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TESTABLE IMPLICATIONS OF GROSS SUBSTITUTES IN DEMAND FOR TWO GOODS

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Abstract. We present a non-parametric "revealed-preference test" for gross substitutes in demand for two goods.

1. Introduction

We study the testable implications of the property that a pair of goods are gross substitutes; strictly speaking, for the joint hypotheses of rationality and gross substitutes. We propose a non-parametric test for gross substitutes using expenditure data; the test is in the spirit of the revealed-preference tests first studied by Samuelson (1947) and Afriat (1967).

Two goods are gross substitutes if when the price of one good increases, demand for the other good increases. The usual way of testing for gross substitutes between goods a and b is by estimating the coefficient of the price of good a in a linear regression for the demand for b. We propose instead a non-parametric test; one that does not require assuming a functional form for the demand function or for the underlying preferences.

Our test is for a *pair* of goods: you can use it to test if coffee and tea are substitutes, for example. You cannot use it to test whether wine, beer, and whisky are substitutes. The parametric test we mentioned above is also a two-good exercise.

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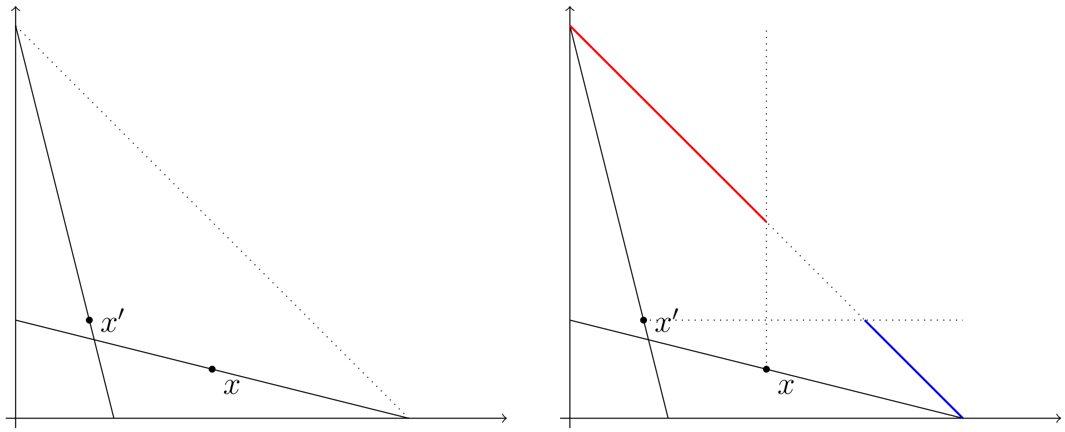
Consumers, of course, buy more than just two goods. But one can isolate a pair of goods, and test for gross substitutes, under some assumptions about the consumers' preferences. The assumptions are made routinely in applied studies of demand, see for example Deaton and Muellbauer (1980).[1] Applied researchers use aggregation and separability to study demand for a subset of broad categories of goods. They will, for example, aggregate different types of coffee into a composite coffee good. They assume that agents' preferences are separable, so that what matters for the purchases of coffee and tea is the money spent on other goods; not how many steaks, salads, or pizzas were bought. We stress that assumptions allowing for aggregation and separability are strong but well understood, and seem to be accepted by the community of researchers on applied demand.

Our test is simple. Given is a finite collection of observed demand choices at given prices. We want to reconcile the data with a demand function that satisfies gross substitutes and comes from a rational consumer. That is, we want to know when the data can be rationalized using a rational demand function with the substitutes property.

Consider the example in Figure 1. We have two observations: x is the bundle purchased at prices p, and x' is purchased at prices p'. These purchases do not violate gross substitutes. The observed choices are also consistent with the weak axiom of revealed preference, so there is an extension of these purchases to a rational demand function that is defined for all prices. There is, however, no extension to a demand function which satisfies gross substitutes: Consider the prices p'' given by the dotted budget line. Gross substitutes and the choice of x at p requires a decrease in the consumption of the good whose price is the same in p and in p'', so demand at p'' should lie in the red segment of the budget line. On the other hand, gross substitutes and x' requires that demand at p'' lies in the blue segment of the budget

[1]Deaton and Muellbauer (1980) explain how most consumption decisions involve in principle an unmanageably large number of goods and, simultaneously, an intertemporal and risk dimensions. They argue that all empirical studies of demand must simplify buy using aggregation and separability



(a) Two observations: (x, p) and (x', p'). (b) Implications of Gross Substitutes.

