore, by Theorem 2, each α-extended economy contains a Pareto-optimal, IR and NJE allocation x^α. We construct a sequence $(x^\alpha)_\alpha$ where α tends to zero.

Wlog such sequence converges to some allocation x^* (if not, a subsequence does.) Such limit is an allocation in the original economy. Obviously, $x_e^{i*} = 0$ for all i. Abusing notation, we ignore in what follows the dimension of the common favorite good.

In order to show feasibility of some useful allocations in the proof, we make use of the following lemma about convex polytopes:

Lemma 5. *Let $P \subset \mathbf{R}^n$ be a convex polytope of dimension n. Let x, y be different elements of P. Let the sequence (x_t) be formed by elements of P and converge to x. Then for each t high enough there exists some $\varepsilon \in (0,1)$ such that $x_t + \varepsilon'(y - x) \in P$ for all $\varepsilon' \in (0, \varepsilon)$.*

Proof. Suppose not. Then, there is a subsequence from (x_t), say (x_τ), such that for every $\varepsilon \in (0,1)$ we have $x_\tau + \varepsilon(y - x) \notin P$. Now, P is the intersection of a finite set of half-spaces, $\{H_1^+, ..., H_m^+, ..., H_M^+\}$, each one linked to a corresponding $n-1$ dimensional hyperplane $H_m \subset H_m^+$. Each hyperplane is a superset of a facet (or $n-1$ dimensional face) of P, denoted with $F_m \subset H_m$.

Since $x_\tau + \varepsilon(y - x) \notin P$ for all $\varepsilon \in (0,1)$, x_τ is not in the interior of P, belonging then to at least one facet $F_m \subset H_m$ for which $x_\tau + \varepsilon(y - x) \notin H_m^+$ for all $\varepsilon \in (0,1)$. Notice that if one point of H_m has this property, all of them do.

Since the number of facets is finite, we can always prune the subsequence such that all of its elements are linked to the same facet $F_m \subset H_m$ and the same property ($x_\tau + \varepsilon(y-x) \notin H_m^+$ for all $\varepsilon \in (0,1)$.) But then, by convergence of the subsequence, $x \in F_m \subset H_m$, and then $x + \varepsilon(y - x) \notin H_m^+$ for all $\varepsilon \in (0,1)$. In contradiction with $y \in P$ and convexity of P. This completes the proof. □

x^* **is consistent weak Pareto optimal.** Suppose not, then there is an allocation x' and a nonempty subset of agents $I' \subset I$ such that 1) $x^{i'} = x^{i*}, \forall i \notin I'$ and 2) $u^i(x^{i'}) > u^i(x^{i*}), \forall i \in I'$.

For any $\varepsilon \in (0,1)$, we explore allocations of the type

$$x_\varepsilon'^{\alpha} = x^\alpha + \varepsilon(x' - x^*).$$

Observe that $x_\varepsilon'^{\alpha} \to x^* + \varepsilon(x' - x^*)$ as $\alpha \to 0$. With semi-strict quasiconcave utilities, $x^* + \varepsilon(x' - x^*)$ has the same properties with respect to x^* as x' does: for the same subset of agents $I' \subset I$ we have 1) $x^{i'} = x^{i*}, \forall i \notin I'$ and 2) $u^i(x^{i'}) > u^i(x^{i*}), \forall i \in I'$.

There is a subsequence $(\alpha, \varepsilon_\alpha)$, with $\alpha \to 0$, such that for every $\varepsilon < \varepsilon_\alpha$, x'^α_ε is an allocation in its corresponding α-extended economy. Lemma 5 shows that for ε small enough, $x^{i\prime\alpha}_\varepsilon \in C^i$ for all i, since each C^i is a full-dimensional convex polytope.

Now, given that

$$\begin{aligned}\sum_i x^{i\prime\alpha}_\varepsilon &= \sum_i x^{i\alpha} + \varepsilon(\sum_i x^{i\prime} - \sum_i x^{i*}) \\ &= Q^\alpha + \varepsilon(Q - Q) = Q^\alpha,\end{aligned}$$

the market clearing aspect of being an allocation is met.

Then, for α close enough to 0 and ε small enough, x'^α_ε is an allocation.