FIGURE 1. An example with two observations.

line. Since the red and blue segments are disjoint, there is no demand function that extends the data and satisfies gross substitutes.

Importantly, the example shows that gross substitutes (and the weak axiom of revealed preference) may be satisfied in the data, but the data may not be rationalizable by a demand function satisfying gross substitutes. Our test is based on expenditure shares. We observe that gross substitutes of a demand function is equivalent to the monotonicity of expenditure shares: Demand satisfies gross substitutes if and only if the share of expenditure corresponding to good a increases as the price of good b increases. Our test is simply to verify that the data satisfies the monotonicity of expenditure shares. That is, if the shares in the data are monotonic then they can be extended to a full demand function defined for all prices, and the demand so defined will be smooth, rational, and induce monotonic expenditure shares.

The problem of gross substitutes is thus much simpler than the problem of testing for complements, which we have studied elsewhere (Chambers, Echenique, and Shmaya, 2008). We note that the current paper only deals with the testable implications of substitutes, not the preferences that generate substitutes. The class of preferences generating substitutes is known from the work of Fisher (1972).

The rest of the paper is organized as follows: Section 2 describes the notation and gives our main definitions; Section 3 presents our result; the proof is in Section 4. Finally, in Section 5 we present a discussion of our results.

2. Preliminaries

Let $\mathbb{R}^2_+$ be the domain of consumption bundles, and $\mathbb{R}^2_{++}$ the domain of possible prices. We use standard notational conventions: $x \leq y$ if $x_i \leq y_i$ in $\mathbb{R}$, for $i = 1, 2$; $x < y$ if $x \leq y$ and $x \neq y$; and $x \ll y$ if $x_i < y_i$ in $\mathbb{R}$, for $i = 1, 2$. We write $x \cdot y$ for the inner product $x_1 y_1 + x_2 y_2$.

A function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is *increasing* if $x \leq y$ implies $u(x) \leq u(y)$. It is *decreasing* if $(-u)$ is increasing. Let $A \subseteq \mathbb{R}^2$ be open. A function $u : A \to \mathbb{R}$ is *smooth* if its partial derivatives of all orders exist.

A function $D : \mathbb{R}^2_{++} \times \mathbb{R}_+ \to \mathbb{R}^2_+$ is a *demand function* if it is homogeneous of degree 0 and satisfies $p \cdot D(p, I) = I$, for all $p \in \mathbb{R}^2_{++}$ and $I \in \mathbb{R}_+$.

Say that a demand function satisfies *gross substitutes* if, for fixed p_1 and I, $p_2 \mapsto D_1((p_1, p_2), I)$ is increasing, and for fixed p_2 and I, $p_1 \mapsto D_1((p_1, p_2), I)$ is increasing.

For all $(p, I) \in \mathbb{R}^2_{++} \times \mathbb{R}_+$, define the *budget* $B(p, I)$ by $B(p, I) = \left\{x \in \mathbb{R}^2_+ : p \cdot x \leq I\right\}$. Note that $B(p, I)$ is compact, by the assumption that prices are strictly positive.

A demand function D is *rational* if there is an increasing function $u : \mathbb{R}^2_+ \to \mathbb{R}$ such that

$$D(p, I) = \text{argmax}_{x \in B(p,I)} u(x). \tag{1}$$

In that case, we say that u is a *rationalization of* (or that it *rationalizes*) D. Note that part of the definition of rationalizability is that $D(p, I)$ is the unique maximizer of u in $B(p, I)$.

3. Result

We shall use homogeneity (and budget balance) to regard demand as only a function of prices: $D(p, I) = D((1/I)p, 1)$, so we can normalize income to 1. In this case, we regard demand as a function $D : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ with $p \cdot D(p) = 1$ for all $p \in \mathbb{R}^2_{++}$.