Note that, no matter α and ε, for all agents $i \notin I'$ we have $x^{i\prime\alpha}_\varepsilon = x^{i\alpha}$. As for agents $i \in I'$, by continuity of preferences, and as $x'^\alpha \to x'$, we have that, for all α low enough, either $x^{i\prime\alpha}_\varepsilon \notin C^i$ or $U^i(x^{i\prime\alpha}_\varepsilon) > U^i(x^{i\alpha})$. Yet we have seen that we can select one of these α and some ε such that x'^α_ε is an allocation. Therefore, there is some α such that x^α is not constrained weak Pareto optimal, contradicting that it is Pareto optimal.

x^* **is individually rational**, since $U^i(x^\alpha_i) \geq \tilde{u}_i$ for all $i \in I$ and all α.

x^* **has no strong justified envy.** Suppose not. Then, some agent i envies some other agent j at x^* and $u^j(x^{i*}) > \tilde{u}^j$. For α low enough, and by continuity of preferences, i envies j at x^α and $U^j(x^{i\alpha}) > \tilde{u}^j$. But this contradicts the fact that x^α satisfies NJE.

If x^* is Pareto optimal then it has no justified envy. Suppose that agent i envies j at x^*. Taking the converging sequence $x^\alpha \to x^*$ as previously defined, for all α low enough we have that i envies j at x^α. Being (x^α, p^α) the corresponding competitive equilibrium obtained by our procedure, by Lemma 6 we have $0 \leq m^i(p^\alpha) < m^j(p^\alpha) = e^j(\tilde{u}^j, p^\alpha)$, where e^j is defined along the consumption space $\Delta^{L+1}_-(c^j)$. Since $\max\{U^j(x^j) : x^j \in \Delta^{L+1}_-(c^j), p^\alpha \cdot x^j \leq m\}$ is continuous in $m > 0$, we obtain $U(x^{j\alpha}) = \tilde{u}^j$, for all α low enough. By continuity of j's utility function, we conclude that $u^j(x^{j*}) = \tilde{u}^j$.

Suppose that such envy is justified. Then $x^i \in C^j$ and $u^j(x^{i*}) \geq \tilde{u}^j$. But then, by letting agents i and j swap their respective bundles, we reach a Pareto improvement, contradicting that x^* is Pareto optimal.

If each u^i is locally either linear or strictly quasi-concave around x^{i*}, then x^* is Pareto optimal and with no justified envy. Suppose that x^* is not Pareto optimal. We know it is consistent weak Pareto optimal. Then there must be an allocation x' and nonempty strict subsets $I'' \subset I' \subset I$ with the following properties: 1) $x^{i'} = x^{i*}, \forall i \in I \setminus I'$, 2) $x^{i'} \neq x^{i*}$ and $u^i(x^{i'}) = u^i(x^{i*}), \forall i \in I' \setminus I''$, 3) $u^i(x^{i'}) > u^i(x^{i*}), \forall i \in I''$.

Let J denote the set of agents in $I' \setminus I''$ with linear preferences on a neighborhood $N^i \subset C^i$ of x^{i*}. By assumption, agents $i \in (I' \setminus I'') \setminus J$ have strict quasi-concave utility functions on a neighborhood $N^i \subset C^i$ of x^{i*}. Note that, by choosing an allocation z with $z^i \in N^i, \forall i \in I$, as a non-degenerate convex combination of x^* and x', we obtain that 1) $z^i = x^{i*}, \forall i \notin I'$, 2) $u^i(z^i) = u^i(x^{i*}), \forall i \in J$, 3) $u^i(z^i) > u^i(x^{i*}), \forall i \in I'' \cup (I' \setminus J)$. (Particularly, this implies that $J = \emptyset$ contradicts that x^* is consistent weak Pareto optimal.) So it is without loss of generality to assume that $J = I' \setminus I''$.

We study allocations of the type

$$x_\varepsilon'^\alpha = x^\alpha + \varepsilon(x' - x^*),$$

with $\varepsilon \in (0,1)$, as in the proof of Statement (1). We had seen there that, for α and ε low enough, $x^\alpha + \varepsilon(x' - x^*)$ is indeed an allocation. Moreover, for α and ε low enough, and for every $i \in I$, both $x_\varepsilon^{i\prime\alpha} \in N^i$ and $x^{i\alpha} \in N^i$.

For $i \notin I'$, $x_\varepsilon^{i\prime\alpha} = x^{i*}$. For $i \in J$, $U^i(x_\varepsilon^{i\prime\alpha}) = U^i(x^{i\alpha})$, given linearity of preferences on N^i. For $i \in I''$, $U^i(x_\varepsilon^{i\prime\alpha}) > U^i(x^{i\alpha})$, by continuity of preferences, when α is low enough. This together contradicts the fact that x^α must be Pareto efficient, hence concluding the proof.

No justified envy follows from the above proven fact that Pareto optimality of x^* implies no justified envy.

11. Proof of Theorem 5

11.1. **Theorem 1.** To prove that Theorem 1 holds as before when NJE is extended, we omit the steps in common and highlight the differences from the previous proof. Let Λ^i be the set of all $\lambda \in \Delta$ at which i has no ε-justified envy by exchange towards any agent at $\phi(\lambda)$. We prove that the collection of Λ^i is still a KKM covering of Δ.

Lemma 6. *For every $i \in I$, Λ^i is closed.*

Proof. Let λ_n be a sequence in Λ^i such that $\lambda_n \to \lambda \in \Delta$. Let $x_n = \phi(\lambda_n)$. By continuity of ϕ, $x_n \to x = \phi(\lambda) \in \mathcal{A}^*$. Now we prove that $\lambda \in \Lambda^i$, that is, i does not have ε-justified envy by exchange towards any other agent at $\phi(\lambda)$. Suppose that this is not the case. Then i has ε-justified envy by exchange towards some agent j, with the sequence $(i_k)_{k=1}^K$ being as in the definition of such envy. By continuity of utility, and since the sequence $(i_k)_{k=1}^K$ is finite, for n large enough we have $u^{i_k}(x_n^{i_{k+1}}) > u^{i_k}(x_n^{i_k})$ for $1 \leq k \leq K-1$ while $u^{i_K}(x_n^{i_1}) > \tilde{u}^{i_K} - \varepsilon$. So i has ε-justified envy by exchange towards j at x_n, which is a contradiction. Therefore, $\lambda \in \Lambda^i$ and Λ^i is closed. □

Lemma 7. *For every $\lambda \in \Delta$, $\lambda \in \cup_{i \in supp(\lambda)} \Lambda^i$.*

Proof. Suppose, towards a contradiction, that for some $\lambda \in \Delta$, $\lambda \notin \cup_{i \in \text{supp}(\lambda)} \Lambda^i$. Let $x = \phi(\lambda)$. Then for every $i \in \text{supp}(\lambda)$, there exists some j such that i has ε-justified envy by exchange towards j at x. Suppose first that there exists such j, with corresponding sequence $(i_k)_{k=1}^K$, in which $\lambda^{i_K} = 0$. Let y be the allocation obtained from x by letting each i_k get $x^{i_{k+1}}$ $(1 \leq k \leq K-1)$ and i_K get x^i. Clearly y is ε-IR, as x was ε-IR and $u^{i_K}(x^i) > \tilde{u}^{i_K} - \varepsilon$. Note that $\lambda^{i_K} = 0$ and $u^{i_k}(y^{i_k}) > u^{i_k}(x^{i_k})$ for all $1 \leq k \leq K-1$ imply that $\sum_{h \in I} \lambda^h u^h(x^h) < \sum_{h \in I} \lambda^h u^h(y^h)$. We also have that $\sum_{h \in I} \|x^h - \mathbf{1}\| = \sum_{h \in I} \|y^h - \mathbf{1}\|$. Hence

$$\sum_{h \in I} \lambda^h u^h(x^h) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(y^h) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\|,$$

which contradicts the definition of $x = \phi(\lambda)$.