A *partial demand function* is a function $D : P \to \mathbb{R}^2_+$ where $P \subseteq \mathbb{R}^2_{++}$ and $p \cdot D(p) = 1$ for every $p \in P$; P is called *the domain* of D. So a demand function is a partial demand function whose domain is $\mathbb{R}^2_{++}$. The concept of the partial demand function allows us to study finite demand observations. We imagine that we have observed demand at all prices in P (see e.g. Afriat (1967), Diewert and Parkan (1983) or Varian (1982)).

Theorem 1. *Let Q be a finite subset of $\mathbb{R}^2_{++}$ and let $D : Q \to \mathbb{R}^2_+$ be a partial demand function. Then D is the restriction to Q of a smooth and rational demand satisfying gross substitutes if and only if $q'_1 D_1(q') \leq q_1 D_1(q)$ for every $q, q' \in Q$ such that $q_1 \leq q'_1$ and $q'_2 \leq q_2$.*

Remark. Let D be a partial demand function. The condition in the theorem is a statement about $q, q' \in Q$ such that $q_1 \leq q'_1$ and $q'_2 \leq q_2$: as such it has bite whenever two price vectors are ordered by good 1 being more expensive and good 2 being cheaper. The definition of substitutes has bite when one good becomes more expensive while the price of other stays the same. The statements are equivalent when $Q = \mathbb{R}^2_{++}$, but not for general partial demand functions. So the condition in the theorem requires a kind of strengthening of the order on prices in order to have bite on arbitrary finite sets of prices.

4. Proof of Theorem 1

4.1. **Demand functions on $\mathbb{R}^2_{++}$.** Say that a partial demand function satisfies the *weak axiom of revealed preference* if $p \cdot D(p') > 1$ whenever $p' \cdot D(p) < 1$. With two goods, the weak axiom is equivalent to the strong axiom of revealed preference, and hence characterizes rational demand.[2]

[2]The weak axiom of revealed preference is often stated as saying if $D(p') \neq D(p)$, and $p' \cdot D(p) \leq 1$, then $p \cdot D(p') > 1$. This condition is equivalent to ours in the case of two goods.

Throughout this section we fix a demand function $D:\mathbb{R}^2_{++}\to\mathbb{R}^2_+$. We introduce the expenditure share function associated to D: for $i\in\{1,2\}$, let $\pi_i=\pi_i^{(D)}:\mathbb{R}^2_{++}\to[0,1]$ be given by $\pi_i(p)=p_iD_i(p)$. Let $\preceq$ be the partial order over $\mathbb{R}^2_{++}$ that is given by

$$p'\preceq p \iff p'_1\geq p_1 \text{ and } p'_2\leq p_2.$$

Say that a function $f:\mathbb{R}^2_{++}\to\mathbb{R}$ is $\preceq$- *increasing* if $x\preceq y$ implies $f(x)\leq f(y)$.

Observe that

$$\pi_i(p_1,p_2)=p_iD_i(p_1,p_2)=1-p_{-i}D_{-i}(p_1,p_2).$$

Hence the property of substitutes is equivalent to $p_i\mapsto\pi_{-i}(p)$ being increasing in p_i for every p_{-i}, by the first equality, and to $p_i\mapsto\pi_i(p)$ is decreasing in p_i for every p_{-i} by the second equality. Thus, D satisfies substitutes if and only if π_1 is $\preceq$-increasing, i.e. $\pi_1(p')\geq\pi_1(p)$ whenever $p'\preceq p$.

Lemma 1. *If D satisfies gross substitutes and $p'\preceq p$ then $D_1(p')\leq D_1(p)$ and $D_2(p')\geq D_2(p)$.*

Proof. Let p'' be given by $p''_1=p'_1$ and $p''_2=p_2$. Then

$$D_1(p')\leq D_1(p'')\leq D_1(p),$$

where the first inequality follows from the definition of substitutes and the second from our observation that π_1 is decreasing in p_1 and from $D_1(p_1,p_2)=\pi_1(p_1,p_2)/p_1$. The second assertion follows from symmetry between the products. ∎

The following lemma was shown by Kehoe and Mas-Colell (1984) for *excess demand* functions, and the case of three goods.