The above argument means that every $i \in \text{supp}(\lambda)$ has ε-justified envy by exchange towards some agent j, with corresponding sequence $(i_k)_{k=1}^K$ in which $\lambda^{i_K} > 0$. Thus, $i_K \in \text{supp}(\lambda)$. But it means that i_K also has ε-justified envy by exchange towards some agent j', with corresponding sequence $(i'_k)_{k=1}^K$ in which $\lambda^{i'_K} > 0$. Since the set of agents in $\text{supp}(\lambda)$ is finite, there must exist a subset of agents $\{h_1, \dots h_M\} \subseteq \text{supp}(\lambda)$ such that h_1 has ε-justified envy by exchange towards some agent with h_2 being the end of the corresponding sequence, h_2 has ε-justified envy by exchange towards some agent with h_3 being the end of the corresponding sequence, and so on until h_M has ε-justified envy by exchange towards some agent with h_1 being the end of the corresponding sequence. We write this situation as the following cycle

$$h_1 \to \cdots \to h_2 \to \cdots \to h_3 \to \cdots \to \cdots \to h_M \to \cdots \to h_1,$$

where $a \to b$ means that a envies b, and $h_k \to \cdots \to h_{k+1}$ is the corresponding sequence of h_k's ε-justified envy by exchange towards some agent. Now note that if an agent h appears more than once in the above cycle, we can shorten the cycle by skipping the agents between any two consecutive positions of h in the cycle. So we can, without loss of generality, focus on the cycle in which each agent appears once. If we carry out the exchange in the cycle as in the proof of Lemma 2, then we obtain an improvement on the objective that defines ϕ. This is a contradiction. $\square$

The remaining part of the proof is the same as before.

11.2. **Theorem 2.** Let (x, p) be an equilibrium in Theorem 2. Suppose some agent i has justified envy by exchange towards some agent j, with the sequence $(i_k)_{k=1}^K$ being as in the definition of such envy. By our construction of income functions, $m^i(p) < m^j(p) < m^{i_2}(p) < \cdots < m^{i_K}(p)$. So it must be that $m^{i_K}(p) = e^{i_K}(\tilde{u}^{i_K}, p)$. It means that x^i is not acceptable to i_K, which is a contradiction. So x satisfies no justified envy by exchange. Then the third statement of Theorem 1 can be proved as before.

11.3. **Theorem 3. x^* has no strong justified envy by exchange.** Suppose not. Then a cycle of agents $(i_k)_{k=1}^K$ exists such that $u^{i_k}(x^{i_{k+1}*}) > u^{i_k}(x^{i_k*})$ for all $1 \leq i < K$, and $u^{i_K}(x^{i_1*}) > \tilde{u}^{i_K}$. For α low enough, and by continuity of preferences, $u^{i_k}(x^{i_{k+1}\alpha}) > u^{i_k}(x^{i_k\alpha})$ for all $1 \leq i < K$, and $u^{i_K}(x^{i_1\alpha}) > \tilde{u}^{i_K}$. Now, x^α is a competitive equilibrium allocation constructed along the lines of Theorem 2. Therefore, we arrive to the same contradiction as in the preceding paragraph.

If x^* is Pareto optimal then it has no justified envy by exchange. Suppose not, thus a cycle of agents $(i_k)_{k=1}^K$ exists such that $u^{i_k}(x^{i_{k+1}*}) > u^{i_k}(x^{i_k*})$ for all $1 \leq i < K$, and $u^{i_K}(x^{i_1*}) \geq \tilde{u}^{i_K}$. Taking the converging sequence $x^\alpha \to x^*$ as defined in the proof of Theorem 3, for all α low enough we have that i_K is envied at x^α. Being (x^α, p^α) the corresponding competitive equilibrium obtained by our procedure, by Lemma 4 we have $0 \leq m^{i_1}(p^\alpha) < m^{i_K}(p^\alpha) = e^{i_K}(\tilde{u}^{i_K}, p^\alpha)$, where e^{i_K} is defined along the consumption space $\Delta_-^{L+1}(c^{i_K})$. Since $\max\{U^{i_K}(x^{i_K}) : x^{i_K} \in \Delta_-^{L+1}(c^j), p^\alpha \cdot x^{i_K} \leq m\}$ is continuous in $m > 0$, we obtain $U^{i_K}(x^{i_K\alpha}) = \tilde{u}^{i_K}$, for all α low enough. By continuity of i_K's utility function, we conclude that $u^{i_K}(x^{i_K*}) = \tilde{u}^{i_K}$.

But then, by letting all agents i_k swap their respective assignments so that each i_k obtains i_{k+1}'s assignment modulo K, we reach a Pareto improvement, contradicting that x^* is Pareto optimal.

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