Lemma 2. *If D satisfies gross substitutes then D satisfies the weak axiom of revealed preference.*

Proof. Assume that $p,p'\in\mathbb{R}^2_{++}$ and let $x=D(p)$ and $x'=D(p')$. Assume that $p'\cdot x<1$. We claim that $p\cdot x'>1$, which proves the weak axiom of revealed preference. Indeed, if $p'\geq p$ then $p'\cdot x\geq p\cdot x=1$, a contradiction. Assume therefore without loss of generality that $p'_2 < p_2$. If $p'_1 < p_1$ then $p' \ll p$ and therefore

$$p \cdot x' > p' \cdot x' = 1,$$

as desired. Assume therefore that $p'_1 \geq p_1$, so that $p' \preceq p$. Then, it follows from Lemma 1 that $x'_1 \leq x_1$ and $x'_2 \geq x_2$. Therefore

$$p \cdot x' = p_1 \cdot x'_1 + p_2 \cdot x'_2 =$$
$$p \cdot x + p' \cdot x' - p' \cdot x + (p_1 - p'_1) \cdot (x'_1 - x_1) + (p'_2 - p_2) \cdot (x_2 - x'_2) > 1,$$

since $p \cdot x = p' \cdot x' = 1$. ∎

4.2. **Partial Demand.** The condition in the theorem is equivalent to $q'_1 D_1(q') \leq q_1 D_1(q)$ for every $q, q' \in Q$ such that $q' \preceq q$. Necessity is evident from our discussion above. For sufficiency, fix $\epsilon > 0$ such that

$$\text{if } q'_i > q_i - 2\epsilon \text{ then } q'_i \geq q_i, \tag{2}$$

for every $q, q' \in Q$ and every $i \in \{1, 2\}$. Let $\pi : \mathbb{R}^2 \to [0, 1]$ be given by

$$\pi(p) = \max\{q_1 \cdot D_1(q) | q \in S(p)\} \tag{3}$$

where $S(p) = \{q \in Q | q_1 > p_1 - \epsilon \text{ and } q_2 < p_2 + \epsilon\}$ and the maximum of the empty set is by definition 0. If $p' \preceq p$ then $S(p') \subseteq S(p)$. Therefore π is $\preceq$-monotone.

Claim 3. *If* $\max(|p_1 - q_1|, |p_2 - q_2|) < \epsilon$ *for some* $p \in \mathbb{R}^2_{++}$ *and* $q \in Q$ *then* $\pi(p) = q_1 D_1(q)$.

Proof. Since $q \in S(p)$ it follows from (3) that $\pi(p) \geq q_1 D_1(q)$. On the other hand, let $q' \in S(p)$. Then $q'_1 > p_1 - \epsilon > q_1 - 2\epsilon$ and therefore $q'_1 \geq q_1$ by (2) and $q'_2 < p_2 + \epsilon < q_2 + 2\epsilon$ and therefore $q'_2 \leq q_2$ by (2). Thus $q' \preceq q$ and therefore $q'_1 D_1(q') \leq q_1 D_1(q)$. Since this is true for every $q' \in S(p)$ it follows from (3) that $\pi(p) \leq q_1 D_1(q)$. ∎

Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function such that $\psi(\tau) \geq 0$ for every $\tau \in \mathbb{R}^2$, $\psi(\tau) = 0$ whenever $\max(|\tau_1|, |\tau_2|) \geq \epsilon$ and

$$\int_{\mathbb{R}^2} \psi = 1. \tag{4}$$

For example, we can choose

$$\psi(x,y) = \begin{cases} \frac{1}{C} e^{-1/(1-(x/\epsilon)^2)-1/(1-(y/\epsilon)^2)}, & \text{if } |x| < \epsilon \text{ and } |y| < \epsilon \\ 0, & \text{otherwise,} \end{cases}$$

for a suitable normalizing factor C.

Let $\tilde{\pi} : \mathbb{R}^2_{++} \to [0,1]$ be given by $\tilde{\pi} = \psi * \pi$, i.e.

$$\tilde{\pi}(p) = \int_{\mathbb{R}^2} \psi(\tau)\pi(p-\tau)d\tau.$$

Then $\tilde{\pi}$ is smooth (as a convolution of a smooth function with a bounded function), $\preceq$-monotone (as a convolution of a nonnegative function with a $\preceq$-monotone function) and $\tilde{\pi}(q) = q_1 D_1(q)$ for every $q \in Q$ by Claim 3 and the properties of ψ. Finally, Let $\tilde{D} : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be the demand function given by $\tilde{D}_1(p) = \tilde{\pi}(p)/p_1$ and $\tilde{D}_2(p) = (1 - \tilde{\pi}(p))/p_2$. Then $\tilde{D}$ is a smooth demand function that satisfies gross substitutes by our argument above, establishing the equivalence between gross substitutes and the monotonicity of π. By Lemma 2 $\tilde{D}$ satisfies the weak axiom, and hence it is rational.

Remark. There is a simple proof of sufficiency if one does not insist on smooth demand. Let D be a partial demand function with domain Q and such that the condition in the theorem is satisfied. Extend the resulting π from Q to $\mathbb{R}^2_{++}$ by

$$\pi_1(p) = \max\{\pi(q) : q \in Q, q \preceq p\}.$$

Then π is $\preceq$-monotone. Observe that π defines a demand function that satisfies substitutes (hence is rational by Lemma 2) and coincides with D on Q.

Remark. In a related context of revealed preferences, Chiappori and Rochet (1987) use a similar smoothing technique to construct a rationalizing differentiable utility. A difference between their paper (and most of the revealed preference literature) and ours is that we do not construct a rationalizing utility: We extend the demand function to satisfy substitutes, and then use an integrability argument to show that it is a rational demand.

5. Conclusion and remarks

The property of gross substitutes is of obvious interest to economists. One indication of this fact is that all the "principles of economics" courses that we are aware of discuss gross substitutes. We believe that the simple test we have developed is of interest as well.

We have determined the testable implications of gross substitutes of a pair of goods. The implications are very simple: the data itself must satisfy the definition in the form of the monotonicity of expenditure shares. One might want to study more than two goods; for example whether beer, wine, and whisky are substitutes. We note that such a study would still require the type of assumptions we implicitly made: one needs to aggregate different types of beer, for example, into a composite beer good. One would also need to separate the three goods from the steaks, fishes, and salads on which the consumer also decides.

Finally, the natural extension of the condition in Theorem 1 for more than two goods is that, for every partition (I, J) of the products set, $q'_I \cdot D_I(q') \leq q_I \cdot D_I(q)$ for every $q, q' \in Q$ such that $q_i \leq q'_i$ for every $i \in I$ and $q'_j \leq q_j$ for every $j \in J$. Following a referee's suggestion, we call this condition *n-good expenditure-share monotonicity* (ECM). It is easy to verify that ECM is satisfied by any demand function that satisfies gross substitutes, the argument is similar to the argument for two goods. On the other hand –unlike in the two good case– if a partial demand function satisfies ECM, it does not need to be extendable to a full demand function that satisfies gross substitutes; we present an example to show this point.

Example 1. We suppose that we have data $\{(x^a, p^a), (x^b, p^b), (x^c, p^c)\}$ given, meaning that we have a partial demand function defined on three observations, where $x^a = D(p^a)$ and $x^b = D(p^b)$, $x^c = D(p^c)$. Thus, $Q = \{p^a, p^b, p^c\}$. All vectors lie in $\mathbb{R}^3$;

the observations are in the table below. For convenience the table is given in terms of expenditure shares, so that $\pi_i = p_i \cdot x_i$ for every product $i \in \{1, 2, 3\}$. The

	p_1	p_2	p_3	π_1	π_2	π_3
(p^a, x^a)	1	2	3	3/10	1/2	2/10
(p^b, x^b)	2	3	1	1/2	2/10	3/10
(p^c, x^c)	3	1	2	2/10	3/10	1/2

data satisfy ECM but cannot be extended to $p = (3/2, 3/2, 3/2)$ such that ECM is satisfied.

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